Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 12 (2019), 217–229
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

The transmuted transmuted-G family: properties and applications

M. M. Mansour[a,b], Enayat M. Abd Elrazik[a,b], Ahmed Z. Afify[b], Mohammad Ahsanullah[c], Emrah Altun[d,*]

[a]Department of MIS, Yanbu, Taibah University, Saudi Arabia.
[b]Department of Statistics, Mathematics and Insurance, Benha University, Egypt.
[c]Department of Management Sciences, Rider University NJ, USA.
[d]Department of Statistics, Bartin University, Bartin 74100, Turkey.

Abstract

This paper introduces a new family of continuous distributions called the transmuted transmuted-G family which extends the quadratic rank transmutation map pioneered by Shaw and Buckley [W. T. Shaw, I. R. Buckley, arXiv preprint, **2007** (2007), 28 pages]. We provide two special models of the new family which can be used effectively to model survival data since they accommodate increasing, decreasing, unimodal, bathtub-shaped and increasing-decreasing-increasing hazard functions. We also provide two new characterization theorems of the proposed family. The estimation of the model parameters is performed by the maximum likelihood method. The flexibility of the proposed family is illustrated by means of two applications to real data.

Keywords: Characterization, maximum likelihood, moments, transmuted family.

2010 MSC: 60E05, 62E10, 62E15.

1. Introduction

Shaw and Buckley [24] proposed the quadratic rank transmutation map, also known as transmuted-G (TG) class. This class of generalized distributions has been receiving considerable attention over the last years, in particular after the recent works of transmuted generalized extreme value due to Aryal and Tsokos [9], transmuted Weibull due to Aryal and Tsokos [10], transmuted additive Weibull due to Elbatal and Aryal [12], transmuted Lindley-geometric due to Merovci and Elbatal [21], transmuted complementary Weibull geometric due to Afify et al. [3], new transmuted Lindley due to Mansour and Mohamed [17], transmuted Marshall-Olkin Fréchet due to Afify et al. [2] and transmuted Weibull-Pareto due to Afify et al. [4] distributions, among others.

Furthermore, there are some extensions for the TG family in the literature. For example, the complementary generalized transmuted Poisson-G due to Alizadeh et al. [7], another generalized transmuted

*Corresponding author
Email addresses: mmmansour@taibahu.edu.sa (M. M. Mansour), enayat15@yahoo.com (Enayat M. Abd Elrazik), ahmed.afify@fcom.bu.edu.eg (Ahmed Z. Afify), ahsan@rider.edu (Mohammad Ahsanullah), emrahaltun@bartin.edu.tr (Emrah Altun)

doi: 10.22436/jnsa.012.04.03

Received: 2018-06-26 Revised: 2018-10-05 Accepted: 2018-11-17

due to Merovci et al. [19], the exponentiated transmuted-G due to Merovci et al. [20] and the beta transmuted-H due to Afify et al. [5], among others.

The cumulative distribution function (cdf) and probability density function (pdf) of the TG family are given by

$$H(x;\lambda,\varphi) = (1+\lambda)G(x;\varphi) - \lambda\left[G(x;\varphi)\right]^2 \tag{1.1}$$

and

$$h(x;\lambda,\varphi) = g(x;\varphi)\left[1+\lambda-2\lambda G(x;\varphi)\right], \tag{1.2}$$

respectively, where $G(x;\varphi)$ is the baseline cdf with a parameter vector φ, where $\varphi = \varphi_k = (\varphi_1, \varphi_2, \ldots)$ and $|\lambda| \leqslant 1$. Further details were explored by Shaw and Buckley [24].

In this paper, we propose the transmuted transmuted-G (TT-G) family of distributions, which extends the TG family by incorporating an additional transmuted parameter to generate more flexible distributions. The proposed distribution gives better fits over a large number of well-known lifetime distributions, including those with three and four parameters. We provide a comprehensive account of some of its mathematical properties. In fact, the TT-G family has a good physical interpretation (see Section 2).

The rest of the paper is organized as follows. In Section 2, we define the TT-G family, provide its special cases and give a very useful representation for its density function. In Section 3, we provide two special models corresponding to the baseline Weibull and Lindley distributions. We derive, in Section 4, some mathematical properties of the TT-G family. We give two important characterization theorems in Section 5. Maximum likelihood estimation of the model parameters is addressed in Section 6. In Section 7, we illustrate the flexibility of the new family by means of two applications to real data. A small simulation study is carried out in Section 8. Finally, some concluding remarks are provided in Section 9.

2. The TT-G family

Let $p(t)$ be the pdf of a random variable $T \in [b, c]$ for $-\infty < b < c < \infty$ and let $W[H(x)]$ be a function of the cdf of a random variable X such that $W[H(x)]$ satisfies the following conditions:

(i) $W[H(x)] \in [b, c]$;
(ii) $W[H(x)]$ is diferentiable and monotonically non-decreasing, and;
(iii) $W[H(x)] \to b$ as $x \to -\infty$ and $W[|H(x)] \to c$ as $x \to \infty$.

Recently, Alzaatreh et al. [8] defined the T-X family of distributions by

$$F(x) = \int_b^{W[H(x)]} p(t)dt, \tag{2.1}$$

where $W[H(x)]$ satisfies conditions (i)-(iii). The pdf corresponding to (2.1) is given by

$$f(x) = \left\{\frac{d}{dx}W[H(x)]\right\} p\left\{W[H(x)]\right\}.$$

For $W[H(x)] = H(x;\lambda,\varphi)$ given in (1.1) and $p(t) = 1 + a - a2t$, $0 < t < 1$, we define the cdf of the new TT-G family of distributions by

$$F(x;a,\lambda,\varphi) = \int_0^{H(x;\lambda,\varphi)} (1+a-a2t)dt = (1+a)H(x;\lambda,\varphi) - a\left[H(x;\lambda,\varphi)\right]^2. \tag{2.2}$$

The corresponding pdf is given by

$$f(x;a,\lambda,\varphi) = h(x;\lambda,\varphi)\left[1+a-2aH(x;\lambda,\varphi)\right], \tag{2.3}$$

where $H(x;\lambda,\varphi)$ is the TG cdf (1.1), $h(x;\lambda,\varphi)$ is the TG pdf (1.2) and $-1 \leqslant a, \lambda \leqslant 1$ are two additional transmuted parameters. We denote by X ~TT-G(a,λ,φ) a random variable having the density function (2.3). Further, we will exclude the reliance on the model parameters and write simply $F(x) = F(x; a, \lambda, \varphi)$, $H(x) = H(x;\lambda,\varphi)$ and $f(x) = f(x; a, \lambda, \varphi)$, etc.. Clearly, for $a = 0$, the TT-G family reduces to the TG family. For $a = 0$ and $\lambda = 0$, the TT-G reduces to the base distribution.

