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The type I half-logistic Burr X distribution: theory and practice

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Abstract

In this paper, we explore the properties and importance of a lifetime distribution so called type I half-logistic Burr X (TIHL_{BX}) in detail (also called type I half logistic generalized Rayleigh (TIHL_{GR})). We investigate some of its mathematical and statistical properties such as the explicit form of the ordinary moments, moment generating function, conditional moments, Bonferroni and Lorenz curves, mean deviations, residual life and reversed residual functions, Shannon entropy and Renyi entropy. The maximum likelihood method is used to estimate the model parameters. Simulation studies were conducted to assess the finite sample behavior of the maximum likelihood estimators. Finally, we illustrate the importance and applicability of the model by the study of two real data sets.

Keywords: Type I half logistic distribution, Burr X distribution, moments, maximum likelihood estimate.

2010 MSC: 62E05, 62F10, 62F12.

1. Introduction

Burr [5] introduced twelve different forms of cumulative distribution functions for modeling lifetime data, where among them is the Burr Type X distribution (BX) . Several authors considered different aspects of the Burr Type X, for example [1, 13, 24, 27], anong others. [28] introduced two-parameter Burr Type X distribution known as the generalized Rayleigh (GR) distribution. The two-parameter generalized Rayleigh distribution is a particular member of the generalized Weibull distribution. [27] showed that the two-parameter Burr X distribution can be effectively used in modeling strength data. The cumulative distribution function (c.d.f) of Burr X distribution is given by

$$G_{BX}(x, \alpha, \theta) = \left[1 - e^{-(\alpha x)^2}\right]^{\theta}, \ x > 0, \tag{1.1}$$

where $\alpha > 0$ is the scale parameter and $\theta > 0$ is the shape parameter. The corresponding probability

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density function (p.d.f) and hazard rate function (h.r.f) are given by

$$g_{BX}(x,\alpha,\theta) = 2\theta\alpha^2 x e^{-(\alpha x)^2}\left[1-e^{-(\alpha x)^2}\right]^{\theta-1} \quad \text{and} \quad h_{BX}(x,\alpha,\theta) = \frac{2\theta\alpha^2 x e^{-(\alpha x)^2}\left[1-e^{-(\alpha x)^2}\right]^{\theta-1}}{1-\left[1-e^{-(\alpha x)^2}\right]^{\theta}},$$

respectively. [24] shows that the hazard function of the BX distribution can be bathtub or an increasing function, depending on the shape parameter θ. If $\theta \leqslant \frac{1}{2}$, the hazard function is bathtub and for $\theta > \frac{1}{2}$ it has an increasing hazard function. There are several extensions and generalizations of the Burr X (BX) distributions (or generalized Rayleigh (GR)), for example, beta generalized Rayleigh (BGR) by [7], the exponentiated generalized Burr type X (EGBX) by [12], beta Burr X by [17], and beta compound Rayleigh by [25], among others.

The method of extending a family of distributions for added flexibility is a well-known technique in the literature. In many applied sciences such as medicine, engineering, finance, economics, biomedical sciences, public health, modeling and analyzing lifetime data are very essential. Several lifetime distributions have been used to analyze such kinds of data, but the quality of the procedure used in a statistical study depends on the assumed probability model. However, there still are various vital problems where the real data does not follow any of the classical probability models. Due to this, significant effort has been spent in the development of new classes of flexible probability distributions along with relevant statistical methodologies over the years. These include the following technique.

Let $G(x)$ be any valid cumulative distribution function defined on $\mathbb{R}$. Various approaches for generating new distributions based on $G(x)$ were proposed in recent years. The well-known generators include beta-G by [10], gamma-G distributions by [30], Kumaraswamy-G distributions by [8], Weibull X distributions by [3], odd-generalized exponential-G by [29], and Poisson odd-generalized exponential-G by [19], among others.

[6] presented a new G class of continuous distributions with an extra positive parameter $\lambda > 0$ called the type I half-logistic family.

The cumulative distribution function (cdf) of the new type I half-logistic (TIHL) family of distributions is given by

$$F(x;\delta) = \int_0^{-\log[1-G(x;\delta)]} \frac{2\lambda e^{-\lambda x}}{(1+e^{-\lambda x})^2}\,dx = \frac{1-[1-G(x;\delta)]^{\lambda}}{1+[1-G(x;\delta)]^{\lambda}}, \tag{1.2}$$

where $G(x;\delta)$ is the baseline cdf depending on a parameter vector δ. For each baseline G, we can generate the type I half-logistic-G $(\mathrm{TIHL}-G)$ distribution by the cdf (1.2), and the corresponding probability density function (pdf) to equation (1.2) is given by

$$f(x;\delta) = \frac{2\lambda g(x;\delta)\,[1-G(x;\delta)]^{\lambda-1}}{\left\{1+[1-G(x;\delta)]^{\lambda}\right\}^2}, \tag{1.3}$$

where $g(x;\delta)$ is the baseline pdf. Equation (1.3) will be more tractable when $G(x;\delta)$ and $g(x;\delta)$ have simple expressions. The failure rate function is

$$h(x,\delta) = \frac{\lambda g(x;\delta)}{[1-G(x;\delta)]\left\{1+[1-G(x;\delta)]^{\lambda}\right\}}.$$

In this paper, we derived the lifetime model so-called the type I half-logistic Burr X (TIHL_{BX}) distribution. The model serves as an excellent alternative to many existing life distributions in modeling positive real data. This model was also mentioned in [23].

The rest of the paper is arranged as follows. In Section 2, we derive the TIHL_{BX} distribution and presented some of its essential mathematical and statistical properties. In Section 3, we established the parameter estimation by the method of maximum likelihood and accessed the maximum likelihood estimators via simulation studies. In Section 4, we illustrate the importance of the TIHL_{BX} distribution by two real data applications. Finally, conclusions in Section 5.

2. The TIHL_{BX} model and Properties

In this section, we derived the three-parameter type I half-logistic Burr X (TIHL_{BX}) distribution. Using (1.1) in (1.2), the cdf of the (TIHL_{BX}) distribution can be written as

$$F(x;\lambda,\alpha,\theta) = \int_0^{-\log\left[1-\left(1-e^{-(\alpha x)^2}\right)^\theta\right]} \frac{2\lambda e^{-\lambda t}}{(1+e^{-\lambda t})^2} dt = \frac{1-\left[1-\left(1-e^{-(\alpha x)^2}\right)^\theta\right]^\lambda}{1+\left[1-\left(1-e^{-(\alpha x)^2}\right)^\theta\right]^\lambda}. \tag{2.1}$$

The corresponding probability density function is given by

$$f(x;\lambda,\alpha,\theta) = 4\lambda\theta\alpha^2 x e^{-(\alpha x)^2}\left[1-e^{-(\alpha x)^2}\right]^{\theta-1} \frac{\left[1-\left(1-e^{-(\alpha x)^2}\right)^\theta\right]^{\lambda-1}}{\left\{1+\left[1-\left(1-e^{-(\alpha x)^2}\right)^\theta\right]^\lambda\right\}^2}. \tag{2.2}$$

The hazard rate function of the TIHL_{BX} takes the form

$$h(x,\phi) = \frac{2\lambda\theta\alpha^2 x e^{-(\alpha x)^2}\left[1-e^{-(\alpha x)^2}\right]^{\theta-1}}{\left[1-\left(1-e^{-(\alpha x)^2}\right)^\theta\right]\left\{1+\left[1-\left(1-e^{-(\alpha x)^2}\right)^\theta\right]\right\}}.$$

A random variable X having pdf (2.2) is denoted by $X \sim \text{TIHL}_{\text{BX}}(\phi)$, where $\phi = (\lambda, \alpha, \theta)$.

