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Strong convergence theorems for mixed equilibrium problems and uniformly Bregman totally quasi-asymptotically nonexpansive mappings in reflexive Banach spaces

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Abstract

The purpose of this paper is to suggest a new algorithm for finding a common solution of a mixed equilibrium problem and a common fixed point of uniformly Bregman totally quasi-asymptotically nonexpansive mappings in reflexive Banach spaces. The strong convergence theorems under suitable control conditions are proven.

Keywords: Mixed equilibrium problems, Bregman totally quasi-asymptotically nonexpansive mappings, reflexive Banach spaces.

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1. Introduction

Let E be a real reflexive Banach space with dual space E^* and C be a nonempty closed convex subset of E. Let $g : C \times C \to \mathbb{R}$ be a bifunction and $\varphi : C \to \mathbb{R}$ be a real-valued function. The mixed equilibrium problem is to find $x \in C$ such that

$$g(x, y) + \varphi(y) - \varphi(x) \geqslant 0, \quad \forall y \in C. \tag{1.1}$$

Problem (1.1) is studied by Ceng and Yao [5]. Denote the set of solutions of problem (1.1) by $MEP(g, \varphi)$, i.e.,

$$MEP(g, \varphi) = \{x \in C : g(x, y) + \varphi(y) - \varphi(x) \geqslant 0, \quad \forall y \in C\}.$$

If $\varphi = 0$, then the mixed equilibrium problem (1.1) turns into the following equilibrium problem, which is to find $x \in C$ such that

$$g(x, y) \geqslant 0, \quad \forall y \in C. \tag{1.2}$$

Problem (1.2) is studied by Blum and Oettli [1]. Denote the set of solutions of problem (1.2) by $EP(g)$, i.e.,

$$EP(g) = \{x \in C : g(x, y) \geqslant 0, \quad \forall y \in C\}.$$

If $g(x, y) = 0$ for all $x, y \in C$, then the mixed equilibrium problem (1.1) turns into the following minimize

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problem, which is to find $x \in C$ such that

$$\varphi(y) - \varphi(x) \geqslant 0, \quad \forall y \in C. \tag{1.3}$$

Denote the set of solutions of problem (1.3) by $\operatorname{Argmin}(\varphi)$, i.e.,

$$\operatorname{Argmin}(\varphi) = \{x \in C : \varphi(y) - \varphi(x) \geqslant 0, \quad \forall y \in C\}.$$

The mixed equilibrium problems include fixed point problems, variational inequality problems, equilibrium problems and optimization problems.

It appeared that the fixed point theory of nonexpansive mappings can be applied to solving solutions of certain evolution equations and solving convex feasibility, variational inequality and equilibrium problems. There are, many researchers that deal with a variety of procedures for finding fixed points of nonexpansive mappings and quasi-nonexpansive mappings in Hilbert spaces, uniformly convex and uniformly smooth Banach spaces.

Whenever the researchers attempted to extend this theory to generalized Banach spaces, they discovered some difficulties and there are a lot of ways to overpower these barriers, for instant, using the Bregman distance in place of the norm, Bregman (quasi-)nonexpansive mappings in place of the (quasi-) nonexpansive mappings and the Bregman projection in place of the metric projection.

In 1967, Bregman [2] encountered a technique using the Bregman distance function $D_f(\cdot,\cdot)$ in calculating and analyzing optimization and feasibility algorithms. Bregman's technique has been applied in a variety of ways.

In 2011, Reich and Sabach [12] introduced the new concept of Bregman nonexpansive mappings, that is Bregman strongly nonexpansive mappings and initiatively studied the convergence theorems of two iterative algorithms for finding common fixed points of finitely many Bregman strongly nonexpansive mappings in reflexive Banach spaces.

In 2012, Suantai et al. [15] also considered and obtained the strong convergence theorems for Bregman strongly nonexpansive mappings by Halpen's iteration in reflexive Banach spaces.

In 2014, Chang et al. [7] extended the notion of Bregman nonexpansive mappings and introduced the new concept of these mappings namely Bregman totally quasi-asymptotically nonexpansive mappings and achieved an iteration for finding common fixed points for Bregman totally quasi-asymptotically nonexpansive mappings in reflexive Banach spaces. Moreover, they also proved strong convergence theorems for the mentioned mappings.

In 2016, Zhu and Huang [17] proposed the iterative methods for finding the common solutions of the equilibrium problems and fixed points for Bregman totally quasi-asymptotically nonexpansive mappings in reflexive Banach spaces. Furthermore, they proved and obtained the strong convergence theorems.

The purpose of this paper is to suggest a new algorithm for finding a common solution of a mixed equilibrium problem and a common fixed point of uniformly Bregman totally quasi-asymptotically nonexpansive mappings in reflexive Banach spaces. The strong convergence theorems under suitable control conditions are proven.