The TT-G family reduces to the TG family when $a = 0$. So, there are more than 50 well-know distributions in the literature as special models of the TT-G family which are listed in Tahir and Cordeiro [25].

The hazard rate function (hrf) associated with (2.2), $\eta(x; a, \lambda, \varphi)$, is given by

$$\eta(x; a, \lambda, \varphi) = \frac{h(x;\lambda,\varphi)\,[1 + a - 2aH(x;\lambda,\varphi)]}{1 - H(x;\lambda,\varphi)\,[1 + a - aH(x;\lambda,\varphi)]}.$$

We give a valuable representation to (2.3) using the concept of exponentiated-G (EG) distributions. The TT-G family is a mixture of the TG and exponentiated-TG (ETG) distributions, the last one with power parameter 2.

Let Y_α be a random variable following the ETG class with parameter $\alpha = 1, 2$, say Y_1 ~TG(λ,φ) and Y_2 ~ETG$(2,\lambda,\varphi)$, i.e., its cdf and pdf are

$$H_1(x) = H(x;\lambda,\varphi), \quad H_2(x) = [H(x;\lambda,\varphi)]^2, \quad \text{and} \quad h_1(x) = h(x;\lambda,\varphi), \quad h_2(x) = 2h(x;\lambda,\varphi)H(x;\lambda,\varphi),$$

respectively. The properties of the TG distributions have been studied by many researchers in the last twenty years as seen from Table 1.

Hence, the TT-G family density in (2.3) can be expressed as

$$f(x; a, \lambda, \varphi) = (1 + a)h_1(x;\lambda,\varphi) - ah_2(x;\lambda,\varphi). \tag{2.4}$$

Equation (2.4) reveals that the TT-G density function is a mixture of two ETG densities. Thus, some mathematical properties of the new family can be derived from those properties of the ETG class. For example, the ordinary and incomplete moments and mgf of X can be obtained directly from those quantities of the ETG class. Further information about exponentiated distributions can be explored in Al-Hussaini and Ahsanullah [6].

3. Special TT-G distributions

In this section, we provide two special cases of the TT-G family. The pdf (2.3) will be most tractable when $G(x;\varphi)$ and $g(x;\varphi)$ have simple analytic expressions. The two special models corresponding to the Weibull (W) and Lindley (Li) distributions. The pdf and cdf of the W distribution with positive parameters α and θ model are given (for $x > 0$) by W: $g(x) = \alpha\theta^\alpha x^{\alpha-1} e^{-(\theta x)^\alpha}$ and $G(x) = 1 - e^{-(\theta x)^\alpha}$.

The pdf and cdf of the Li distribution with positive parameter θ are given (for $x > 0$) by Li: $g(x) = \frac{\theta^2}{1+\theta}(1 + x)e^{-\theta x}$ and $G(x) = 1 - \frac{1+\theta+\theta x}{1+\theta} e^{-\theta x}$.

3.1. The TTW distribution

The TTW pdf follows from (2.3) as

$$f(x) = \alpha\theta^\alpha x^{\alpha-1} e^{-(\theta x)^\alpha} \left[1 - \lambda + 2\lambda e^{-(\theta x)^\alpha}\right] \left(1 + a - 2a \left\{(1 + \lambda) \left[1 - e^{-(\theta x)^\alpha}\right] - \lambda \left[1 - e^{-(\theta x)^\alpha}\right]^2 \right\}\right).$$

The TTW distribution includes the TW distribution with $a = 0$. For $\alpha = 2$, we obtain the TT-Rayleigh (TTR) distribution. For $\alpha = 1$, we obtain the TT-exponential (TTEx) distribution. For $a = 0$ and $\alpha = 2$, we obtain the TR distribution. For $a = 0$ and $\alpha = 1$, we obtain the TEx distribution. For $a = 0$ and $\lambda = 0$, we have the W distribution. For $a = 0$, $\lambda = 0$, and $\alpha = 2$, we have the R distribution. For $a = 0$, $\lambda = 0$, and

$\alpha = 1$, we have the Ex distribution. Plots of the pdf and hrf of the TTW distribution for some parameter values are displayed in Figure 1.

The pdf plots reveals that the TTW model can be reversed J-shape, unimodal or left skewed. The TTW hrf can be decreasing, increasing, increasing then bathtub, upside down bathtub or reversed J-shape.

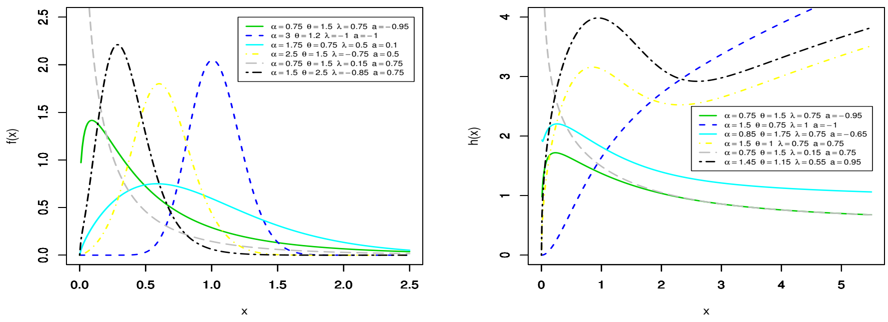

Figure 1: Plots of the pdf and hrf of the TTW for some parameter values.

3.2. *The TTLi distribution*

The TTLi pdf is given by

$$f(x) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}\left(1-\lambda+2\lambda d e^{-\theta x}\right)\left\{1+a-2a\left[(1+\lambda)\left(1-de^{-\theta x}\right)-\lambda\left(1-de^{-\theta x}\right)^2\right]\right\},$$

where $d = (1+\theta+\theta x)/(1+\theta)$.

