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Relation theoretic contraction results in $\mathcal{F}$-metric spaces

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Abstract

Jleli and Samet in [M. Jleli, B. Samet, J. Fixed Point Theory Appl., **20** (2018), 20 pages] introduced a new metric space named as $\mathcal{F}$-metric space. They presented a new version of the Banach contraction principle in the context of this generalized metric spaces. The aim of this article is to define relation theoretic contraction and prove some generalized fixed point theorems in $\mathcal{F}$-metric spaces. Our results extend, generalize, and unify several known results in the literature.

Keywords: $\mathcal{F}$-metric space, relation theoretic contractions, fixed point, binary relation.

2010 MSC: 47H10, 47H06.

1. Introduction and preliminaries

The famous Banach contraction principle, which establishes that each single-valued contraction self map on a complete metric space has a unique fixed point, plays a central role in nonlinear analysis. Due to its significance and importance, many authors have established many fascinating generalizations of the Banach contraction principle; see [1–13, 15, 16, 24] and references therein.

Very recently, Jleli and Samet [14] presented a fascinating generalization of a metric space in the following way.

Assume that $\mathcal{F}$ be the set of continuous functions $f : (0, +\infty) \to \mathbb{R}$ which satisfies the conditions given below:

($\mathcal{F}_1$) f is non-decreasing, i.e., $0 < s < t \Longrightarrow f(s) \leqslant f(t)$;

($\mathcal{F}_2$) For every sequence $\{t_n\} \subseteq R^+$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} f(\alpha_n) = -\infty$.

Definition 1.1 ([14])**.** Let X be a nonempty set, and let $D : X \times X \to [0, +\infty)$ be a given mapping. Assume that there exists $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$ such that

(D_1) $(x, y) \in X \times X$, $D(x, y) = 0 \Longleftrightarrow x = y$;

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(D_2) $D(x,y) = D(y,x)$ for all $(x,y) \in X \times X$;

(D_3) for each $(x,y) \in X \times X$ and for every $N \in \mathbb{N}$, $N \geqslant 2$, and for each $(u_i)_{i=1}^{N} \subset X$, with $(u_1, u_N) = (x,y)$, we get

$$D(x,y) > 0 \Rightarrow f(D(x,y)) \leqslant f(\sum_{i=1}^{N-1} D(x_i, x_{i+1})) + \alpha.$$

Then D is said to be an $\mathcal{F}$-metric on X, and the pair (X,D) is called an $\mathcal{F}$-metric space.

Remark 1.2. They showed that any metric space is an $\mathcal{F}$-metric space but the converse is not true. It confirms that this conception is more prevalent than the standard metric definition.

Example 1.3 ([14])**.** Let $\mathbb{R}$ be the set of real numbers. $\mathbb{R}$ is called an $\mathcal{F}$-metric if we define D by

$$D(x,y) = \begin{cases} (x-y)^2, & \text{if } (x,y) \in [0,3] \times [0,3], \\ |x-y|, & \text{if } (x,y) \notin [0,3] \times [0,3], \end{cases}$$

with $f(s) = \ln(s)$ and $\alpha = \ln(3)$.

Definition 1.4 ([14])**.** Assume that (X,D) be an $\mathcal{F}$-metric space.

(i) Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is $\mathcal{F}$-convergent to $x \in X$ if $\{x_n\}$ is convergent to x with reference to the $\mathcal{F}$-metric D.

(ii) The sequence $\{x_n\}$ is $\mathcal{F}$-Cauchy, if

$$\lim_{n,m\to\infty} D(x_n, x_m) = 0.$$

(iii) If every $\mathcal{F}$-Cauchy sequence in X is $\mathcal{F}$-convergent to a precise element of X, then the pair (X,D) is called $\mathcal{F}$-complete.

Theorem 1.5 ([14])**.** *Let us assume that (X,D) be an $\mathcal{F}$-metric space and $F : X \to X$ be a given mapping. Assume that the following conditions fulfilled.*

(i) *The pair (X,D) is $\mathcal{F}$-complete.*

(ii) *There exists $k \in (0,1)$ which implies that*

$$D(F(x), F(y)) \leqslant kD(x,y).$$

Then F has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$, the sequence $\{x_n\} \subset X$ defined by

$$x_{n+1} = F(x_n), \quad n \in \mathbb{N},$$

is $\mathcal{F}$-convergent to x^*.