2. Preliminaries

In this section, we commence by recalling some preliminaries and lemmas which will be used for proving our main results.

In every part of this paper, we let E be a real reflexive Banach space and let E^* be its dual, a function $f : E \to (-\infty, +\infty]$ be a proper and the Fenchel conjugate of f be the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}, \quad \forall x^* \in E^*,$$

where $\langle\cdot,\cdot\rangle : E^* \to E$ is a duality pairing and $\{x \in E : f(x) < \infty\}$ is the set of the domain of f, written by dom(f). For any $x \in \text{int}(\text{dom}(f))$ and $y \in E$ the right-hand derivative of f at x in the direction y defined by

$$f^0(x,y) = \lim_{t \to 0^+} \frac{f(x+ty)-f(x)}{t}.$$

The function f is called Gâteaux differentiable at x if $\lim_{t \to 0^+} \frac{f(x+ty)-f(x)}{t}$ exists for any y.

In this case, $f^0(x,y)$ corresponds to $\nabla f(x)$, the value of the gradient ∇f of f at x. The function f is called Gâteaux differentiable, if it is Gâteaux differentiable for any $x \in \text{int}(\text{dom}(f))$. The function f is called Frêchet differentiable at x, if this limit is attained uniformly in $\|y\| = 1$. Finally, f is called uniformly Frêchet differentiable on a subset C of E, if the above limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

The following important lemma was proved by Reich and Sabach [12].

Lemma 2.1 ([12]). *Let K be a bounded subset of a real reflexive Banach space E and $f : E \to (-\infty, +\infty]$ be uniformly Frêchet differentiable and bounded on $K \subset E$. Then f is uniformly continuous on bounded subset $K \subset E$ and ∇f is uniformly continuous on bounded subset $K \subset E$ from the strong topology of E to the strong topology of E^*.*

Definition 2.2. A function $f : E \to (-\infty, +\infty]$ is said to be "Legendre", if the following statements are satisfied:

(L1) the interior of the domain of f, $\text{int}(\text{dom}(f))$ is nonempty, f is Gâteaux differentiable on $\text{int}(\text{dom}(f))$ and $\text{dom}\nabla f = \text{int}(\text{dom}(f))$;

(L2) the interior of the domain of f^*, $\text{int}(\text{dom}(f)^*)$ is nonempty, f^* is Gâteaux differentiable on $\text{int}(\text{dom}(f)^*)$ and $\text{dom}\nabla f^* = \text{int}(\text{dom}(f)^*)$.

Recall the subdifferential of f at $x \in \text{int}(\text{dom}(f))$ is the convex set defined by

$$\partial f(x) := \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leqslant f(y), \quad \forall y \in E\}.$$

Remark 2.3. If E is a real reflexive Banach space and $f : E \to (-\infty, +\infty]$ is the Legendre function, then all of the following conditions are true:

(a) the function f is the Legendre function if and only if the function f^* is the Legendre function;
(b) an inverse of subdifferential of f is equal to subdifferential of f^*, $(\partial f)^{-1} = \partial f^*$;
(c) $\nabla f = (\nabla f^*)^{-1}$, $\text{ran}\nabla f = \text{dom}\nabla f^* = \text{int}(\text{dom}(f)^*)$ and $\text{ran}\nabla f^* = \text{dom}\nabla f = \text{int}(\text{dom}(f))$;
(d) the functions f and f^* are strictly convex on $\text{int}(\text{dom}(f))$ and $\text{int}(\text{dom}(f)^*)$, respectively.

Definition 2.4 ([6]). Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Define the function $D_f : \text{dom}(f) \times \text{int}(\text{dom}(f)) \to [0, +\infty)$ by

$$D_f(y,x) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle.$$

In this situation, $D_f(\cdot,\cdot)$ is called the Bregman distance with respect to f.

It ought to be noted that the Bregman distance is not a distance in the usual sense of the term. By previous definition, in general sense of $D_f(\cdot,\cdot)$, we can conclude that it is not symmetric and has no triangle inequality property.