Figures 1 and 2 represent some plots of the probability density and hazard rate function of the TIHL_{BX} distribution for some different parameter values.

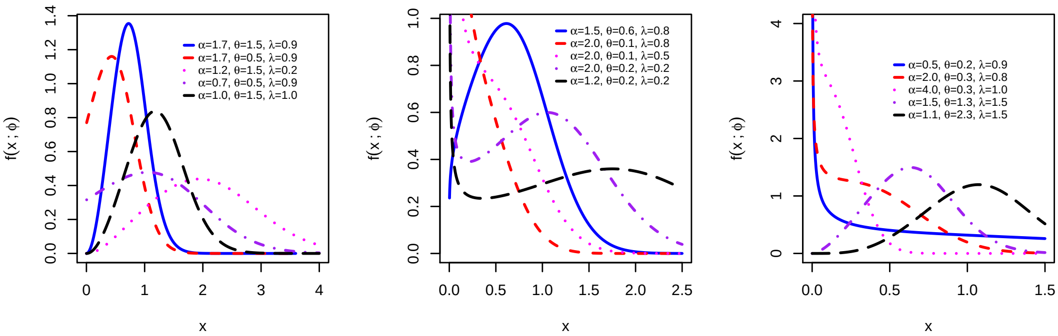


Figure 1: Plots of the probability density function of TIHL_{BX} for some parameter values.

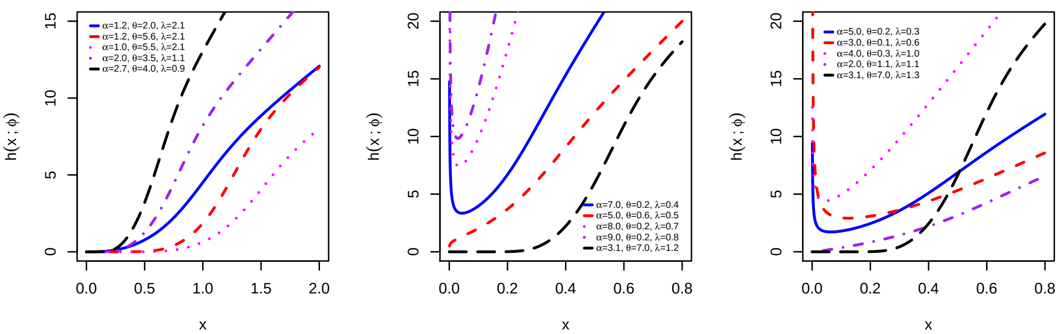


Figure 2: Plots of the hazard rate function of TIHL_{BX} for some parameter values.

2.1. Expansion of the density function.

In this subsection, we can express the type I half-logistic Burr X as an infinite mixture of Burr X distributions . If $|z| < 1$, $k > 0$, and $b > 0$, we have the series representations

$$(1+z)^{-k} = \sum_{j=0}^{\infty} \binom{-k}{j} z^j, \tag{2.3}$$

and

$$(1-z)^{b-1} = \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} z^i. \tag{2.4}$$

Expanding $\left\{1 + \left[1 - \left(1 - e^{-(\alpha x)^2}\right)^{\theta}\right]^{\lambda}\right\}^{-2}$ as in (2.3), we can write (2.2) as

$$f(x;\lambda,\theta,\alpha) = 4\lambda\theta\alpha^2 x e^{-(\alpha x)^2} \sum_{j=0}^{\infty} \binom{-2}{j} \left[1 - e^{-(\alpha x)^2}\right]^{\theta-1} \left[1 - \left(1 - e^{-(\alpha x)^2}\right)^{\theta}\right]^{\lambda(j+1)-1}, \tag{2.5}$$

by considering (2.4) in the expansion equation (2.5) and after some algebra, the pdf of type I half-logistic Burr X can be written as

$$\begin{aligned} f(x;\lambda,\theta,\alpha) &= 4\lambda\theta\alpha^2 x e^{-(\alpha x)^2} \sum_{j,i=0}^{\infty} (-1)^i \binom{-2}{j} \binom{\lambda(j+1)-1}{i} \left(1 - e^{-(\alpha x)^2}\right)^{\theta(i+1)-1} \\ &= \sum_{j,i=0}^{\infty} \vartheta_j \; 2\theta\alpha^2 (i+1) x e^{-(\alpha x)^2} \left(1 - e^{-(\alpha x)^2}\right)^{\theta(i+1)-1} = \sum_{i=0}^{\infty} \vartheta_j f(x;\alpha,\theta(i+1)), \end{aligned}$$

where

$$\vartheta_j = \sum_{j=0}^{\infty} \frac{2\lambda(-1)^i \binom{-2}{j} \binom{\lambda(j+1)-1}{i}}{(i+1)},$$

and $f(x;\alpha,\theta(i+1))$ denotes the Burr X (BX) density function with parameters α and $\theta(i+1)$. Thus, the TIHL_{BX} density function can be expressed as an infinite linear combination of BX densities and then some of its basic mathematical and statistical properties can be obtained from those of BX properties. Another series form of pdf for type I half-logistic Burr X is

$$\begin{aligned} f(x;\lambda,\theta,\alpha) &= 4\lambda\theta\alpha^2 x e^{-(\alpha x)^2} \sum_{j,i=0}^{\infty} (-1)^i \binom{-2}{j} \binom{\lambda(j+1)-1}{i} \left(1 - e^{-(\alpha x)^2}\right)^{\theta(i+1)-1} \\ &= 4\lambda\theta\alpha^2 \sum_{j,i,k=0}^{\infty} (-1)^{i+k} \binom{-2}{j} \binom{\lambda(j+1)-1}{i} \binom{\theta(i+1)-1}{k} x e^{-(k+1)(\alpha x)^2} = w_{j,i,k} \, x e^{-(k+1)(\alpha x)^2}, \end{aligned} \tag{2.6}$$

where $w_{j,i,k} = 4\lambda\theta\alpha^2 \sum_{j,i,k=0}^{\infty} (-1)^{i+k} \binom{-2}{j} \binom{\lambda(j+1)-1}{i} \binom{\theta(i+1)-1}{k}$.

2.2. Quantile and moments

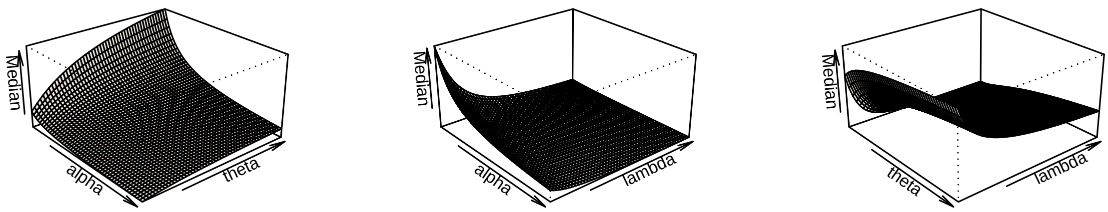
The inverse of the cdf in (2.1) yields the quantile function of the TIHL_{BX} as

$$Q(u) = \left[-\frac{1}{\alpha^2} \log\left(1 - \left(1 - \left(\frac{(1-u)}{(1+u)}\right)^{1/\lambda}\right)^{1/\theta}\right)\right]^{1/2}, \quad u \in (0,1).$$

Since the uniform random variables are easily generated in most statistical packages, the above scheme is very useful to generate $TIHL_{BX}$ random variates and therefore can be easily implemented. In particular, the median (M) of X is Q(0.5) given by

$$M = \left[-\frac{1}{\alpha^2} \log \left(1 - \left(\left(\frac{2}{3} \right)^{1/\lambda} \right)^{1/\theta} \right) \right]^{1/2}.$$

Figure 3 shows the plots of the Median of the $TIHL_{BX}$, (i) for fixed value of $\lambda = 0.5$ the median is increasing in θ and decreasing in α (ii) for $\theta = 0.2$ the median is decreasing in both α and λ, and (iii) for $\alpha = 0.2$ the median is increasing in θ and decreasing in λ.