Afterward, Hussain et al. [11] considered the notion of α-ψ-contraction in the setting of $\mathcal{F}$-metric spaces and established a fixed point theorem given below.

Theorem 1.6 ([11])**.** *Assume that (X,D) be an $\mathcal{F}$-metric space and $F : X \to X$ be β-admissible mapping. Suppose that the following conditions are satisfied.*

(i) *(X,D) is $\mathcal{F}$-complete.*

(ii) *There exist two functions $\beta : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that*

$$\beta(x,y)D(F(x), F(y)) \leqslant \psi(M(x,y)),$$

where

$$M(x,y) = \max\{D(x,y), D(x,Fx), D(y,Fy)\}$$

for $x, y \in X$.

(iii) *There exists* $x_0 \in X$ *such that* $\beta(x_0, F(x_0))$.

Then F *contain a unique fixed point* $x^* \in X$.

Recently, Sawangsup et al. [25] defined $F_{\mathfrak{R}}$-contraction and established some fixed point results in which they included a binary relation. Now we give some definitions regarding binary relation.

Definition 1.7 ([17]). A binary relation on X is a nonempty subset $\mathfrak{R}$ of $X \times X$. $\mathfrak{R}$ is called transitive binary relation if $(x, z) \in \mathfrak{R}$ for all $x, y, z \in X$ such that $(x, y) \in \mathfrak{R}$ and $(y, z) \in \mathfrak{R}$.

If $(x, y) \in \mathfrak{R}$, we also denote it by $x\mathfrak{R}y$, and we say that "x is related to y".

Definition 1.8 ([3]). Given a mapping $F : X \to X$, the binary relation $\mathfrak{R}$ which is defined on X is F-closed if for any $x, y \in X$, $(x, y) \in \mathfrak{R} \Rightarrow (Fx, Fy) \in \mathfrak{R}$.

The above definition is equal to say that F is nondecreasing (see, for instance, [23]).

Proposition 1.9. *Let* $F : X \to X$ *a self-mapping on a nonempty set* X, *and* $\mathfrak{R}$ *a binary relation which is defined on* X. *If* $\mathfrak{R}$ *is* F*-closed, then* $\mathfrak{R}^s$ *is also* F*-closed.*

Definition 1.10 ([15]). Let X be a nonempty set and $\mathfrak{R}$ be a binary relation on X. Let k is a natural number, a path in $\mathfrak{R}$ from x to y is a finite sequence $\{z_0, z_1, z_2, \ldots, z_k\} \subseteq X$ which satisfies the following conditions:

(i) $z_0 = x$ and $z_k = y$;
(ii) $(z_i, z_{i+1}) \in \mathfrak{R}$ for all $i \in \{0, 1, 2, \ldots, k-1\}$ for all $x, y \in X$.

Let us denote the family of all paths in $\mathfrak{R}$ from x to y by $\Upsilon(x, y, \mathfrak{R})$.

Notice that a path of length k includes $k+1$ elements of X, it is not necessary that these elements are distinct.

Definition 1.11 ([23]). A metric space (X, d) endowed with a binary relation $\mathfrak{R}$ is $\mathfrak{R}$-nondecreasing-regular if for any sequence $\{x_n\} \subseteq X$,

$$\left.\begin{array}{r}(x_n, x_{n+1}) \in \mathfrak{R}, \forall n \in \mathbb{N}, \\ x_n \to x^* \in X,\end{array}\right\} \Longrightarrow (x_n, x^*) \in \mathfrak{R}, \forall n \in \mathbb{N}.$$

We denote by $X(F, \mathfrak{R})$ the set of all points $x \in X$ satisfying $(x, Fx) \in \mathfrak{R}$, where $\mathfrak{R}$ be a binary relation on a nonempty set X and $F : X \to X$ a self mapping.

Definition 1.12. Assume that the pair (X, D) be an $\mathcal{F}$-metric space. The binary relation $\mathfrak{R}$ defined on X is known as D-self-closed if $\{x_n\}$ is an $\mathfrak{R}$-preserving sequence and

$$x_n \overset{D}{\to} x^*,$$

then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x^*] \in \mathfrak{R}$ for all $k \in \mathbb{N}_0$.