Definition 2.5 ([2]). Let C be a nonempty closed convex subset of dom(f), $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The Bregman projection of $x \in \text{int}(\text{dom}(f))$ onto $C \subset \text{dom}(f)$ is the necessarily unique vector $\text{proj}_C^f(x) \in C$ satisfying the following:

$$D_f(\text{proj}_C^f(x), x) = \inf\{D_f(y,x) : y \in C\}.$$

Definition 2.6 ([3]). Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function and $\nu_f : \text{int}(\text{dom}(f)) \times [0, +\infty) \to [0, +\infty)$, define the modulus of total convexity of the function f at x by

$$\nu_f(x, t) := \inf\{D_f(y, x) : y \in \text{dom}(f), \|y - x\| = t\}.$$

Then function f is called to be

(a) totally convex at a point $x \in \text{int}(\text{dom}(f))$, if the modulus of total convexity of the function f at x is positive, $\nu_f(x, t) > 0$ whenever $t > 0$;

(b) totally convex, if it is totally convex at every point $x \in \text{int}(\text{dom}(f))$.
Let B be a nonempty bounded subset of E, define the modulus of total convexity of the function f on the set B by

$$\nu_f(B, t) := \inf\{\nu_f(x, t) : x \in B \cap \text{dom}(f)\};$$

(c) totally convex on bounded sets, if the modulus of total convexity of the function f on the set B is positive, $\nu_f(B, t) > 0$ for any nonempty bounded subset B of E and $t > 0$.

Lemma 2.7 ([12]). *If x is an element in* $\text{int}(\text{dom}(f))$, *then the following conditions are equivalent:*

(1) *the function f is totally convex at x;*
(2) *for any sequence* $\{x_n\} \subset \text{dom}(f)$ *such that if* $\lim_{n\to\infty} D_f(x_n, x) = 0$, *then* $\lim_{n\to\infty} \|x_n - x\| = 0$;
(3) *for any* $\varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon) > 0$ *such that if* $y \in \text{dom}(f)$ *and* $D_f(y, x) \leqslant \delta$, *then* $\|x - y\| \leqslant \varepsilon$.

Recall that the function $f : E \to (-\infty, +\infty]$ is called sequentially consistent, if for any two sequences and $\{x_n\}$ and $\{y_n\}$ in $\text{int}(\text{dom}(f))$ and $\text{dom}(f)$, respectively such that sequence $\{x_n\}$ is bounded, then

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \ \text{ implies } \ \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 2.8 ([3]). *The function* $f : E \to (-\infty, +\infty]$ *is totally convex on bounded sets if and only if it is sequentially consistent.*

Lemma 2.9 ([13]). *Let* $f : E \to (-\infty, +\infty]$ *be a Gâteaux differentiable and totally convex function. If* $x_0 \in E$ *and the sequence* $\{D_f(x_n, x_0)\}$ *is bounded, then the sequence* $\{x_n\}$ *is also bounded.*

Lemma 2.10 ([4]). *Let* $f : E \to (-\infty, +\infty]$ *be a Gâteaux differentiable function and totally convex on* $\text{int}(\text{dom}(f))$, *C be a nonempty closed convex subset of* $\text{int}(\text{dom}(f))$ *and x be an element in* $\text{int}(\text{dom}(f))$. *If* $x \in C$, *then the following conditions are equivalent:*

(1) $z \in C$ *is the Bregman projection of x onto* $C \subset \text{int}(\text{dom}(f))$ *with respect to f denoted by* $z = \text{proj}_C^f(x)$;
(2) *the vector z is the unique solution of the variational inequality:*

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \geqslant 0, \quad \forall y \in C;$$

(3) *the vector z is the unique solution of the inequality:*

$$D_f(y, z) + D_f(z, x) \leqslant D_f(y, x), \quad \forall y \in C.$$

Definition 2.11. Let C be a subset of E and T be a mapping from C into itself. Denote the set of all fixed points of T by $F(T) = \{x \in C : Tx = x\}$. A point $p \in C$ is called an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote $\hat{F}(T)$ the set of asymptotic fixed points of T. A mapping T is called to be

(a) nonexpansive, if $\|Tx - Ty\| \leqslant \|x - y\|$;
(b) quasi-nonexpansive, if F(T) is nonempty and $\|Tx - p\| \leqslant \|x - p\|, \ \forall x \in C, p \in F(T)$;
(c) closed, if for any sequence $\{x_n\} \subset C$ with $x_n \to x \in C$ and $Tx_n \to y \in C$, then $Tx = y$;

(d) uniformly asymptotically regular on C, if $\lim_{n\to\infty} \sup_{x\in C}\|T^{n+1}x - T^n x\| = 0$;

(e) Bregman firmly nonexpansive, if

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leqslant D_f(Tx, y) + D_f(Ty, x),\ \forall x, y \in C;$$