For $a = 0$, the TTLi distribution reduces to the TLi distribution. For $a = 0$ and $\lambda = 0$, we have the Li distribution. Plots of the pdf and hrf of the TTLi model for some selected parameter values are displayed in Figure 2. Figure 2 reveals that the pdf of the TTLi model can be reversed J-shape, unimodal or left skewed. The TTLi hrf can be decreasing, increasing, upside down bathtub or bathtub failure rate shapes.

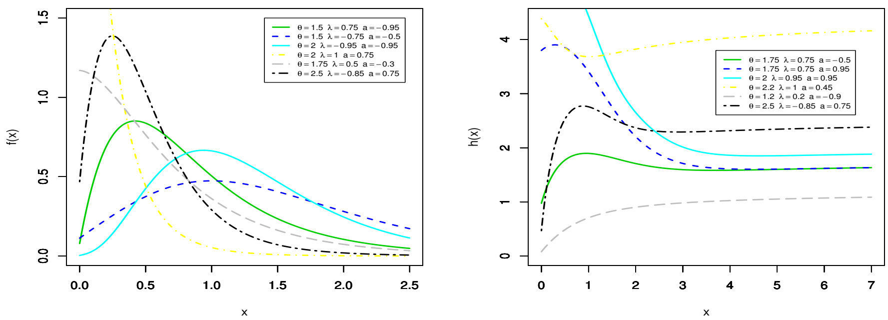

Figure 2: Plots of the pdf and hrf of the TTLi model for some parameter values.

4. Mathematical properties

In this section, we investigate mathematical properties of the TT-G family of distributions. The numerical commutations of the formulae derived in this section and also in other sections of this paper can easily be derived using Mathlab, Mathematica, and Maple.

4.1. Ordinary and incomplete moments

The r^{th} ordinary moment of X, say $\acute{\mu}_r$, follows from (2.4) as

$$\acute{\mu}_r = E(X^r) = (1+a)E(Y_1^r) - aE(Y_2^r), \tag{4.1}$$

where $E(Y_1^r)$ and $E(Y_2^r)$ can be computed numerically in terms of the baseline quantile function (qf) $Q_H(u, \lambda, \varphi) = H^{-1}(x, \lambda, \varphi)$ as

$$E(Y_1^r) = \int_0^1 Q_H(u; \lambda, \varphi)^r du, \quad E(Y_2^r) = 2\int_0^1 Q_H(u; \lambda, \varphi)^r u du.$$

Setting $r = 1$ in (4.1) gives mean of X.

The n^{th} incomplete moment of X is defined by $m_n(y) = \int_{-\infty}^{y} x^n f(x)dx$ and we have

$$m_n(y) = (1+a)m_{1n}(y) - am_{2n}(y),$$

where

$$m_{1n}(y) = \int_0^{H(y,\lambda,\varphi)} Q_H(u; \lambda, \varphi)du, \quad m_{2n}(y) = 2\int_0^{H(y,\lambda,\varphi)} Q_H(u; \lambda, \varphi)u du.$$

The integrals $m_{1n}(y)$ and $m_{2n}(y)$ can be determined analytically for special models with closed-form expressions for $Q_H(u, \lambda, \varphi)$ or computed at least numerically for most baseline distributions. The incomplete moments play an important role in measuring inequality, for example, income quantiles and Lorenz and Bonferroni curves, which depend upon the incomplete moments of a distribution.

5. Characterization

We will use the following assumption in our characterization.

Assumption 5.1. *Suppose the random variable* X *has an absolutely continuous distribution with cdf* $F(x)$ *and pdf* $f(x)$ *for* $\gamma < x < \delta$. *We assume further* $E(X)$ *exists.*

Lemma 5.2. *Under the assumption A for the random variable* X, *if*

$$E(X|X \leqslant x) = g_1(x)\frac{f(x)}{F(x)},$$

where $g_1(x)$ *is a continuous differentiable function in* $\gamma < x < \delta$, *then*

$$f(x) = ce^{\int \frac{x - g_1'(x)}{g_1(x)}dx},$$

where c *is determined by the condition* $\int_\gamma^\delta f(x)dx = 1$.

Lemma 5.3. *Under the Assumption 5.1 for the random variable* X, *if*

$$E(X|X \geqslant x) = g_2(x)\frac{f(x)}{1 - F(x)},$$

where $g_2(x)$ *is a continuous differentiable function in* $\gamma < x < \delta$, *then*

$$f(x) = ce^{-\int \frac{x + g_2'(x)}{g_2(x)}dx},$$

where c *is determined by the condition* $\int_\gamma^\delta f(x)dx = 1$.

The proofs of the Lemmas are easy to establish.

Theorem 5.4. *Under the Assumption 5.1 for the random variable* X*, if*

$$E(X|X \leqslant x) = g_1(x)\frac{f(x)}{F(x)},$$

where

$$g_1(x) = \frac{1}{f(x)}[(1+a)m_1(x) - 2am_2(x)], \quad m_1(x) = \int_\gamma^x uh(u,\lambda,\phi)du, \quad m_2(x) = \int_\gamma^x uh(u,\lambda,\phi)H(u,\lambda,\phi)du,$$

and where $h(x,\lambda,\phi)$ *is a pdf in* $\gamma < x < \delta$ *and* $H(u,\lambda,\phi)$ *is the corresponding cdf, then*

$$f(x) = h(x,\lambda,\phi)\left[1 + a - 2aH(u,\lambda,\phi)\right].$$

Proof. We will write $f(x)$ as

$$f(x) = (1+a)h(x,\lambda,\phi) - 2ah(x,\lambda,\phi)H(u,\lambda,\phi).$$

Then

$$g_1(x) = \frac{\int_\gamma^x uf(u)du}{f(x)} = \frac{1}{f(x)}\left[(1+a)m_1(x) - 2am_2(x)\right].$$

Suppose that

$$g_1(x) = \frac{1}{f(x)}\left[(1+a)m_1(x) - 2am_2(x)\right].$$

Hence

$$g_1'(x) = x - g_1(x)\left\{\frac{(1+a)h'(x,\lambda,\phi) - 2a\left[h'(x,\lambda,\phi)H(u,\lambda,\phi)\right] + h'(x,\lambda,\phi)^2}{(1+a)h(x,\lambda,\phi) - 2ah(x,\lambda,\phi)H(u,\lambda,\phi)}\right\}.$$