Proposition 1.13. *Let* (X, D) *be an* $\mathcal{F}$*-metric space and* $\mathfrak{R}$ *a binary relation on* X, $F : X \to X$ *is a self-mapping and* $\lambda \in (0, 1)$, *then the contractivity conditions given below are equivalent:*

(1) $D(F(x), F(y)) \leqslant \lambda D(x, y)$ *for all* $x, y \in X$ *with* $(x, y) \in \mathfrak{R}$;
(2) $D(F(x), F(y)) \leqslant \lambda D(x, y)$ *for all* $x, y \in X$ *with* $[x, y] \in \mathfrak{R}$.

Proof. The implication (2)⇒(1) is trivial. Conversely, let us assume that (1) holds. Let $x, y \in X$ such that $[x, y] \in \mathfrak{R}$. If $(x, y) \in \mathfrak{R}$, then (2) directly follows from (1). But, if $(y, x) \in \mathfrak{R}$, then by the symmetry of D and (1), we get $D(Fx, Fy) = d(Fy, Fx) \leqslant D(y, x) = D(x, y)$, which shows that $(1) \Rightarrow (2)$. □

2. Main results

We are going to state our main result in the following way.

Theorem 2.1. *Assume that* (X, D) *be an* $\mathcal{F}$*-metric space and* $\mathfrak{R}$ *a binary relation on* X*. Let* $F : X \to X$ *be a self mapping satisfying the assertions given below*

(i) (X, D) *is* $\mathcal{F}$*-complete;*
(ii) $X(F, \mathfrak{R})$ *is nonempty;*
(iii) $\mathfrak{R}$ *is* F*-closed;*
(iv) *either* F *is continuous or* $\mathfrak{R}$ *is* D*-self-closed;*
(v) *there exists* $\lambda \in (0, 1)$ *such that*

$$D(F(x), F(y)) \leqslant \lambda D(x, y), \tag{2.1}$$

Then F *has a fixed point. Moreover, if*

(vi) $\mathcal{Y}(x, y, \mathfrak{R}^s)$ *is nonempty for each* $x, y \in X$*,*

then F *has a unique fixed point.*

Proof. Assume that $x_0 \in X(F, \mathfrak{R})$ be an arbitrary point. For such x_0, we define the sequence $\{x_n\}$ by $x_n = F^n(x_0) = F(x_{n-1})$ for all $n \in \mathbb{N}$. Since $(x_0, F(x_0)) \in \mathfrak{R}$ and $\mathfrak{R}$ is F-closed, so we have

$$(F(x_0), F^2(x_0)), (F^2(x_0), F^3(x_0)), \dots, (F^n(x_0), F^{n+1}(x_0)), \dots \in \mathfrak{R}$$

so that

$$(x_n, x_{n+1}) \in \mathfrak{R} \quad \text{for all } n \in \mathbb{N}.$$

Therefore the sequence $\{x_n\}$ is $\mathfrak{R}$-preserving. By (2.1), we get

$$D(x_n, x_{n+1}) = D(F(x_{n-1}), F(x_n)) \leqslant \lambda D(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$. By applying induction, we get

$$D(x_n, x_{n+1}) \leqslant \lambda^n D(x_0, F(x_0)),$$

which yields that

$$\sum_{i=n}^{m-1} D(x_i, x_{i+1}) \leqslant \frac{\lambda^n}{1-\lambda} D(x_0, F(x_0)), \quad m > n. \tag{2.2}$$

Since

$$\lim_{n\to\infty} \frac{\lambda^n}{1-\lambda} D(x_0, F(x_0)) = 0. \tag{2.3}$$

There exists some $N \in \mathbb{N}$ such that

$$0 < \frac{\lambda^n}{1-\lambda} D(x_0, F(x_0)) < \delta, \; n \geqslant N. \tag{2.4}$$

Next, let $(f, \alpha) \in F \times [0, +\infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Longrightarrow f(t) < f(\delta) - \alpha. \tag{2.5}$$