(f) Bregman strongly nonexpansive with respect to a nonempty $\hat{F}(T)$, if

$$D_f(p, Tx) \leqslant D_f(p, x),\quad \forall x \in C, p \in \hat{F}(T);$$

(g) Bregman relatively nonexpansive, if $F(T)$ is nonempty, $F(T) = \hat{F}(T)$ and

$$D_f(p, Tx) \leqslant D_f(p, x),\quad \forall x \in C, p \in F(T);$$

(h) Bregman quasi-nonexpansive, if $F(T)$ is nonempty and

$$D_f(p, Tx) \leqslant D_f(p, x), \forall x \in C, p \in F(T);$$

(i) Bregman quasi-asymptotically nonexpansive, if $F(T)$ is nonempty and there exists a real sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ such that

$$D_f(p, T^n x) \leqslant k_n D_f(p, x),\quad \forall x \in C, p \in F(T);$$

(j) Bregman totally quasi-asymptotically nonexpansive, if $F(T)$ is nonempty and there exist real sequences $\{\lambda_n\}, \{\mu_n\}$ in $\mathbb{R}^+$ with $\lambda_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\delta(0) = 0$ such that

$$D_f(p, T^n x) \leqslant D_f(p, x) + \lambda_n \delta(D_f(p, x)) + \mu_n,\quad \forall n \geqslant 1,\quad \forall x \in C, p \in F(T).$$

Definition 2.12. A countable family of mappings $\{T_i\} : C \to C$ is called to be uniformly Bregman totally quasi-asymptotically nonexpansive, if $\mathcal{F} := \bigcap_{i=1}^{\infty} F(T_i)$ is nonempty and there exist real sequences $\{\lambda_n\}, \{\mu_n\}$ in $\mathbb{R}^+$ with $\lambda_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\delta(0) = 0$ such that

$$D_f(p, T_i^n x) \leqslant D_f(p, x) + \lambda_n \delta(D_f(p, x)) + \mu_n,\quad \forall n \geqslant 1, i \geqslant 1,\quad \forall x \in C, p \in \mathcal{F}.$$

Remark 2.13. Consequences of previous definitions, we can explain the correlation as follows:

(1) each Bregman relatively nonexpansive mapping can be extended to Bregman quasi-nonexpansive mapping;

(2) each Bregman quasi-nonexpansive mapping can be extended to Bregman quasi-asymptotically nonexpansive mapping. Indeed, if we take $k_n = 1$, then we have

$$D_f(p, T^n x) \leqslant k_n D_f(p, Tx) \leqslant k_n D_f(p, x),\quad \forall x \in C, p \in F(T);$$

(3) each Bregman quasi-asymptotically nonexpansive mapping can be extended to Bregman totally quasi-asymptotically nonexpansive mapping, but the converse maybe not guaranteed. Indeed, if we take $\delta(t) = t, t \geqslant 0, \lambda_n = k_n - 1$ and $\mu_n = 0$, then equation (2.3) can be rewritten as

$$D_f(p, T^n x) \leqslant D_f(p, x) + \lambda_n \delta(D_f(p, x)) + \mu_n,\quad \forall x \in C, p \in F(T).$$

As a direct consequence of Remark 2.13, we obtain that each Bregman relatively nonexpansive mapping must be a Bregman totally quasi-asymptotically nonexpansive mapping, but the converse is not true.

Lemma 2.14 ([14]). *Let K be a bounded subset of a real reflexive Banach space E and $f : E \to (-\infty, +\infty]$ be a Legendre function which is uniformly Fréchet differentiable and bounded on $K \subset E$. Let C be a nonempty closed convex subset of E and $T : C \to C$ be a Bregman firmly nonexpansive mapping with respect to f. Then $F(T) = \hat{F}(T)$.*

Let C be a nonempty closed convex subset of a real reflexive Banach space E, and $g : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

(C1) $g(x,x) = 0, \quad \forall x \in C$;
(C2) g is monotone, i.e., $g(x,y) + g(y,x) \leqslant 0, \quad \forall x, y \in C$;
(C3) $\forall x, y, z \in C, \limsup_{t \to 0^+} g(tz + (1-t)x, y) \leqslant g(x,y)$;
(C4) $\forall x \in C, g(x, \cdot)$ is convex and lower semicontinuous.

Definition 2.15. Let $f : E \to (-\infty, +\infty]$. We say that f is a strong coercive function if $\lim_{\|x\| \to \infty} \frac{f(x)}{\|x\|} = +\infty$.

The following crucial lemma was proved by Vahid [8].