Thus

$$\frac{x - g_1'(x)}{g(x)} = \frac{(1+a)h'(x,\lambda,\phi) - 2a\left[h'(x,\lambda,\phi)H(u,\lambda,\phi)\right] + h'(x,\lambda,\phi)^2}{(1+a)h(x,\lambda,\phi) - 2ah(x,\lambda,\phi)H(u,\lambda,\phi)}.$$

By Lemma 5.2

$$\frac{f'(x)}{f(x)} = \frac{(1+a)h'(x,\lambda,\phi) - 2a\left[h'(x,\lambda,\phi)H(u,\lambda,\phi)\right] + h'(x,\lambda,\phi)^2}{(1+a)h(x,\lambda,\phi) - 2ah(x,\lambda,\phi)H(u,\lambda,\phi)}.$$

On integrating the above equality with respect to x, we obtain

$$f(x) = c\left[(1+a)h(x,\lambda,\phi) - 2ah(x,\lambda,\phi)H(u,\lambda,\phi)\right].$$

Using the boundary condition $\int_\gamma^\delta f(x)dx = 1$, we obtain $c = 1$. □

Theorem 5.5. *Under the Assumption 5.1 for the random variable* X*, if*

$$E(X|X \geqslant x) = g_2(x)\frac{f(x)}{1 - F(x)},$$

where

$$g_2(x) = \frac{1}{f(x)}\left[(1+a)m_3(x) - 2am_4(x)\right], \quad m_3(x) = \int_x^\delta uh(u,\lambda,\phi)du, \quad m_4(x) = \int_x^\delta uh(u,\lambda,\phi)H(u,\lambda,\phi)du,$$

and where $h(x,\lambda,\phi)$ *is a pdf in* $\gamma < x < \delta$ *and* $H(u,\lambda,\phi)$ *is the corresponding cdf. Then*

$$f(x) = h(x,\lambda,\phi)\left[1 + a - 2aH(u,\lambda,\phi)\right].$$

Proof. We will write $f(x)$ as

$$f(x) = (1+a)h(x,\lambda,\phi) - 2ah(x,\lambda,\phi)H(u,\lambda,\phi).$$

Then

$$g_2(x) = \frac{\int_x^\delta uf(u)du}{f(x)} = \frac{1}{f(x)}\left[(1+a)m_3(x) - 2am_4(x)\right].$$

Suppose that

$$g_2(x) = \frac{1}{f(x)}\left[(1+a)m_3(x) - 2am_4(x)\right].$$

Then

$$g_2'(x) = -x - g_2(x)\left\{\frac{(1+a)h'(x,\lambda,\phi) - 2a\left[h'(x,\lambda,\phi)H(u,\lambda,\phi)\right] + h'(x,\lambda,\phi)^2}{(1+a)h(x,\lambda,\phi) - 2ah(x,\lambda,\phi)H(u,\lambda,\phi)}\right\}.$$

Thus

$$\frac{x+g_2'(x)}{g_2(x)} = -\frac{(1+a)h'(x,\lambda,\phi) - 2a\left[h'(x,\lambda,\phi)H(u,\lambda,\phi)\right] + h'(x,\lambda,\phi)^2}{(1+a)h(x,\lambda,\phi) - 2ah(x,\lambda,\phi)H(u,\lambda,\phi)}.$$

By Lemma 5.3

$$\frac{f'(x)}{f(x)} = \frac{(1+a)h'(x,\lambda,\phi) - 2a\left[h'(x,\lambda,\phi)H(u,\lambda,\phi)\right] + h'(x,\lambda,\phi)^2}{(1+a)h(x,\lambda,\phi) - 2ah(x,\lambda,\phi)H(u,\lambda,\phi)}.$$

On integrating the above equality with respect to x, we obtain

$$f(x) = c\left[(1+a)h(x,\lambda,\phi) - 2ah(x,\lambda,\phi)H(u,\lambda,\phi)\right].$$

Using the boundary condition $\int_\gamma^\delta f(x)dx = 1$, we obtain $c = 1$,

$$g'(x) = x - g(x)\left[\frac{(1+a)h'(x) - 2ah_1'(x)}{(1+a)h(x) - 2ah_1(x)}\right], \quad \frac{d}{dx}\ln\left[(1+a)h(x) - 2ah_1(x)\right] = \frac{(1+a)h'(x) - 2ah_1'(x)}{(1+a)h(x) - 2ah_1(x)}. \quad \square$$

6. Maximum likelihood estimation

In this section, we obtain the maximum likelihood estimates (MLEs) of the parameters of the TT-G distribution from complete samples only. Let $X_1, X_2, \ldots, X_n$ be observed values from the TT-G distribution with parameters a, λ, and φ . Let $\Theta = (a, \lambda, \varphi)^T$ be the $p \times 1$ parameter vector. Then, the all log-likelihood function for Θ is given by

$$\ell = \ell(\Theta) = \sum_{i=1}^{n} \log h(x;\lambda,\varphi) + \sum_{i=1}^{n} \log\left[1 + a - 2aH(x;\lambda,\varphi)\right].$$

The components of the score function $U_n(\theta) = (\partial\ell_n/\partial a, \partial\ell_n/\partial\lambda, \partial\ell_n/\partial\varphi)$ are

$$\frac{\partial\ell_n}{\partial a} = \sum_{i=1}^{n} \frac{-2H(x;\lambda,\varphi)}{[1 + a - 2aH(x;\lambda,\varphi)]},$$

$$\frac{\partial\ell_n}{\partial\lambda} = \sum_{i=1}^{n} \frac{h'_\lambda(x;\lambda,\varphi)}{h(x;\lambda,\varphi)} + \sum_{i=1}^{n} \frac{-2aH'_\lambda(x;\lambda,\varphi)}{[1 + a - 2aH(x;\lambda,\varphi)]},$$

and

$$\frac{\partial\ell_n}{\partial\varphi} = \sum_{i=1}^{n} \frac{h'_\varphi(x;\lambda,\varphi)}{h(x;\lambda,\varphi)} + \sum_{i=1}^{n} \frac{-2aH'_\varphi(x;\lambda,\varphi)}{[1 + a - 2aH(x;\lambda,\varphi)]},$$

where $h'_\varphi(.)$ means the derivative of the function h with respect to φ.