Hence, by (2.2), (2.3), (2.5), and $(\mathcal{F}_2)$, we have

$$f(\sum_{i=n}^{m-1} D(x_i, x_{i+1})) \leqslant f(\frac{\lambda^n}{1-\lambda} D(x_0, F(x_0))) < f(\epsilon) - \alpha \tag{2.6}$$

for $m > n \geqslant N$. Using (D_3) and (2.6), we obtain $D(x_n, x_m) > 0$, for $m > n \geqslant N$ implies

$$f(D(x_n, x_m)) \leqslant f(\sum_{i=n}^{m-1} D(x_i, x_{i+1})) + \alpha < f(\epsilon),$$

which implies by ($\mathcal{F}_1$) that $D(x_n, x_m) < \epsilon$, $m > n \geqslant N$. Thus it proved that $\{x_n\}$ is $\mathcal{F}$-Cauchy. Since (X, D) is $\mathcal{F}$-complete, there exists $x^* \in X$ such that $\{x_n\}$ is $\mathcal{F}$-convergent to x^*, i.e.,

$$\lim_{n\to\infty} D(x_n, x^*) = 0. \tag{2.7}$$

Now using the continuity of F, we obtain $F(x^*) = x^*$ and so x^* is a fixed point of F.

Alternately, let us assume that $\mathfrak{R}$ is D-self-closed. As $\{x_n\}$ is an $\mathfrak{R}$ preserving sequence and

$$x_n \xrightarrow{D} x^*$$

as $n \to \infty$. So there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x^*] \in \mathfrak{R}$ for all $k \in \mathbb{N}$. Using (v), Proposition 1.13, $[x_{n_k}, x^*] \in \mathfrak{R}$, and $x_{n_k} \xrightarrow{D} x^*$, we obtain

$$f(D(x^*, F(x^*))) \leqslant f(D(x^*, F(x_{n_k})) + D(F(x_{n_k}), F(x^*))) + \alpha.$$

Using (ii) and ($\mathcal{F}_1$), we obtain

$$f(D(x^*, F(x^*))) \leqslant f(D(x^*, x_{n_{k+1}})) + D(x_{n_k}, x^*)) + \alpha.$$

Using ($\mathcal{F}_2$) and (2.7), we have

$$\lim_{n\to\infty} f(D(x^*, x_{n_{k+1}})) + D(x_{n_k}, x^*)) + \alpha = -\infty.$$

We get a contradiction. Therefore, we have $D(x^*, F(x^*))$ that is $x^* = F(x^*)$. To prove uniqueness, take $x^*, x^{/}$ as two fixed points of F, i.e., $x^* = F(x^*)$ and $x^{/} = F(x^{/})$. By assumption (vi), there exists a path (say $\{y_0, y_1, y_2, \ldots, y_k\}$) of some finite length k in $\mathfrak{R}^s$ from x^* to $x^{/}$ so that

$$y_0 = x^*, \quad y_k = x^{/}, \quad [y_i, y_{i+1}] \in \mathfrak{R} \text{ for each } i\ (0 \leqslant i \leqslant k-1).$$

As $\mathfrak{R}$ is F-closed, by using Proposition 1.9, we have

$$[F^n y_i, F^n y_{i+1}] \in \mathfrak{R}$$

for each $i (0 \leqslant i \leqslant k-1)$ and for each $n \in \mathbb{N}$. We suppose that $x^* \neq x^{/}$, then $D(x^*, x^{/}) > 0$. Now

$$\begin{aligned} f(D(x^*, x^{/})) = f(D(F^n y_0, F^n y_k)) &\leqslant f(\sum_{i=0}^{k-1} D(F^n y_i, F^n y_{i+1})) + \alpha \\ &\leqslant f(\lambda \sum_{i=0}^{k-1} D(F^{n-1} y_i, F^{n-1} y_{i+1})) + \alpha \\ &\leqslant f(\lambda^2 \sum_{i=0}^{k-1} D(F^{n-1} y_i, F^{n-1} y_{i+1})) + \alpha \leqslant \cdots \leqslant f(\lambda^n \sum_{i=0}^{k-1} D(y_i, y_{i+1})) + \alpha. \end{aligned}$$

Using ($\mathcal{F}_2$) and (2.7), we have

$$\lim_{n\to\infty} f(\lambda^n \sum_{i=0}^{k-1} D(y_i, y_{i+1})) + \alpha = -\infty,$$

which is a contradiction. Therefore, we have $D(x^*, x^{/})$ that is $x^* = x^{/}$. Hence F has a unique fixed point.