(i) (ii) (iii)

Figure 3: Plots of the Median of $TIHL_{BX}$ distribution.

Here, we provide the ordinary moments and moment generating function of the $TIHL_{BX}$ distribution.

Theorem 2.1. *If* $X \sim TIHL_{BX}$, *then the* r^{th} *moment of* X *is given by*

$$\mu_r'(x) = w_{i,j,k} \frac{\Gamma(\frac{r}{2}+1)}{2\alpha^{r+2}(k+1)^{\frac{r}{2}+1}}. \tag{2.7}$$

Proof. Let X be a random variable following the $TIHL_{BX}$ distribution, the r^{th} ordinary moment of X can obtained using (2.6) as

$$\mu_r'(x) = E(X^r) = \int_0^\infty x^r f(x,\phi)dx = w_{i,j,k} \int_0^\infty x^{r+1} e^{-(k+1)(\alpha x)^2} dx = w_{i,j,k} \frac{\Gamma(\frac{r}{2}+1)}{2\alpha^{r+2}(k+1)^{\frac{r}{2}+1}}.$$

□

Theorem 2.2. *If* $X \sim TIHL_{BX}$, *then the moment generating function (mgf) of* X *is given as*

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} w_{i,j,k} \frac{\Gamma(\frac{r}{2}+1)}{2\alpha^{r+2}(k+1)^{\frac{r}{2}+1}}.$$

Proof. We start with the well known definition of the moment generating function given by

$$M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^\infty x^r f(x)dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'(x),$$

thus, substituting (2.7) into (2.2) we get

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} w_{i,j,k} \frac{\Gamma(\frac{r}{2}+1)}{2\alpha^{r+2}(k+1)^{\frac{r}{2}+1}}.$$

□

Further, the central moments $\mu_r^{'}$ in (2.7) can be used to obtain the higher order moments by substituting $r = 1, 2, 3, \ldots$.

Corollary 2.3. *Let* $X \sim \text{TIHL}_{BX}$ *with pdf in* (2.2)*, then the variance*(σ^2)*, coefficient of variation (CV), skewness* (γ^3)*, and kurtosis* (γ^4) *could be determined from*

$$\sigma^2 = \mu_2^{'} - \mu_1^{'2}, \qquad CV = \sqrt{\frac{\mu_2^{'}}{\mu_1^{'2}} - 1}, \qquad \gamma^3 = \frac{\mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2\mu_1^{'3}}{(\mu_2^{'} - \mu_1^{'2})^{3/2}}, \qquad \gamma^4 = \frac{\mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}\mu_1^{'} + 3\mu_1^{'4}}{(\mu_2^{'} - \mu_1^{'2})^2}.$$

Table 1 provides some numerical values of the first six moments, σ^2, CV, γ^3, and γ^4 for some parameter values. Observe from the table, as the λ, α, and θ increase the first six moments, CV and skewness decrease while the kurtosis increases.

Table 1: First six moments, variance (σ^2), coefficient of variation (CV), skewness γ^3, and kurtosis γ^4 of TIHL_{BX} for some parameter values.

λ	α	θ	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	σ^2	CV	γ^3	γ^4
1.0	0.1	0.1	3.1888	27.558	322.73	4524.4	72052.5	1266445	7.3896	1.3077	1.7091	4.1185
1.1	0.2	0.3	2.9181	13.975	83.816	582.97	4532.2	38537.1	5.4594	0.8007	0.8761	2.2426
1.3	0.4	0.5	1.6959	4.0522	11.668	38.284	139.04	549.10	1.1761	0.6395	0.6327	18.203
1.6	0.6	0.9	1.3444	2.2224	4.2073	8.8257	20.133	49.334	0.4150	0.4792	0.3875	80.885
2.1	0.9	1.2	0.9025	0.9520	1.1216	1.4414	1.9921	2.9329	0.1375	0.4108	0.2788	240.15
2.3	1.5	1.8	0.6271	0.4361	0.3286	0.2645	0.2256	0.2024	0.0428	0.3301	0.1569	841.90
2.5	1.8	1.9	0.5192	0.2969	0.1832	0.1204	0.0836	0.0609	0.0274	0.3188	0.1330	1177.1
3.2	2.1	2.0	0.4191	0.1919	0.0942	0.0490	0.0268	0.01535	0.0162	0.3037	0.0837	1775.1
3.5	2.5	2.5	0.3777	0.1528	0.0654	0.0294	0.0138	0.0067	0.0102	0.2672	0.0226	3258.0
4.5	3.0	3.1	0.3220	0.1092	0.0387	0.0143	0.0054	0.0021	0.0056	0.2319	-0.0599	6682.9
5.5	4.5	4.1	0.2290	0.0544	0.0134	0.0034	0.0009	0.0002	0.0020	0.1956	-0.1345	18430.5
6.5	5.5	5.1	0.1965	0.0397	0.0083	0.0018	0.0004	$8.40e^{-5}$	0.0011	0.1713	-0.1878	36298.5

It is also of interest to compute the conditional moments of this lifetime model. One of the important applications of the first incomplete moment refers to the Bonferroni and Lorenz curves. The s^{th} lower and upper incomplete moments of X are defined by $\nu_s(t) = E(X^s \mid X < t) = \int_0^t x^s f(x, \phi)dx$ and $\eta_s(t) = E(X^s \mid X > t) = \int_t^{\infty} x^s f(x, \phi)dx$, respectively. For any $s \in \mathbb{N}$, the s^{th} lower incomplete moment of TIHL_{BX} distribution is

$$\nu_s(t) = \int_0^t x^s f(x)dx = w_{i,j,k} \int_0^t x^{s+1} e^{-(k+1)(\alpha x)^2} dx = w_{i,j,k} \left[\frac{\gamma(\frac{s}{2}+1, (k+1)(\alpha t)^2)}{2\alpha^{s+2}(k+1)^{\frac{s}{2}+1}} \right],$$

where $\gamma(s, t) = \int_0^t x^{s-1} e^{-x} dx$ is the lower incomplete gamma function. Similarly, the s^{th} upper incomplete moment of TIHL_{BX} distribution is

$$\eta_s(t) = \int_t^{\infty} x^s f(x)dx = w_{i,j,k} \int_t^{\infty} x^{s+1} e^{-(k+1)(\alpha x)^2} dx = w_{i,j,k} \left[\frac{\Gamma(\frac{s}{2}+1, (k+1)(\alpha t)^2)}{2\alpha^{s+2}(k+1)^{\frac{s}{2}+1}} \right],$$

where $\Gamma(s, t) = \int_t^{\infty} x^{s-1} e^{-x} dx$ is the upper incomplete gamma function.