Lemma 2.16 ([8]). *Let $f : E \to (-\infty, +\infty]$ be a strong coercive Legendre function and C be a nonempty closed convex subset of $\mathrm{int}(\mathrm{dom}(f))$. Let $\varphi : C \to \mathbb{R}$ be a proper lower semi-continuous and convex function. Assume that $g : C \times C \to \mathbb{R}$ satisfies conditions (C1)-(C4). For $x \in E$ define a mapping $\mathrm{Res}^f_{g,\varphi} : E \to 2^C$ as follows:*

$$\mathrm{Res}^f_{g,\varphi}(x) = \{z \in C : g(z,y) + \varphi(y) - \varphi(z) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geqslant 0, \quad \forall y \in C\}.$$

Then the following statements are true:

(1) $\mathrm{Res}^f_{g,\varphi}$ *is single-valued and* $\mathrm{dom}(\mathrm{Res}^f_{g,\varphi}) = E$;
(2) $\mathrm{Res}^f_{g,\varphi}$ *is Bregman firmly nonexpansive;*
(3) $\mathrm{MEP}(g,\varphi)$ *is a closed convex subset of* C *and* $\mathrm{MEP}(g,\varphi) = F(\mathrm{Res}^f_{g,\varphi})$;
(4) *for all* $x \in E, u \in F(\mathrm{Res}^f_{g,\varphi})$,

$$D_f(u, \mathrm{Res}^f_{g,\varphi}x) + D_f(\mathrm{Res}^f_{g,\varphi}x, x) \leqslant D_f(u, x).$$

Lemma 2.17 ([10]). *Let K be a bounded subset of $\mathrm{int}(\mathrm{dom}(f))$ and $f : E \to \mathbb{R}$ be a Legendre function such that ∇f^* is bounded on $K \subset \mathrm{int}(\mathrm{dom}(f))$ and x be an element in E. If $\{D_f(x, x_n)\}$ is bounded, then sequence $\{x_n\}$ is also bounded.*

The following important lemma was proved by Chang et al. [7].

Lemma 2.18 ([7]). *Let K be a bounded subset of a real reflexive Banach space E, C be a nonempty closed convex subset of E and $f : E \to (-\infty, +\infty]$ be a Legendre function which is total convex on $K \subset E$. Let $T : C \to C$ be a closed and Bregman totally quasi-asymptotically nonexpansive mapping with real sequences $\{\lambda_n\}, \{\mu_n\}$ in $\mathbb{R}^+$ and a strictly increasing continuous function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lambda_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and $\delta(0) = 0$. Then $F(T)$ is a closed convex subset of C.*

Lemma 2.19 ([13]). *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function, x_0 be an element in E and C be a nonempty closed convex subset of E. Suppose that the sequence $\{x_n\}$ is bounded and the weak limits of any subsequence of a sequence $\{x_n\}$ belong to $C \subset E$. If $D_f(x_n, x_0) \leqslant D_f(\mathrm{proj}^f_C(x_0), x_0)$ for any $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $\mathrm{proj}^f_C(x_0)$.*

3. Main result

In this section, we introduce a new algorithm for finding a common solution of a mixed equilibrium problem and a common fixed point of uniformly Bregman totally quasi-asymptotically nonexpansive mappings in reflexive Banach spaces. We also prove the strong convergence theorems under suitable control conditions.

In the following, we illustrate the main result of this paper.

Theorem 3.1. *Let K be a bounded subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ be a strong coercive Legendre function which is bounded, uniformly Frêchet differentiable and totally convex on bounded subset $K \subset E$. Let $C \subset E$ be a nonempty closed convex subset of $\operatorname{int}(\operatorname{dom}(f))$, $\{T_m : C \to C\}_{m=1}^{\infty}$ be a countable family of closed and uniformly Bregman totally quasi-asymptotically nonexpansive mappings with real sequences $\{\lambda_n\}, \{\mu_n\}$ in $\mathbb{R}^+$ and a strictly increasing continuous function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lambda_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and $\delta(0) = 0$. Let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions* (C1)-(C4) *and $\varphi : C \to \mathbb{R}$ be a proper lower semicontinuous and convex function.*

Assume that $\{T_m\}_{m=1}^{\infty}$ is a countable family of uniformly asymptotically regular mappings and $\Omega := \bigcap_{m=1}^{\infty} F(T_m) \cap MEP(g, \varphi) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 = u \in C, \quad \text{chosen arbitrarily}, \\ u_n^m : g(u_n^m, y) + \varphi(y) - \varphi(u_n^m) + \langle \nabla f(u_n^m) - \nabla f(T_m^n x_n), y - u_n^m \rangle \geqslant 0, \quad \forall y \in C, \quad \forall m \geqslant 1, \\ C_n = \{z \in C : \sup_{m \geqslant 1} D_f(z, u_n^m) \leqslant D_f(z, x_n) + \xi_n\}, \\ D_n = \bigcap_{i=1}^{n} C_i, \\ x_{n+1} = \operatorname{proj}_{D_n}^f u, \end{cases} \tag{3.1}$$

where $\xi_n = \lambda_n \sup \delta(D_f(\nu, x_n)) + \mu_n$ and $\operatorname{proj}_{D_n}^f$ is the Bregman projection of E onto D_n.