If either $\mathfrak{R}$ is $\mathcal{F}$-complete or X is $\mathfrak{R}^s$-directed, then the following consequence is worth recording. □

Corollary 2.2. *Theorem 2.1 remains true if we replace condition* (vi) *by one of the following conditions (besides retaining the rest of the hypotheses):*

(vii) $\mathfrak{R}$ *is* $\mathcal{F}$*-complete;*
(viii) X *is* $\mathfrak{R}^s$*-directed.*

Proof. If $\mathfrak{R}$ is $\mathcal{F}$-complete, then for each $x, y \in X$, $[x, y] \in \mathfrak{R}$ which is equivalent to say that $\{x, y\}$ is a path of length 1 in $\mathfrak{R}^s$ from x to y so that $\mathcal{Y}(x, y, \mathfrak{R}^s)$ is nonempty. Hence Theorem 2.1 gives rise to the conclusion.

Otherwise, if X is $\mathfrak{R}^s$-directed, then for each $x, y \in X$, there exists $z \in X$ such that $[x, z] \in \mathfrak{R}$ and $[y, z] \in \mathfrak{R}$ so that $\{x, z, y\}$ is a path of length 2 in $\mathfrak{R}^s$ from x to y. Hence $\mathcal{Y}(x, y, \mathfrak{R}^s)$ is nonempty, for each $x, y \in X$ and again by Theorem 2.1 the conclusion is immediate. □

Example 2.3. Let $X = \mathbb{R}$ endowed with $\mathcal{F}$-complete $\mathcal{F}$-metric D given by

$$D(x, y) = \begin{cases} e^{|x-y|}, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Take $f(t) = \frac{-1}{t}$ and $\alpha = 1$. Define a binary relation $\mathfrak{R} = \{(x, y) \in \mathbb{R}^2 : x - y \geqslant 0, x \in Q\}$ on X and $F : X \to X$ by

$$F(x) = 2 + \frac{1}{2}x.$$

Clearly, $\mathfrak{R}$ is F-closed and F is continuous. Now, for $x, y \in X$ and $(x, y) \in \mathfrak{R}$, we have

$$D(F(x), F(y)) = e^{\frac{1}{2}|x-y|} < \frac{1}{2}e^{|x-y|} = \lambda D(x, y).$$

Hence F is a relation theoretical contraction for $\lambda = \frac{1}{2}$. Thus all the hypotheses (i)-(vi) of Theorem 2.1 are satisfied. Consequently F has a unique fixed point which is $x^* = 4$.

Notice that the underlying binary relation $\mathfrak{R}$ is a near-order. Indeed, $\mathfrak{R}$ is nonreflexive, nonirreflexive as well as nonsymmetric and hence it is not a preorder, partial order, strict order or tolerance and also never turns out to be a symmetric closure of any binary relation.

Corollary 2.4 ([14])**.** *Let the pair* (X, D) *be an* $\mathcal{F}$*-metric space and* $F : X \to X$ *be a given mapping. Assume that the following conditions satisfied:*

(i) (X, D) *is* $\mathcal{F}$*-complete;*
(ii) *there exists* $\lambda \in (0, 1)$ *such that*
$$D(F(x), F(y)) \leqslant \lambda D(x, y).$$

Then F *has a unique fixed point* $x^* \in X$*. Moreover, for any* $x_0 \in X$*, the sequence* $\{x_n\} \subset X$ *defined by*

$$x_{n+1} = F(x_n), \quad n \in \mathbb{N}$$

is $\mathcal{F}$*-convergent to* x^**.*

Proof. Taking the universal relation that is $\mathfrak{R} = X^2$ in Theorem 2.1, it is clear that under the universal relation, the conditions (ii), (iii), (iv), and (vi) of Theorem 2.1 hold trivially. □

If we set $\mathfrak{R} = \leqslant$, the partial order in Theorem 2.1, then we get the results given below. Clearly, presumption (iii) (i.e., $\leqslant$ is F-closed) is equivalent to the increasing property of F.