The maximum likelihood estimator $\hat{\Theta}$ is obtained by solving the nonlinear system of equations $\frac{\partial \ell_n}{\partial a} = \frac{\partial \ell_n}{\partial \lambda} = 0$ and $\frac{\partial \ell_n}{\partial \varphi_k} = 0$. It is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function.

7. Applications

In this section, we illustrate the importance and potentiality of the TTW and TTLi models, presented in Section 3, by means of two real data sets. The fitted models are compared using goodness-of-fit criteria namely: the Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC), Bayesian information criterion (BIC), $-2\widehat{\ell}$, where $\widehat{\ell}$ is the maximized log-likelihood, Anderson-Darling (A^*), and Cramér-von Mises (W^*). The smaller these statistics are, the better the fits.

7.1. Data set 1: failure times of 84 aircraft windshields

We consider the data on failure and service times for a particular model windshield given in Murthy et al. [22]. These data were recently studied by Ramos et al. [23]. The data consist of 153 observations, of which 88 are classified as failed windshields, and the remaining 65 are service times of windshields that had not failed at the time of observation. The unit for measurement is 1000 h. For this data set, we shall compare the fits of the TTW model with other models namely: the Kumaraswamy transmuted exponential (Kw-TE) by Afify et al. [1], McDonald Weibull (McW) by Cordeiro et al. [11], beta Weibull (BW) by Lee et al. [15], modified beta Weibull (MBW) by Khan [14], and transmuted exponentiated generalized Weibull (TExGW) by Yousof et al. [26] distributions, whose pdfs (for $x > 0$) are given by

$$\begin{aligned}
\text{Kw-TE:}\ \ f(x) &= \alpha a b e^{-\alpha x}\left\{1-\lambda+2\lambda e^{-\alpha x}\right\}\left\{\left(1-e^{-\alpha x}\right)\left[1+\lambda e^{-\alpha x}\right]\right\}^{a-1} \\
&\quad \times \left[1-\left\{\left(1-e^{-\alpha x}\right)\left[1+\lambda e^{-\alpha x}\right]\right\}^{a}\right]^{b-1}, \\
\text{McW:}\ \ f(x) &= \frac{\beta c \alpha^{\beta}}{B\left(a/c, b\right)} x^{\beta-1} e^{-(\alpha x)^{\beta}}\left[1-e^{-(\alpha x)^{\beta}}\right]^{a-1}\left\{1-\left[1-e^{-(\alpha x)^{\beta}}\right]^{c}\right\}^{b-1}, \\
\text{BW:}\ \ f(x) &= \frac{\beta \alpha^{\beta}}{B\left(a, b\right)} x^{\beta-1} e^{-b(\alpha x)^{\beta}}\left[1-e^{-(\alpha x)^{\beta}}\right]^{a-1}, \\
\text{MBW:}\ \ f(x) &= \frac{\beta \gamma^{a} \alpha^{-\beta}}{B\left(a, b\right)} x^{\beta-1} e^{-b\left(\frac{x}{\alpha}\right)^{\beta}}\left[1-e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right]^{a-1}\left\{1-(1-\gamma)\left[1-e^{-b\left(\frac{x}{\alpha}\right)^{\beta}}\right]\right\}^{-a-b}, \\
\text{TExGW:}\ \ f(x) &= a b \beta \alpha^{\beta} x^{\beta-1} e^{-a(\alpha x)^{\beta}}\left[1-e^{-a(\alpha x)^{\beta}}\right]^{b-1}\left\{1+\lambda-2\lambda\left[1-e^{-a(\alpha x)^{\beta}}\right]^{b}\right\}.
\end{aligned}$$

The parameters of the above densities are all positive real numbers except for the TExGW distributions for which $|\lambda| \leqslant 1$.

7.2. Data set 2: cancer patients data

The second data set on the remission times (in months) of a random sample of 128 bladder cancer patients, reported by Lee and Wang [16]. For this data set, we compare the fits of the TTLi distribution with some other competitive models namely: the new transmuted Lindley (NTLi) by Mansour and Mohamed [17], transmuted Lindley (TLi) by Merovci [18], power Lindley (PLi) by Ghitany et al. [13], Lindley (Li), and exponential (Ex) distributions. The pdfs of the NTLi, TLi, PLi, Li, and Ex distributions are give (for $x > 0$) by

$$\text{NTLi:}\ \ f(x) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}\left[\delta(1+\lambda)\left(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)^{\delta-1} - \alpha\lambda\left(1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)^{\alpha-1}\right],$$

$$\text{TLi:}\quad f(x) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}\left[1-\lambda+2\lambda\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right],$$

$$\text{PLi:}\quad f(x) = \frac{\alpha\theta^2}{1+\theta}x^{\alpha-1}(1+x^{\alpha})e^{-\theta x^{\alpha}},$$

$$\text{Li:}\quad f(x) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x},$$

$$\text{Ex:}\quad f(x) = \theta e^{-\theta x}.$$

The parameters of the above densities are all positive real numbers except for the NTLi and TLi distributions for which $|\lambda| \leqslant 1$.

Tables 1 and 3 list the values of $-2\hat{\ell}$, AIC, CAIC, BIC, HQIC, A^*, and W^* whereas the MLEs and their corresponding standard errors (in parentheses) of the model parameters are given in Tables 2 and 4. These numerical results are obtained using the R package.

The fitted pdf, cdf, sf, and Q-Q plots of the TTW and TTLi distributions are shown in Figures 3 and 4, respectively.

Table 1: Goodness-of-fit statistics for failure times data.

Model	$-2\hat{\ell}$	AIC	CAIC	BIC	HQIC	A^*	W^*
TTW	257.485	265.485	265.991	275.208	269.394	0.4755	0.0479
Kw-TE	257.596	265.596	266.103	275.32	269.505	0.5485	0.0578
McW	273.899	283.899	284.669	296.053	288.785	1.5906	0.1986
BW	297.028	305.028	305.534	314.751	308.937	3.2197	0.4652
MBW	299.573	309.573	310.342	321.727	314.459	3.2656	0.4717
TExGW	352.594	362.594	363.363	374.748	367.48	6.2332	1.0079

Table 2: MLEs and their standard errors (in parentheses) for failure times data.