2.3. *Mean deviation*

In this subsection, we compute the mean deviation about the mean and the mean deviation about the median. If X has the $TIHL_{BX}$ distribution, then we can derive the mean deviations about the mean $\mu = E(X)$ and the mean deviations about the median M as

$$\delta_1(x) = \int_0^\infty |x-\mu| f(x)dx = 2\left[\mu F(\mu) - J(\mu)\right] \quad \text{and} \quad \delta_2(x) = \int_0^\infty |x-M| f(x)dx = \mu - 2J(M),$$

respectively. The measures $\delta_1(x)$ and $\delta_2(x)$ can be calculated using the relationships $J(.)$ as

$$J(d) = \int_0^d xf(x)dx = w_{i,j,k}\left[\frac{\gamma(\frac{1}{2}+1,(k+1)(\alpha d)^2)}{2\alpha^3(k+1)^{\frac{1}{2}+1}}\right].$$

2.4. *Bonferroni and Lorenz curves*

Bonferroni and Lorenz curves are income inequality measures that are also useful and applicable to other areas including reliability, demography, medicine and insurance.

The Lorenz curve for a positive random variable X is defined as

$$L(p) = \frac{1}{\mu}\int_0^q xf(x)dx = \frac{J(q)}{\mu} = \frac{w_{i,j,k}}{\mu}\left[\frac{\gamma(\frac{1}{2}+1,(k+1)(\alpha d)^2)}{2\alpha^3(k+1)^{\frac{1}{2}+1}}\right],$$

where $q = F^{-1}(p)$. Also Bonferroni curve of X is given by

$$B(p) = \frac{1}{\mu p}\int_0^q xf(x)dx = \frac{J(q)}{\mu p} = \frac{w_{i,j,k}}{\mu p}\left[\frac{\gamma(\frac{1}{2}+1,(k+1)(\alpha d)^2)}{2\alpha^3(k+1)^{\frac{1}{2}+1}}\right].$$

3. Residual life and reversed residual life functions

Suppose that a component survives up to time $t \geqslant 0$, the residual life is the period beyond t until the time of failure and is defined by the conditional random variable $X-t|X>t$. In theory of reliability, it is well known that the mean residual life function and ratio of two consecutive moments of residual life determine the distribution uniquely (see [11]). Therefore, we obtain the r^{th}-order moment of the residual life via the general formula

$$\mu R_r(t) = E((X-t)^r \mid X>t) = \frac{1}{\bar{F}(t)}\int_t^\infty (x-t)^r f(x,\varphi)dx.$$

Applying the binomial expansion of $(x-t)^r$ and substituting $f(x,\varphi)$ given by (2.6) into the above formula gives

$$\mu R_r(t) = \frac{w_{i,j,k}}{F(t)}\sum_{h=0}^{r}(-t)^h\binom{r}{h}\int_t^\infty x^{r-h+1}e^{-(k+1)(\alpha x)^2}dx,$$

$$\mu R_r(t) = \frac{w_{i,j,k}}{\bar{F}(t)}\sum_{h=0}^{r}(-t)^h\binom{r}{h}\left[\frac{\Gamma(\frac{r-h}{2}+1,(k+1)(\alpha t)^2)}{2\alpha^{r-h+2}(k+1)^{\frac{r-h}{2}+1}}\right],$$

where $\Gamma(s,t) = \int_t^\infty x^{s-1}e^{-x}dx$ is the upper incomplete gamma function. The mean residual life of the $TIHL_{BX}$ distribution is given by

$$\mu R(t) = \frac{w_{i,j,k}}{\bar{F}(t)}\left[\frac{\gamma(\frac{1}{2}+1,(k+1)(\alpha t)^2)}{2\alpha^3(k+1)^{\frac{1}{2}+1}}\right] - t.$$

The variance of the residual life of the $\mathrm{TIHL_{BX}}$ distribution can be obtained easily by using $\mu R_2(t)$ and $\mu R(t)$. The reversed residual life can be defined as the conditional random variable $t - X|X \leqslant t$ which denotes the time elapsed from the failure of a component given that its life is less than or equal to t. This random variable may also be called the inactivity time (or time since failure). The r^{th} order moment of the reversed residual life can be obtained by

$$mR_r(t) = E((t-X)^r \mid X \leqslant t) = \frac{1}{F(t)} \int_0^t (t-x)^r f(x,\varphi)dx.$$

Applying the binomial expansion of $(t-x)^r$ and substituting $f(x,\varphi)$ given by (2.6) into the above formula gives

$$\begin{aligned} mR_r(t) &= \frac{w_{i,j,k}}{F(t)} \sum_{h=0}^{r} (-t)^h \binom{r}{h} \int_0^t x^{r-h+1} e^{-(k+1)(\alpha x)^2} dx \\ &= \frac{w_{i,j,k}}{F(t)} \sum_{h=0}^{r} (-t)^h \binom{r}{h} \left[\frac{\gamma(\frac{r-h}{2}+1, (k+1)(\alpha t)^2)}{2\alpha^{r-h+2}(k+1)^{\frac{r-h}{2}+1}} \right], \end{aligned}$$

where $\gamma(s,t) = \int_0^t x^{s-1} e^{-x} dx$ is the lower incomplete gamma function. Thus, the mean reversed residual life (or mean waiting time) of the $\mathrm{TIHL_{BX}}$ distribution is given by

$$mR(t) = t - \frac{w_{i,j,k}}{F(t)} \left[\frac{\gamma(\frac{1}{2}+1, (k+1)(\alpha t)^2)}{2\alpha^3 (k+1)^{\frac{1}{2}+1}} \right].$$

Using $mR(t)$ and $mR_2(t)$ one can obtain the variance and the coefficient of variation of the reversed residual life of the $\mathrm{TIHL_{BX}}$ distribution.

3.1. *Entropy*

Entropy of a random variable X can be defined as a measure of variation of uncertainty. In this subsection, we consider the two most important and popular entropies known as the Shannon and Renyi entropies. The Shannon entropy measure of a random variable X with $\mathrm{TIHL_{BX}}$ distribution can be defined by $E[-\log f(x)]$. The Shannon entropy of $\mathrm{TIHL_{BX}}$ can be computed by considering the following lemmas.