Theorem 2.5. *Assume that* $(X, \leqslant)$ *be a partially ordered set and let* D *be an* $\mathcal{F}$*-metric on* X *such that* (X, D) *is a* $\mathcal{F}$*-metric space. If* $F : X \to X$ *be a continuous and nondecreasing mapping such that:*

(i) *there exists* $k \in (0, 1)$ *such that* $D(F(x), F(y)) \leqslant \lambda D(x, y)$ *for all* $x \geqslant y$*;*

(ii) (X, D) *is* $\mathcal{F}$*-complete;*
(iii) *there exists* $x_0 \in X$ *such that* $x_0 \leqslant F(x_0)$,

then F *has a fixed point.*

Theorem 2.6. *Let us assume that* $(X, \leqslant)$ *be a partially ordered set and* D *be an* $\mathcal{F}$*-metric on* X *such that* (X, D) *is a* $\mathcal{F}$*-metric space. Assume that* X *satisfies*

$$\text{if a non-decreasing sequence } \{x_n\}_{n\in\mathbb{N}} \to x^* \text{ in } X, \text{ then } x_n \leqslant x^* \text{ for all } n \in \mathbb{N}.$$

If $F : X \to X$ *be a monotone nondecreasing mapping such that:*

(i) *there exists* $k \in (0, 1)$ *such that* $D(F(x), F(y)) \leqslant kD(x, y)$ *for all* $x \geqslant y$;
(ii) (X, D) *is* $\mathcal{F}$*-complete;*
(iii) *there exists* $x_0 \in X$ *such that* $x_0 \leqslant F(x_0)$,

then F *has a fixed point.*

Theorem 2.7. *Adding the following condition*

every pair of elements has a lower bound or an upper bound

to the hypotheses of Theorem 2.5 (Resp. 2.6), we obtain uniqueness of the fixed point of F.

If we set $\mathfrak{R} = \geqslant$, the dual relation associated with a partial order $\leqslant$ in Theorem 2.1, we obtain the following theorems in the context of $\mathcal{F}$-metric space. Clearly, assumption (iii) (i.e., $\geqslant$ is F-closed) is equivalent to the increasing property of F.

Theorem 2.8. *Let* $(X, \leqslant)$ *be a partially ordered set and let* D *be an* $\mathcal{F}$*-metric on* X *such that* (X, D) *is a* $\mathcal{F}$*-metric space. If* $F : X \to X$ *be a nondecreasing mapping such that*

(i) *there exists* $k \in (0, 1)$ *such that* $D(F(x), F(y)) \leqslant kD(x, y)$ *for all* $x \geqslant y$;
(ii) (X, D) *is* $\mathcal{F}$*-complete;*
(iii) *either* F *is continuous or* X *is such that*

$$\text{if a non-decreasing sequence } \{x_n\}_{n\in\mathbb{N}} \to x^* \text{ in } X, \text{ then } x^* \leqslant x_n \text{ for all } n \in \mathbb{N};$$

(iv) *there exists* $x_0 \in X$ *such that* $x_0 \geqslant F(x_0)$,

then F *has a fixed point.*

Theorem 2.9. *Adding condition*

every pair of elements has a lower bound or an upper bound

to the hypotheses of Theorem 2.8, we obtain uniqueness of the fixed point of F.

Corollary 2.10 ([14]). *Assume that* (X, D) *be an* $\mathcal{F}$*-metric space and* $F : X \to X$ *be a given mapping. Let us assume that the conditions given below are satisfied:*

(i) (X, D) *is* $\mathcal{F}$*-complete;*
(ii) *there exists* $k \in (0, 1)$ *such that*

$$D(F(x), F(y)) \leqslant kD(x, y).$$

Then F *has a unique fixed point* $x^* \in X$. *Moreover, for any* $x_0 \in X$, *the sequence* $\{x_n\} \subset X$ *defined by*

$$x_{n+1} = F(x_n), \quad n \in \mathbb{N}$$

is $\mathcal{F}$*-convergent to* x^*.

By Remark 1.2, we can deduces several well-known fixed point theorems of the existing literature from our main results as special cases such as

(1) the classical Banach contraction principle [7] from Corollary 2.4;
(2) Theorems 2.1, 2.2, and 2.3 of Nieto and Rodríguez-López [19] from Theorems 2.5, 2.6, and 2.7;
(3) Theorems 2.4 and 2.5 of Nieto and Rodríguez-López [19] from Theorems 2.8 and 2.9.

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