Model	Estimates				
TTW	1.7205	0.4806	-0.5230	-0.6047	
$(\alpha, \theta, \lambda, a)$	(0.2899)	(0.0739)	(0.4106)	(0.3554)	
Kw-TE	0.0965	-0.8971	1.6346	65.0082	
(α, λ, a, b)	(0.053)	(0.129)	(0.35)	(76.536)	
McW	1.9401	0.306	17.686	33.6388	16.7211
(α, β, a, b, c)	(1.011)	(0.045)	(6.222)	(19.994)	(9.622)
MBW	10.1502	0.1632	57.4167	19.3859	2.0043
(α, β, a, b, c)	(18.697)	(0.019)	(14.063)	(10.019)	(0.662)
TExGW	4.2567	0.1532	0.0978	5.2313	1173.33
$(\alpha, \beta, \lambda, a, b)$	(33.401)	(0.017)	(0.609)	(9.792)	(129.165)
BW	1.36	0.2981	34.1802	11.4956	
(α, β, a, b)	(1.002)	(0.06)	(14.838)	(6.73)	

Table 3: Goodness-of-fit statistics for cancer data.

Model	$-2\hat{\ell}$	AIC	CAIC	BIC	HQIC	A^*	W^*
TTLi	825.603	831.603	831.796	840.159	835.079	0.4016	0.0684
NTLi	831.160	839.160	839.480	850.570	843.790	0.9330	0.1530
TLi	830.310	834.310	834.400	840.014	836.620	0.9530	0.1590
PLi	826.700	830.700	830.800	836.411	833.020	0.7000	0.1700
Li	839.050	841.050	841.090	843.910	842.210	1.0250	0.1710
Ex	828.680	830.680	830.710	833.530	831.840	0.7150	0.1190

Table 4: MLEs and their standard errors (in parentheses) for cancer data.

Model	Estimates			
TTLi	0.5382	0.5372	0.1298	
(a, λ, θ)	(0.2108)	(0.2278)	(0.0172)	
NTLi	-0.0317	0.1766	0.7548	23.5316
$(\lambda, \delta, \alpha, \theta)$	(0.0352)	(0.0233)	(0.1000)	(18.5785)
PLi	0.294	0.8302		
(θ, α)	(0.0370)	(0.047)		
TLi	0.6180	0.1550		
(λ, θ)	(0.168)	(0.0149)		
Li	0.1960			
(θ)	(0.0123)			
Ex	0.1060			
(θ)	(0.009)			

The figures in Tables 1 and 3 reveal that the TTW and TTLi models have the lowest values for goodness-of-fit statistics among all fitted models. So, They could be chosen to model the two data sets.

In Table 3, we compare the fits of the TTLi model with the NTLi, TLi, PLi, Li, and Ex models. It is noted that the TTLi model has the lowest values among the fitted models for goodness-of-fit statistics. We prove that the special models of new family can provide better fits than other competitive lifetime models. Figures 3 and 4 display the fitted pdf, cdf, sf, and Q-Q plots of TTW and TTLi distributions.

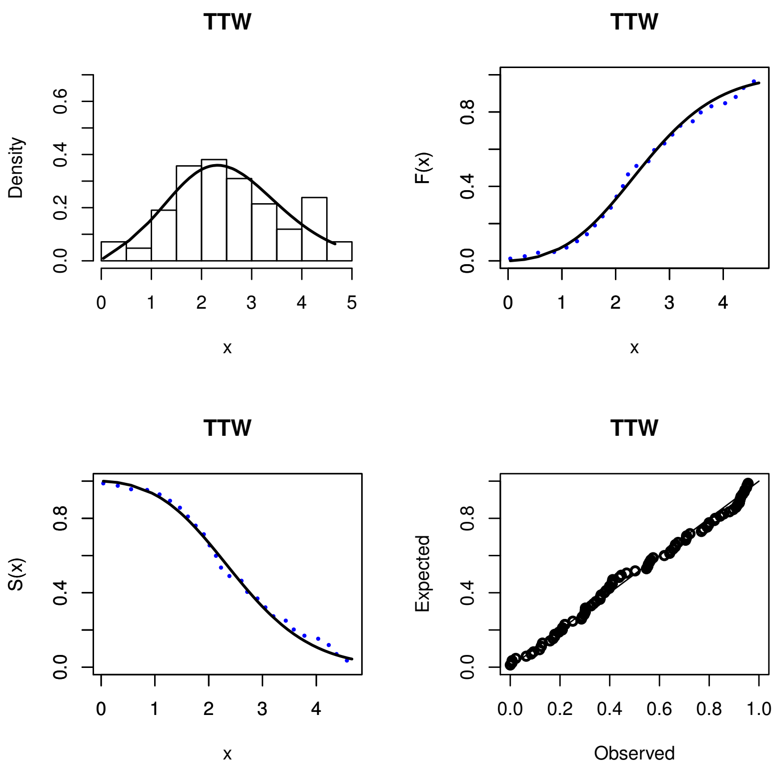

Figure 3: Fitted pdf, cdf, sf and Q-Q plots of the TTW distribution for failure times data.

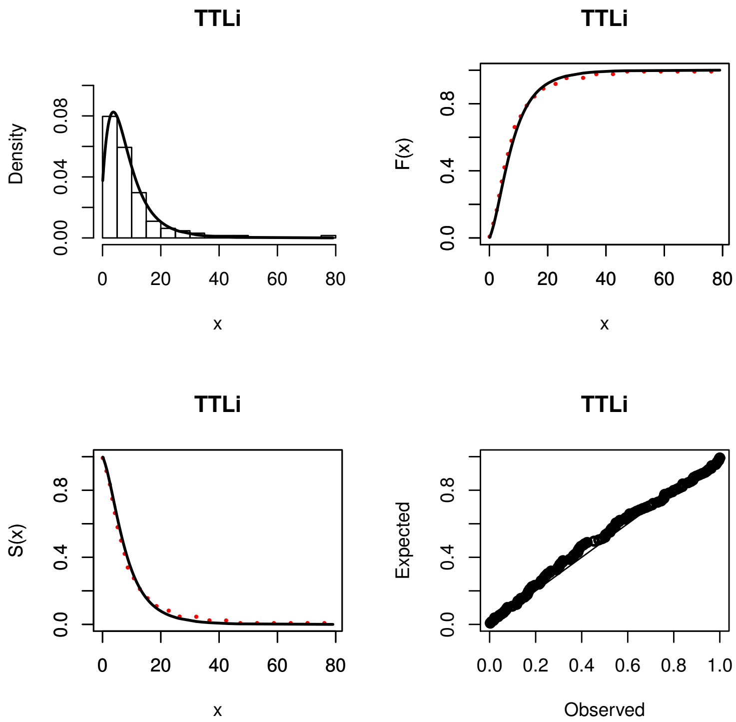

Figure 4: Fitted pdf, cdf, sf and Q-Q plots of the TTLi distribution for cancer data.