Lemma 3.1. *For* $t \in \mathbb{R}$, *let* X *be a random variable with* $\mathrm{TIHL_{BX}}$, *let,* $\psi(t) = E\left[\left(1-e^{-(\alpha x)^2}\right)^t\right]$, *then,*

$$\psi(t) = w_{i,j,k} \sum_{l=0}^{\infty} \frac{(-1)^l \binom{t}{k} \sqrt{\pi}}{2[(k+l+1)\alpha^2]^{\frac{3}{2}}}.$$

Proof.

$$\begin{aligned} \psi(t) &= w_{i,j,k} \int_0^\infty (1-e^{-(\alpha x)^2})^t \, x \, e^{-(k+1)(\alpha x)^2} dx \\ &= w_{i,j,k} \sum_{l=0}^{\infty} (-1)^l \binom{t}{k} \int_0^\infty x \, e^{-(k+l+1)(\alpha x)^2} dx = w_{i,j,k} \sum_{l=0}^{\infty} \frac{(-1)^l \binom{t}{k} \sqrt{\pi}}{2[(k+l+1)\alpha^2]^{\frac{3}{2}}}. \end{aligned}$$

□

Lemma 3.2. *Let X be a random variable with* $\mathrm{TIHL_{BX}}$ *distribution and pdf given by* (2.2), *then,*

$$E\left[\log\left(1-e^{-(\alpha x)^2}\right)\right] = \frac{\partial}{\partial t}\psi(t)|_{t=0},$$

$$E[\log x] = \frac{\partial}{\partial t}\mu_t^{'}(x)|_{t=0},$$
$$E\left[\log\left(1-\left(1-e^{-(\alpha x)^2}\right)^{\theta}\right)\right] = \sum_{m=1}^{\infty}\frac{(-1)^{2m+1}\psi(\theta k)}{m},$$
$$E\left[\log\left(1+\left(1-\left(1-e^{-(\alpha x)^2}\right)^{\theta}\right)^{\lambda}\right)\right] = \sum_{s=1}^{\infty}\sum_{w=0}^{\infty}\frac{(-1)^{s+w-1}\binom{\lambda s}{w}\psi(\theta w)}{s},$$

where $\mu_t^{'}(x)$ is the t^{th} moments of X by considering (2.7) and $\psi(.)$ is given by Lemma 3.1.

Now, the Shannon entropy of X can be expressed as

$$E[-\log f(x)] = -\log(4\lambda\theta\alpha^2) - E[\log x] + \alpha^2 E[x^2] - (\theta-1)\log(1-e^{(\alpha x)^2})$$
$$-(\lambda-1)E\left[\log\left(1-\left(1-e^{-(\alpha x)^2}\right)^{\theta}\right)\right] + 2E\left[\log\left(1+\left(1-\left(1-e^{-(\alpha x)^2}\right)^{\theta}\right)^{\lambda}\right)\right].$$

By using the Lemma 3.2 we get

$$E[-\log f(x)] = -\log(4\lambda\theta\alpha^2) - \frac{\partial}{\partial t}\mu_t^{'}(x)|_{t=0} + \alpha^2\mu_2^{'}(x) - (\theta-1)\frac{\partial}{\partial t}\psi(t)|_{t=0}$$
$$-(\lambda-1)\sum_{m=1}^{\infty}\frac{(-1)^{2m+1}\psi(\theta k)}{m} + 2\sum_{s=1}^{\infty}\sum_{w=0}^{\infty}\frac{(-1)^{s+w-1}\binom{\lambda s}{w}\psi(\theta w)}{s}.$$

The Renyi entropy of a random variable X is defined by $I_{R(\rho)} = \frac{1}{1-\rho}\log\left[\int_a^b f(x)^{\rho}dx\right]$, where $\rho > 0$ and $\rho \neq 1$. The Renyi entropy of X with $TIHL_{BX}$ distribution can be obtained as follows

$$\int_0^{\infty} f^{\rho}(x)dx = w_{i,j,k}^{*}(\rho)\int_0^{\infty} x^{\rho}e^{-(k+\rho)\alpha^2x^2}dx = \frac{w_{i,j,k}^{*}(\rho)\,\Gamma(\frac{\rho}{2}+1)}{[(k+\rho)\alpha^2]^{\frac{\rho}{2}+1}},$$

where $w_{i,j,k}^{*}(\rho) = 4^{\rho}\theta^{\rho}\alpha^{2\rho}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\binom{-2\rho}{j}\binom{\rho(\lambda-1)+\lambda j}{i}\binom{\rho(\theta-1)+\theta i}{k}(-1)^{i+k}$. Thus,

$$I_{R(\rho)} = (1-\rho)^{-1}\log\left[\frac{w_{i,j,k}^{*}(\rho)\,\Gamma(\frac{\rho}{2}+1)}{[(k+\rho)\alpha^2]^{\frac{\rho}{2}+1}}\right].$$

4. Maximum likelihood estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $TIHL_{BX}(\varphi)$, let $\varphi = (\lambda, \alpha, \theta)^T$ be the parameter vector. The log likelihood function for the vector of parameters can be written as

$$\log\ L = n\log(4\lambda) + n\log\alpha^2 + \sum_{i=1}^{n}x_i - \alpha^2\sum_{i=1}^{n}x_i^2 + (\theta-1)\sum_{i=1}^{n}\log z_i^{\theta}$$
$$+(\lambda-1)\sum_{i=1}^{n}\log\left[1-z_i^{\theta}\right] - 2\sum_{i=1}^{n}\log\left\{1+\left[1-z_i^{\theta}\right]^{\lambda}\right\}, \tag{4.1}$$

where $z_i = \left(1-e^{-(\alpha x_i)^2}\right)$. The associated score function is given by

$$U_n(\varphi) = \left[\frac{\partial\log L}{\partial\lambda}, \frac{\partial\log L}{\partial\alpha}, \frac{\partial\log L}{\partial\theta}\right]^T.$$

The log-likelihood can be maximized by solving the nonlinear likelihood equations obtained by differentiating (4.1). The components of the score vector are given by

$$\frac{\partial \log\ L}{\partial \lambda} = \frac{2}{\lambda} + \sum_{i=1}^{n} \log\left[1 - z_i^{\theta}\right] - 2\sum_{i=1}^{n} \frac{\left[1 - z_i^{\theta}\right]^{\lambda} \log\left[1 - z_i^{\theta}\right]}{\left\{1 + \left[1 - z_i^{\theta}\right]^{\lambda}\right\}}, \tag{4.2}$$

$$\begin{aligned} \frac{\partial \log\ L}{\partial \alpha} = \frac{2n}{\alpha} - 2\alpha \sum_{i=1}^{n} x_i^2 + 2\alpha(\theta - 1) \sum_{i=1}^{n} \frac{x_i^2 e^{-(\alpha x)^2}}{\left[1 - e^{-(\alpha x)^2}\right]} \\ + \theta(\lambda - 1) \sum_{i=1}^{n} \frac{x_i^2 e^{-(\alpha x)^2} z_i^{\theta - 1}}{\left[1 - z_i^{\theta}\right]} + 2\lambda\theta \sum_{i=1}^{n} \frac{x_i^2 e^{-(\alpha x)^2} z_i^{\theta-1} \left[1 - z_i^{\theta}\right]^{\lambda - 1}}{\left\{1 + \left[1 - z_i^{\theta}\right]^{\lambda}\right\}}, \end{aligned} \tag{4.3}$$

and

$$\frac{\partial \log\ L}{\partial \theta} = \sum_{i=1}^{n} \log z_i - (\lambda - 1) \sum_{i=1}^{n} \frac{z_i^{\theta} \log z_i}{1 - z_i^{\theta}} + 2\lambda \sum_{i=1}^{n} \frac{z_i^{\theta} \log z_i \left[1 - z_i^{\theta}\right]^{\lambda - 1}}{\left\{1 + \left[1 - z_i^{\theta}\right]^{\lambda}\right\}}. \tag{4.4}$$