If $\Omega := \bigcap_{m=1}^{\infty} F(T_m) \cap MEP(g, \varphi)$ is bounded, then the sequence $\{x_n\}$ converges strongly to $\bar{x} = \operatorname{proj}_{\Omega}^f u$.

Proof. We separate the proof into five steps.

Step 1: We will show that Ω is a closed convex subset of E, by using Lemmas 2.16 and 2.18, and we shall show that D_n is a closed convex subset of E, by substantiation that C_n is a closed and convex subset of E. It follows from Lemma 2.16 that $MEP(g, \varphi)$ is closed and convex. By Lemma 2.18, we have $F(T_m), m \geqslant 1$ is closed and convex. This implies that Ω is also closed and convex.

In the next process, we prove that C_n is closed and convex. Let $\nu \in \Omega$ be given. Since $\operatorname{Res}_{g,\varphi}^f$ is single-valued, $u_n^m = \operatorname{Res}_{g,\varphi}^f(T_m^n(x_n)), m \geqslant 1$ and by Lemma 2.16 (4), we obtain that

$$D_f(\nu, \operatorname{Res}_{g,\varphi}^f(T_m^n(x_n))) + D_f(\operatorname{Res}_{g,\varphi}^f(T_m^n(x_n)), T_m^n(x_n)) \leqslant D_f(\nu, T_m^n(x_n)).$$

It follows that

$$D_f(\nu, \operatorname{Res}_{g,\varphi}^f(T_m^n(x_n))) \leqslant D_f(\nu, T_m^n(x_n)) - D_f(\operatorname{Res}_{g,\varphi}^f(T_m^n(x_n)), T_m^n(x_n)) \leqslant D_f(\nu, T_m^n(x_n)), \quad \forall m \geqslant 1.$$

This implies that

$$D_f(\nu, u_n^m) = D_f(\nu, \operatorname{Res}_{g,\varphi}^f(T_m^n(x_n))) \leqslant D_f(\nu, T_m^n(x_n)) \leqslant D_f(\nu, x_n) + \nu_n \zeta(D_f(\nu, x_n)) + \mu_n.$$

Therefore

$$D_f(\nu, u_n^m) \leqslant D_f(\nu, x_n) + \xi_n, \quad \forall m \geqslant 1,$$

where $\xi_n = \lambda_n \sup \delta(D_f(\nu, x_n)) + \mu_n$. It follows that

$$\sup_{m \geqslant 1} D_f(\nu, u_n^m) \leqslant D_f(\nu, x_n) + \xi_n.$$

This yields $v \in C_n$ for all $n \geqslant 1$. Hence $\Omega \subset C_n$. Therefore $\Omega \subset D_n$. Suppose that $u, v \in C_n, s \in (0,1)$. Setting $z = su + (1-s)v$, we will prove that $z \in C_n$. Since $u, v \in C_n$, we obtain that $\sup_{m\geqslant 1} D_f(u, u_n^m) \leqslant D_f(u, x_n) + \xi_n$ and $\sup_{m\geqslant 1} D_f(v, u_n^m) \leqslant D_f(v, x_n) + \xi_n$. By definition of $D_f(\cdot,\cdot)$, we have

$$\sup_{m\geqslant 1} (f(u) - f(u_n^m) - \langle \nabla f(u_n^m), u - u_n^m\rangle) \leqslant f(u) - f(x_n) - \langle \nabla f(x_n), u - x_n\rangle + \xi_n.$$

This implies that

$$f(u) - f(u_n^m) - \langle \nabla f(u_n^m), u - u_n^m\rangle \leqslant f(u) - f(x_n) - \langle \nabla f(x_n), u - x_n\rangle + \xi_n, \quad \forall m \geqslant 1.$$

This yields

$$f(x_n) - f(u_n^m) \leqslant \langle \nabla f(u_n^m), u - u_n^m\rangle - \langle \nabla f(x_n), u - x_n\rangle + \xi_n, \quad \forall m \geqslant 1.$$

It follows that

$$f(x_n) - f(u_n^m) \leqslant \langle \nabla f(u_n^m), u - u_n^m\rangle - \langle \nabla f(x_n), u - x_n\rangle + \xi_n.$$

Similarly, we also obtain that

$$f(x_n) - f(u_n^m) \leqslant \langle \nabla f(u_n^m), v - u_n^m\rangle - \langle \nabla f(x_n), v - x_n\rangle + \xi_n.$$

Therefore

$$f(x_n) - f(u_n^m) \leqslant \langle \nabla f(u_n^m), su + (1-s)v - u_n^m\rangle + \langle \nabla f(x_n), su + (1-s)v - x_n\rangle + \xi_n.$$