Moreover, the Kolmogorov-Smirnov (KS) statistic and its p-value for the TTW and TTLi distributions are obtained for both data sets. The KS statistic of TTW distribution is 0.06416 and its p-value is 0.8796. The KS statistic of TTLi distribution is 0.0724 and the p-value is 0.5129.

8. Simulation study

The following simulation procedure is implemented.

1. Set the sample size n and the vector of parameters $\boldsymbol{\tau} = (a, \lambda, \alpha, \theta)$.
2. Generate random observations of size n from the TTW$(a, \lambda, \alpha, \theta)$ distribution.
3. Using the generated random observations in Step 2, estimate $\hat{\boldsymbol{\tau}}$ by means of MLE method.
4. Repeat steps 2 and 3, N times.
5. Using $\hat{\boldsymbol{\tau}}$ and $\boldsymbol{\tau}$ compute the averages of estimates (AEs), biases, and mean square errors (MSEs) via the following equations (for $i = 1, 2, 3, 4$):

$$\text{AEs} = \frac{1}{N}\sum_{j=1}^{N}\left(\hat{\boldsymbol{\tau}}_{i,j}\right), \quad \text{Bias} = \frac{1}{N}\sum_{j=1}^{N}\left(\hat{\boldsymbol{\tau}}_{i,j} - \boldsymbol{\tau}_i\right), \quad \text{and} \quad \text{MSE} = \sum_{j=1}^{N}\frac{\left(\hat{\boldsymbol{\tau}}_{i,j} - \boldsymbol{\tau}_i\right)^2}{N}.$$

The statistical software R is used to obtain simulation results. The chosen parameter values for simulation study are $\boldsymbol{\tau} = (-0.9, 0.9, 2, 2)$, $N = 10{,}000$, and $n = (50, 55, 60, \ldots, 1000)$. We expect that AEs are closer to nominal values for large sample sizes. Figure 5 displays the estimated AEs, biases and MSEs. Figure 5 reveals that when n is sufficiency large, the estimated MSEs for all parameters tend to zero and the values of AEs are closer to nominal values. The biases for the parameters a, α and θ are positive whereas the biases for the parameter λ is negative. The biases for all the parameters tend to zero for large sample sizes. It is clear that the estimates of parameters are asymptotically unbiased. Therefore, the MLE is an appropriate method for estimating parameters of the TTW distribution.

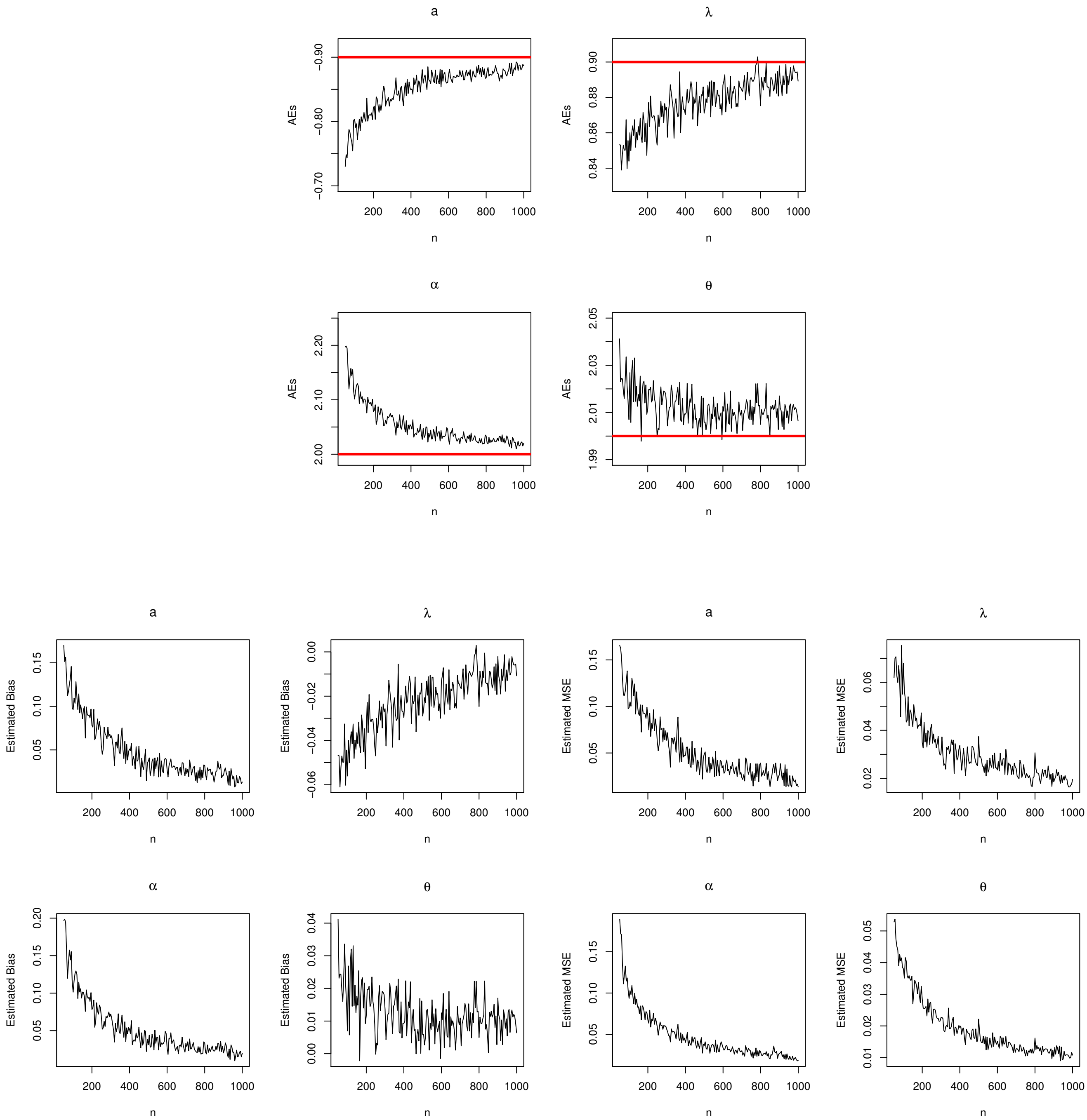

Figure 5: Estimated AEs, biases, and MSEs for the chosen parameter values.