The maximum likelihood estimation (MLE) of φ, say $\widehat{\varphi}$, is obtained by solving the nonlinear system $U_n(\varphi) = 0$. These equations cannot be solved analytically, but statistical software can be used to solve them numerically via iterative methods. For interval estimation and hypothesis tests on the model parameters, we require the information matrix. The 3×3 observed information matrix is given by

$$I_n(\varphi) = -\begin{bmatrix} I_{\lambda\lambda} & I_{\lambda\alpha} & I_{\lambda\theta} \\ I_{\alpha\lambda} & I_{\alpha\alpha} & I_{\alpha\theta} \\ I_{\theta\lambda} & I_{\theta\alpha} & I_{\theta\theta} \end{bmatrix}$$

and $I_n(\varphi) = -\left(\frac{\partial^2(\log L)}{\partial \varphi \partial \varphi^T}\right)$. Applying the usual large sample approximation, MLE of φ, i.e $\widehat{\varphi}$ can be treated as being approximately $N_3(\varphi, J_n(\varphi)^{-1})$, where $J_n(\varphi) = E\left[I_n(\varphi)\right]$. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\widehat{\varphi} - \varphi)$ is $N_3(0, J(\varphi)^{-1})$, where $J(\varphi) = \lim_{n\to\infty} n^{-1} I_n(\varphi)$ is the unit information matrix. This asymptotic behavior remains valid if $J(\varphi)$ is replaced by the average sample information matrix evaluated at $\widehat{\varphi}$, say $n^{-1} I_n(\widehat{\varphi})$. The estimated asymptotic multivariate normal $N_3(\varphi, I_n(\widehat{\varphi})^{-1})$ distribution of $\widehat{\varphi}$ can be used to construct approximate confidence intervals for the parameters and for the hazard rate and survival functions. An $100(1 - \xi)$ asymptotic confidence interval for each parameter φ_r is given by

$$ACI_r = \left(\widehat{\varphi}_r - z_{\frac{\eta}{2}} \sqrt{\widehat{I_{rr}}}, \widehat{\varphi}_r + z_{\frac{\eta}{2}} \sqrt{\widehat{I_{rr}}}\right),$$

where $\widehat{I_{rr}}$ is the (r, r) diagonal element of $I_n(\widehat{\varphi})^{-1}$ for $r = 1, 2, 3$ and $z_{\frac{\eta}{2}}$ is the quantile $1 - \frac{\eta}{2}$ of the standard normal distribution.

4.1. Simulation

Simulations have been performed to assess the proposed method of maximum likelihood estimate. The simulation study was performed base on ten thousand (10, 000) samples of size 30, 50, 100, 200, and 300 each of which is randomly sampled from $TIHL_{BX}$ distribution for some different values of λ, α, and θ. The MLEs are obtained by solving the nonlinear equations (4.2) to (4.4) using `mlninb` package in R software. Moreover, no restriction was imposed on the number of the iterations performed. The MLEs $\hat{\lambda}$, $\hat{\alpha}$, and $\hat{\theta}$ and their standard deviations $sd(\hat{\lambda}), sd(\hat{\alpha})$, and $sd(\hat{\theta})$ of the parameters are given in Table 2 below. The results show that the maximum likelihood performed consistently, as the sample size increases the standard deviations of the MLEs decrease and the MLE approaches their true values in most cases.

Table 2: MLEs and Standard deviations for some various values of parameters.

Sample size	Actual values			Estimated values			Standard deviations		
n	λ	α	θ	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$sd(\hat{\lambda})$	$sd(\hat{\alpha})$	$sd(\hat{\theta})$
30	0.3	0.5	0.1	0.6587	0.4308	0.0939	0.8395	0.1493	0.0330
	0.3	0.6	0.2	3.2204	0.3466	0.1918	19.8349	0.2023	0.0711
	0.1	0.3	0.4	3.1268	0.3971	0.3745	27.5131	0.3490	0.1076
	0.1	0.1	0.2	0.2488	0.1241	0.1900	5.0583	0.0463	0.0476
	0.1	0.1	0.1	0.1567	0.1205	0.1743	0.2554	0.0504	0.0493
	0.2	0.2	0.2	1.0767	0.4001	0.1351	3.7793	0.3937	0.0765
	0.1	1.1	0.1	0.5260	0.8577	0.1146	6.1836	0.2664	0.0646
	1.0	2.0	3.0	6.3094	4.9136	4.0987	99.5357	1.0961	7.7823
	0.1	4.0	0.2	1.3531	3.4673	0.1959	18.2509	0.8912	0.0910
50	0.3	0.5	0.1	0.4312	0.1925	0.0972	0.3718	0.1436	0.0188
	0.3	0.6	0.2	1.5788	0.3212	0.1832	4.2086	0.1550	0.0570
	0.1	0.3	0.4	1.3101	0.3353	0.3756	14.7392	0.2407	0.0782
	0.1	0.1	0.2	0.1952	0.1134	0.1911	1.3772	0.0355	0.0442
	0.1	0.1	0.1	0.1478	0.1114	0.1789	0.1062	0.0360	0.0444
	0.2	0.2	0.2	0.8497	0.3559	0.1402	1.7679	0.3607	0.0728
	0.1	1.1	0.1	0.2444	0.9041	0.1054	0.3243	0.2113	0.0449
	1.0	2.0	3.0	3.6648	4.4701	4.5363	57.4154	0.9987	4.3836
	0.1	4.0	0.2	0.2964	3.6873	0.1828	2.4830	0.6372	0.0591
100	0.3	0.5	0.1	0.2794	0.1333	0.0986	0.2156	0.0921	0.0077
	0.3	0.6	0.2	1.1991	0.3026	0.1760	0.6835	0.1167	0.0429
	0.1	0.3	0.4	0.3674	0.2962	0.3843	4.8677	0.1101	0.0554
	0.1	0.1	0.2	0.1376	0.1051	0.1916	0.1150	0.0226	0.0341
	0.1	0.1	0.1	0.1201	0.1025	0.1900	0.0618	0.0217	0.0335
	0.2	0.2	0.2	0.6437	0.2971	0.1522	0.4319	0.2923	0.0625
	0.1	1.1	0.1	0.1497	0.9646	0.1006	0.1263	0.1284	0.0346
	1.0	2.0	3.0	0.5845	3.9230	4.6716	6.9230	0.6615	2.1013
	0.1	4.0	0.2	0.1485	3.8931	0.1842	0.1650	0.3192	0.0398
200	0.3	0.5	0.1	0.3256	0.4896	0.0993	0.1156	0.0532	0.0063
	0.3	0.6	0.2	1.1007	0.3376	0.1777	0.3973	0.1081	0.0285
	0.1	0.3	0.4	0.2167	0.2972	0.3947	0.1620	0.0345	0.0316
	0.1	0.1	0.2	0.1106	0.1010	0.1971	0.0544	0.0101	0.0207
	0.2	0.2	0.2	0.4121	0.2688	0.1647	0.2825	0.1856	0.0424
	0.1	0.1	0.1	0.1080	0.9952	0.0997	0.0486	0.0542	0.0084
	1.0	2.0	3.0	0.3564	3.8934	4.3273	0.1357	0.4554	1.2545
	0.1	4.0	0.2	0.1110	3.9780	0.1949	0.0454	0.1175	0.0283
300	0.3	0.5	0.1	0.3072	0.4970	0.0998	0.0667	0.0291	0.0030
	0.3	0.6	0.2	1.0538	0.3452	0.1781	0.3672	0.1093	0.0258
	0.1	0.3	0.4	0.2032	0.2992	0.3981	0.0606	0.0108	0.0186
	0.1	0.1	0.2	0.1033	0.1002	0.1991	0.0280	0.0056	0.0097
	0.1	0.1	0.1	0.1007	0.1000	0.1998	0.0150	0.0037	0.0047
	0.2	0.2	0.2	0.3272	0.2601	0.1712	0.2001	0.1272	0.0308
	0.1	1.1	0.1	0.1017	0.9987	0.1001	0.0228	0.0233	0.0089
	1.0	2.0	3.0	0.3544	3.8510	4.1248	0.0939	0.3743	0.9402
	0.1	4.0	0.2	0.1030	3.9955	0.1983	0.0199	0.0389	0.0138