This implies that

$$f(x_n) - f(u_n^m) \leqslant \langle \nabla f(u_n^m), z - u_n^m\rangle - \langle \nabla f(x_n), z - x_n\rangle + \xi_n, \quad \forall m \geqslant 1.$$

It follows that $z \in C_n$ and thus C_n is convex. Therefore D_n is also convex. Suppose that $\{w_i\}$ is a sequence in C_n and $w_i \to w$ (as $i \to \infty$). Therefore

$$\begin{aligned} f(x_n) - f(u_n^m) &\leqslant \langle \nabla f(u_n^m), w_i - u_n^m\rangle - \langle \nabla f(x_n), w_i - x_n\rangle + \xi_n \\ &= \langle \nabla f(u_n^m), w_i - w + w - u_n^m\rangle - \langle \nabla f(x_n), w_i - w + w - x_n\rangle + \xi_n \\ &= \langle \nabla f(u_n^m), w_i - w\rangle + \langle \nabla f(u_n^m), w - u_n^m\rangle - \langle \nabla f(x_n), w_i - w\rangle - \langle \nabla f(x_n), w - x_n\rangle + \xi_n. \end{aligned}$$

By taking the limit as $i \to \infty$, we obtain that

$$f(x_n) - f(u_n^m) \leqslant \langle \nabla f(u_n^m), w - u_n^m\rangle - \langle \nabla f(x_n), w - x_n\rangle + \xi_n, \quad \forall m \geqslant 1.$$

This yields $w \in C_n$. It follows that C_n is closed and convex for any $n \geqslant 1$. Therefore D_n is also closed and convex for any $n \geqslant 1$. This implies that the sequence $\{x_n\}$ is well-defined.

Step 2: We will prove that $\{x_n\}$ is a bounded sequence. Since $x_{n+1} = \mathrm{proj}^f_{D_n} u$ and by Lemma 2.10 (3), we obtain that

$$D_f(x_{n+1}, u) = D_f(\mathrm{proj}^f_{D_n} u, u) \leqslant D_f(v, u) - D_f(v, \mathrm{proj}^f_{D_n} u) \leqslant D_f(v, u)$$

for all $v \in \Omega$. This implies that $\{D_f(x_{n+1}, u)\}$ is a bounded sequence. By Lemma 2.9, we have $\{x_n\}$ is also a bounded sequence.

Step 3: We will prove that $\{x_n\}$ is a Cauchy sequence, that is, to show that $\lim_{m,n\to\infty}\|x_m - x_n\| = 0$. Since $x_{n+1} = \text{proj}^f_{D_n} u$, $x_{n+2} = \text{proj}^f_{D_{n+1}} u \in D_{n+1} \subset D_n$ and by Lemma 2.10 (3), we obtain that

$$D_f(x_{n+2}, \text{proj}^f_{D_n} u) + D_f(\text{proj}^f_{D_n} u, u) \leqslant D_f(x_{n+2}, u).$$

It follows that

$$D_f(x_{n+2}, x_{n+1}) + D_f(x_{n+1}, u) \leqslant D_f(x_{n+2}, u).$$

Therefore $\{D_f(x_n, u)\}$ is an increasing sequence. This implies that $\lim\limits_{n\to\infty} D_f(x_n, u)$ exists. By definition of D_n, we have that $D_p \subset D_n$ for any positive integer $p \geqslant n$. This yields $x_p = \text{proj}^f_{D_{p-1}} u \in D_{p-1} \subset D_{n-1}$ for any positive integer $p \geqslant n$. For each positive integer $p \geqslant n$, we obtain that

$$D_f(x_p, x_n) = D_f(x_p, \text{proj}^f_{D_{n-1}} u) \leqslant D_f(x_p, u) - D_f(\text{proj}^f_{D_{n-1}} u, u) = D_f(x_p, u) - D_f(x_n, u).$$

By taking $p, n \to \infty$, we have

$$D_f(x_p, x_n) \to 0.$$

Using Lemma 2.8, we obtain that

$$\lim_{p,n\to\infty} \|x_p - x_n\| = 0.$$

Step 4: We show that $\{x_n\}$ converges to a point in $\Omega := \bigcap\limits_{m=1}^{\infty} F(T_m) \cap MEP(g, \varphi)$. Since $\{x_n\}$ is a Cauchy sequence in a Banach space E, we can assume that

$$\lim_{n\to\infty} x_n = x^* \in C.$$

We now show that $x^* \in F(T_m)$ for all $m \geqslant 1$. By taking $p = n+1$, we obtain that $\lim\limits_{n\to\infty} D_f(x_{n+1}, x_n) = 0$. Using Lemma 2.8, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Since $x_{n+1} = \text{proj}^f_{D_n} \in D_n \subset C_n$, we obtain that