9. Concluding remarks

We introduce and study a new class of distributions called the transumted transumted-G (TT-G) family, which extends the transmuted class (TC) proposed by Shaw and Buckley [24] and includes the TC as special case. We define two special models of the TT-G family. We provide some mathematical properties of this family including explicit expansions for the ordinary and incomplete moments. We characterize the TT-G family by means of two characterization theorems. The maximum likelihood estimation of the model parameters is investigated. By means of two real data sets, we verify that special case of the TT-G family can provide better fits than other models generated from well-known families.

References

[1] A. Z. Afify, G. M. Cordeiro, H. M. Yousof, A. Alzaatreh, Z. M. Nofal, *The Kumaraswamy transmuted-G family of distributions: properties and applications*, J. Data Sci., **14** (2016), 245–270. 7.1

[2] A. Z. Afify, G. G. Hamedani, I. Ghosh, M. E. Mead, *The transmuted Marshall-Olkin Fréchet distribution: properties and applications*, International Journal of Statistics and Probability, **4** (2015), 132–184. 1

[3] A. Z. Afify, Z. M. Nofal, N. S. Butt, *Transmuted complementary Weibull geometric distribution*, Pak. J. Stat. Oper. Res., **10** (2014), 435–454. 1

[4] A. Z. Afify, H. M. Yousof, N. S. Butt, G. G. Hamedani, *The transmuted Weibull-Pareto distribution*, Pakistan J. Statist., **32** (2016), 183–206. 1

[5] A. Z. Afify, H. M. Yousof, S. Nadarajah, *The beta transmuted-H family for lifetime data*, Stat. Interface, **10** (2017), 505–520. 1

[6] E. k. Al-Hussaini, M. Ahsanullah, *Exponentiated Distributions*, Atlantis Press, Paris, (2015). 2

[7] M. Alizadeh, H. M. Yousof, A. Z. Afify, G. M. Cordeiro, M. Mansoor, *The complementary generalized transmuted Poisson-G family of distributions*, Austrian J. Statist., **47** (2018), 51–71. 1

[8] A. Alzaatreh, C. Lee, F. Famoye, *A new method for generating families of continuous distributions*, Metron, **71** (2013), 63–79. 2

[9] G. R. Aryal, C. P. Tsokos, *On the transmuted extreme value distribution with application*, Nonlinear Anal., **71** (2009), 1401–1407. 1

[10] G. R. Aryal, C. P. Tsokos, *Transmuted Weibull distribution: a generalization of the Weibull probability distribution*, Eur. J. Pure Appl. Math., **4** (2011), 89–102. 1

[11] G. M. Cordeiro, E. M. Hashimoto, E. M. M. Ortega, *The McDonald Weibull model*, Statistics, **48** (2014), 256–278. 7.1

[12] I. Elbatal, G. Aryal, *On the transmuted additive Weibull distribution*, Austrian J. Statist., **42** (2013), 117–132. 1

[13] M. E. Ghitany, D. K. Al-Mutairi, N. Balakrishnan, L. J. Al-Enezi, *Power Lindley distribution and associated inference*, Comput. Statist. Data Anal., **64** (2013), 20–33. 7.2

[14] M. N. Khan, *The modified beta Weibul distribution*, Hacettepe J. Math. Statist., **44** (2015), 1553–1568. 7.1

[15] C. Lee, F. Famoye, O. Olumolade, *Beta-Weibull distribution: some properties and applications to censored data*, J. Modern Appl. Statist. Methods, **6** (2007), 173–186. 7.1

[16] E. T. Lee, J. W. Wang, *Statistical Methods for Survival Data Analysis*, John Wiley & Sons, Hoboken, (2003). 7.2

[17] M. M. Mansour, S. M. Mohamed, *A new generalized of transmuted Lindley distribution*, Appl. Math. Sci., **9** (2015), 2729–2748. 1, 7.2

[18] F. Merovci, *Transmuted lindley distribution*, Int. J. Open Problems Comput. Sci. Math., **6** (2013), 63–72. 7.2

[19] F. Merovci, M. Alizadeh, M. G. G. Hamedani, *Another generalized transmuted family of distributions: properties and applications*, Austrian J. Statist., **45** (2016), 71–93. 1

[20] F. Merovci, M. Alizadeh, H. M. Yousof, G. G. Hamedani, *The exponentiated transmuted-G family of distributions: theory and applications*, Comm. Statist. Theory Methods, **46** (2017), 10800–10822. 1

[21] F. Merovci, I. Elbatal, *Transmuted Lindley-geometric distribution and its applications*, J. Statist. Appl. Prob., **3** (2014), 77–91. 1

[22] D. N. Prabhakar Murthy, M. Xie, R. Jiang, *Weibull Models*, John Wiley & Sons, Hoboken, (2004). 7.1

[23] M. W. A. Ramos, P. R. D. Marinho, R. V. da Silva, G. M. Cordeiro, *The exponentiated Lomax Poisson distribution with an application to lifetime data*, Adv. Appl. Stat., **34** (2013), 107–135. 7.1

[24] W. T. Shaw, I. R. Buckley, *The alchemy of probability distributions: beyond gram-charlier expansions and a skew-kurtotic-normal distribution from a rank transmutation map*, arXiv preprint, **2007** (2007), 28 pages. 1, 1, 9

[25] M. H. Tahir, G. M. Cordeiro, *Compounding of distributions: a survey and new generalized classes*, J. Statist. Distribut. Appl., **3** (2016), 1–35. 2

[26] H. M. Yousof, A. Z. Afify, M. Alizadeh, N. S. Butt, G. G. Hamedani, M. M. Ali, *The transmuted exponentiated generalized-G family of distributions*, Pak. J. Stat. Oper. Res., **11** (2015), 441–464. 7.1