5. Applications

In this section, we illustrate the performance of the $TIHL_{BX}$ distribution as compared to some alternative distributions using two real data applications. For each data set, the estimates of the model parameters by maximum likelihood estimation, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Consistent Akaike Information Criterion (CAIC) are calculated to compare the fitted models. The model with the smallest values of these measures represents the data better than the others. The numerical values of these measures for the $TIHL_{BX}$ and the other competing distributions of the first data are provided in Table 3 and for the second data set in Table 4. The competing distributions includes the beta BurrX (BBX) by [17], beta compound Rayleigh (BCR) [25], transmuted generalized Rayleigh (TGR) [15], Weibull Rayleigh (WR) [16], exponentiated generalized inverse Weibull (EGIW) [9], generalized half-logistic Poisson (GHLP) [20], half-logistic Poisson (HLP) [22], Kumaraswamy Exponentiated Inverse Rayleigh (KwEIR) [2], Transmuted Rayleigh (TR) [14], generalized BurrXII Poisson (GBXIIP) [18], type I half-logistic Frechet (TIHL-Fr) [6], complimentary exponentiated BurrXII Poisson (CEBXIIP) [21], Burr-XII (BXII) and Burr-X (BX) (or generalized Rayleigh (GR)) by [5] and Rayleigh (R) distributions.

The first data set consists of 63 observations of the strengths of 1.5 cm glass fibers obtained by workers at the UK National Physical Laboratory. The data are:

0.55, 0.74, 0.77, 0.81, 0.84,0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49,1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61,1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76,1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24. Also analyzed by [17, 25, 26].

The second data set is the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed. The data was provided and studied in [4]. The data set are: 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

Figure 4 is the TTT (total time on test) plot of: (i) First data set, it show that the data exhibit an increasing failure rate function and $TIHL_{BX}$ is capable of accommodating increasing failure rates. (ii) Second data set, it shows that the data exhibit an upside down bath-tube failure rate functions but we illustrate how $TIHL_{BX}$ accommodating the failure rate of the data. We also used the `muhaz` package in R software to obtain the empirical hazard function of each of the two data set and then fitted with the estimated hazard function of the $TIHL_{BX}$.

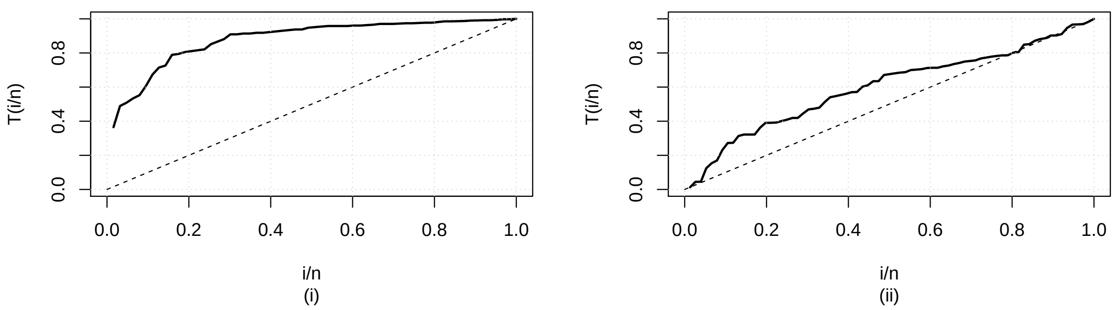


Figure 4: TTT-plots of (i) first data set, (ii) second data set.

The results presented in Tables 3 and 4 show that $TIHL_{BX}$ fit the two data set better than the other competing distributions since $TIHL_{BX}$ has the smallest values of the AIC, BIC, and CAIC.

Table 3: MLEs, L, AIC, BIC, and CAIC of the competing distributions for the first data set.

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$	L	AIC	BIC	CAIC
$TIHL_{BX}$	0.3407	–	2.8431	64.7396	–	–	−14.293	34.586	41.0153	34.993
BBX	0.3833	37.7339	7.8336	0.5352	–	–	−14.858	37.717	46.289	38.406
BCR	96.1649	–	489.55	–	5.0878	9.0725	−21.022	50.043	58.616	50.733
TGR	0.8302	–	4.8452	0.9000	–	–	−21.962	49.923	56.353	50.330
BX(GR)	0.9869	–	5.4860	–	–	–	−23.929	51.858	56.144	52.058
WR	1.2665	2.1534	0.4751	–	–	–	−14.643	35.286	41.716	35.693
EGIW	0.2896	81.371	9.7617	0.2651	–	–	−47.414	102.83	111.40	103.52
GHLP	$9.26e^{-3}$	30.136	–	500.64	–	–	−31.384	68.770	75.198	69.176
HLP	2.7427	–	–	−20.417	–	–	−29.294	62.587	66.874	62.787
CEBXIIP	2.9820	2.2085	4.3066	4.2033	–	–	−27.501	63.001	71.574	63.691
GBXIIP	7.0099	0.5258	$3.14e^{-9}$	2.0287	–	–	−42.323	92.647	101.219	93.336
BXII	7.4821	0.3207	–	–	–	–	−48.721	105.44	114.02	106.13
R	0.6490	–	–	–	–	–	−49.791	101.582	103.725	101.647

Table 4: MLEs, L, AIC, BIC and CAIC of the competing distributions for the second data set.

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$	L	AIC	BIC	CAIC
$TIHL_{BX}$	0.0135	–	0.5563	76.2386	–	–	−122.71	251.42	258.41	251.75
BBX	1.5863	27.8447	0.4957	0.0268	–	–	−122.38	252.76	262.08	253.32
TGR	0.2598	–	0.6237	0.7546	–	–	−122.84	251.68	258.67	252.01
TR	0.3406	–	–	0.7035	–	–	−130.87	265.75	270.41	265.91
TIHLFr	64.8083	0.3223	–	32.1303	–	–	−125.11	256.23	263.22	256.56
KwEIR	0.2041	–	0.1986	–	0.1987	0.1803	−190.89	387.79	394.78	388.12
BX(GR)	0.3165	–	0.5325	–	–	–	−125.50	255.00	259.67	255.17
EGIW	22.6658	0.5637	0.4344	36.8006	–	–	−128.51	265.02	274.34	265.58
HLP	0.0359	–	–	28.4939	–	–	−127.15	258.29	262.95	258.46
BXII	2.2306	0.6656	–	–	–	–	−128.55	265.11	274.43	265.67
R	0.3989	–	–	–	–	–	−137.32	276.64	278.97	276.69

Figure 5 provides the plots of the (i) histogram and estimated density (ii) empirical and estimated cdfs of the $TIHL_{BX}$ distribution for the first data set, and Figure 6 shows the (i) quantile-quantile plot, (ii) empirical and estimated hazard functions of the $TIHL_{BX}$ distribution for the first data set. While Figure 7 presents the plots of the (i) histogram and estimated density (ii) empirical and estimated cdfs of the $TIHL_{BX}$ distribution for the second data set, and Figure 8 gives the (i) quantile-quantile plot, (ii) empirical and estimated hazard functions of the $TIHL_{BX}$ distribution for the second data set.

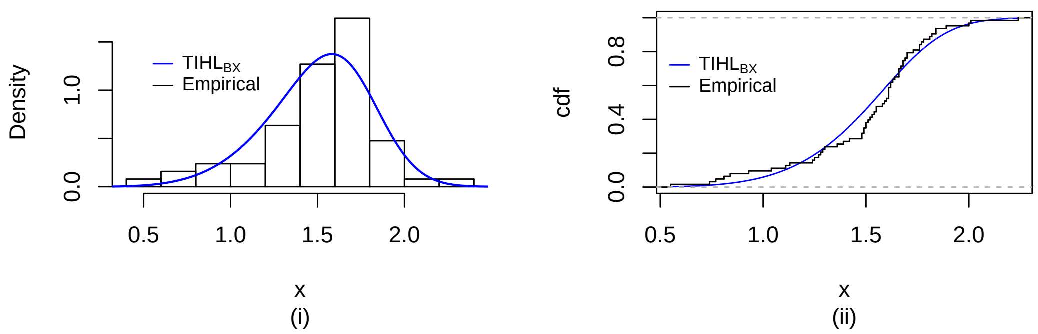


Figure 5: Plots of the (i) histogram and estimated density, (ii) empirical and estimated cdfs of the $TIHL_{BX}$ distribution for the first data set.

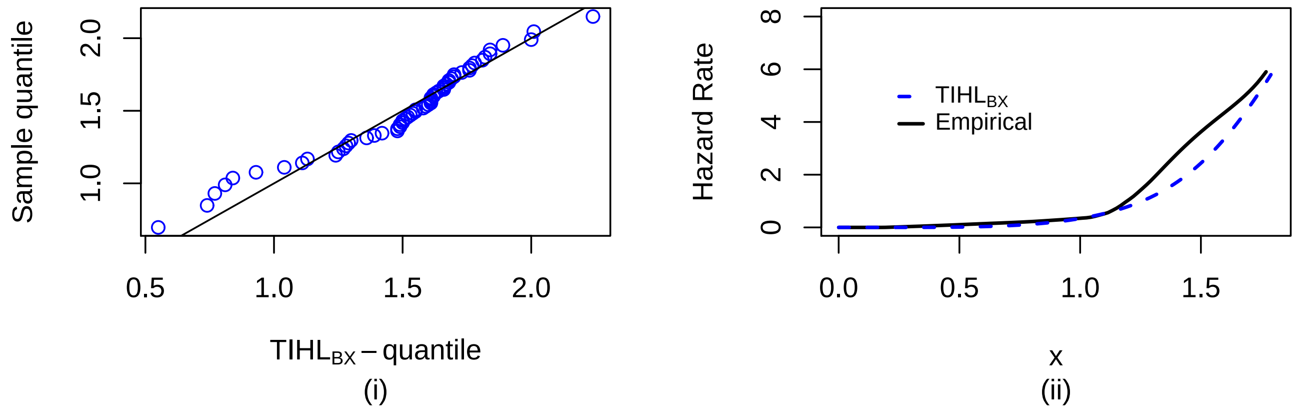


Figure 6: Plots of the (i) quantile-quantile, (ii) empirical and estimated hazard functions of the TIHL$_{BX}$ distribution for the first data set.

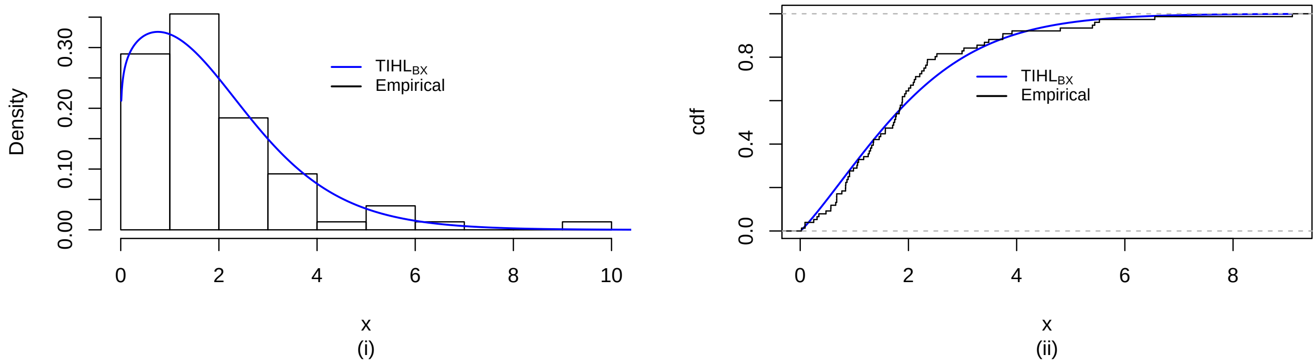


Figure 7: Plots of the (i) histogram and estimated density, (ii) empirical and estimated cdfs of the TIHL$_{BX}$ distribution for the second data set.

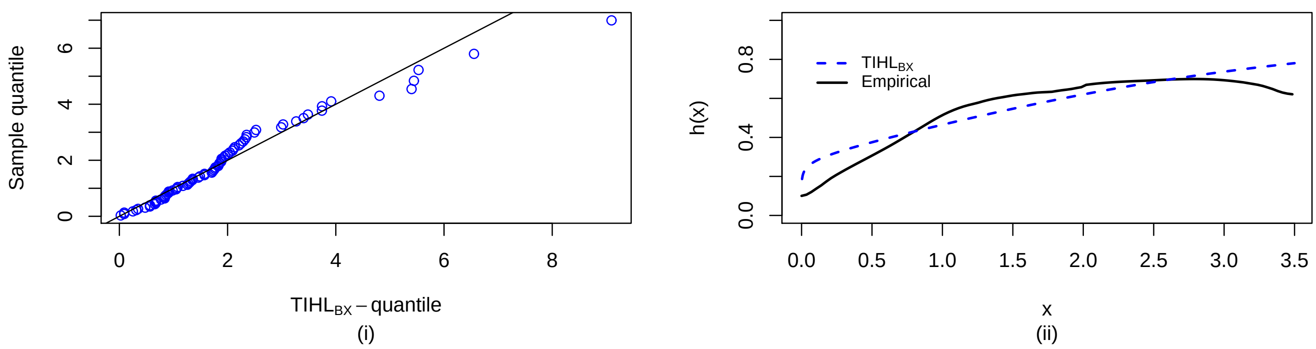


Figure 8: Plots of the (i) quantile-quantile, (ii) empirical and estimated hazard functions of the TIHL$_{BX}$ distribution for the second data set.

6. Conclusion

In this paper, we have derived and studied the properties and applications of the type I half-logistic Burr X ($TIHL_{BX}$) also called type I half-logistic generalized Rayleigh ($TIHL_{GR}$). The model extends the Burr X (BX) (or generalized Rayleigh (GR)) distribution. We provide an explicit mathematical expression for the moments, moment generating function, conditional moment, Bonferroni and Lorenz curves, mean deviations, residual life and reversed residual life functions, Shannon and Renyi entropies. We estimated the model parameters by the method of maximum likelihood and assessed by simulation studies. Finally, we fit the model to two real data set to demonstrate its usefulness and flexibility; the results show that the type I half-logistic Burr X ($TIHL_{BX}$) distribution provides a better fit than some other popular distributions as measured regarding the AIC, BIC, and CAIC. We hope that this distribution will attract wider applications in the areas of sciences and applied sciences.

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