$$\sup_{m\geqslant 1} D_f(x_{n+1}, u^m_n) \leqslant D_f(x_{n+1}, x_n) + \xi_n,$$

where $\xi_n = \lambda_n \sup \delta(D_f(\nu, x_n)) + \mu_n$. It follows from $\lim\limits_{n\to\infty} D_f(x_{n+1}, x_n) = 0$, $\lambda_n \to 0$, and $\mu_n \to 0$ (as $n \to \infty$), that

$$\lim_{n\to\infty} \left(\sup_{m\geqslant 1} D_f(x_{n+1}, u^m_n) \right) = 0.$$

Since f is lower semicontinuous and $\sup\limits_{m\geqslant 1} D_f(\nu, u^m_n) \leqslant D_f(\nu, x_n) + \xi_n$, we obtain that $D_f(\nu, x_n) + \xi_n$ is a bounded sequence. This implies that $\{D_f(\nu, u^m_n)\}_{n=1}^{\infty}$ is also bounded. Using Lemma 2.17, we obtain that $\{u^m_n\}_{n=1}^{\infty}$ is a bounded sequence. Therefore

$$\lim_{n\to\infty} \|x_{n+1} - u^m_n\| = 0 \text{ for all } m \geqslant 1.$$

For each $m \geqslant 1$, we have

$$\|x_n - u_n^m\| \leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - u_n^m\|.$$

This yields

$$\lim_{n\to\infty} \|x_n - u_n^m\| = 0 \text{ for all } m \geqslant 1.$$

Since f is uniformly Frêchet differentiable and by Lemma 2.1, we obtain that

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(u_n^m)\| = 0 \text{ for all } m \geqslant 1.$$

Using Lemma 2.1, we have f is also uniformly continuous and then

$$\lim_{n\to\infty} |f(x_n) - f(u_n^m)| = 0 \text{ for all } m \geqslant 1.$$

Furthermore, we have

$$\begin{aligned}
D_f(v, x_n) - D_f(v, u_n^m) &= f(v) - f(x_n) - \langle \nabla f(x_n), v - x_n\rangle - [f(v) - f(u_n^m) - \langle \nabla f(u_n^m), v - u_n^m\rangle] \\
&= f(u_n^m) - f(x_n) + \langle \nabla f(u_n^m), v - u_n^m\rangle - \langle \nabla f(x_n), v - x_n\rangle \\
&= f(u_n^m) - f(x_n) + \langle \nabla f(u_n^m), x_n - u_n^m + v - x_n\rangle - \langle \nabla f(x_n), v - x_n\rangle \\
&= f(u_n^m) - f(x_n) + \langle \nabla f(u_n^m), x_n - u_n^m\rangle + \langle \nabla f(u_n^m), v - x_n\rangle - \langle \nabla f(x_n), v - x_n\rangle \\
&= f(u_n^m) - f(x_n) + \langle \nabla f(u_n^m), x_n - u_n^m\rangle + \langle \nabla f(u_n^m) - \nabla f(x_n), v - x_n\rangle.
\end{aligned}$$

Since $\{u_n^m\}_{n=1}^\infty$ is bounded, we have $\nabla f(u_n^m)$ is also bounded. This implies that

$$\lim_{n\to\infty} (D_f(v, x_n) - D_f(v, u_n^m)) = 0 \quad \text{for all } m \geqslant 1.$$

Since $u_n^m = \text{Res}^f_{g,\varphi}(T_m^n(x_n))$ and by Lemma 2.16 (4) we have

$$\begin{aligned}
D_f(u_n^m, T_m^n(x_n)) &= D_f(\text{Res}^f_{g,\varphi}(T_m^n(x_n)), T_m^n(x_n)) \\
&\leqslant D_f(v, T_m^n(x_n)) - D_f(v, \text{Res}^f_{g,\varphi}(T_m^n(x_n))) \\
&\leqslant D_f(v, x_n) + \lambda_n \delta(D_f(v, x_n)) + \mu_n - D_f(v, u_n^m).
\end{aligned}$$

Since $\{D_f(v, x_n)\}$ is bounded, we have $\delta(D_f(v, x_n))$ is also bounded. From $\lambda_n \to 0, \mu_n \to 0$ (as $n \to \infty$), we obtain that

$$\lim_{n\to\infty} D_f(u_n^m, T_m^n(x_n)) = 0 \quad \text{for all } m \geqslant 1.$$

Using Lemma 2.8, this yields

$$\lim_{n\to\infty} \|u_n^m - T_m^n(x_n)\| = 0 \quad \text{for all } m \geqslant 1.$$

Since

$$\|x_n - T_m^n(x_n)\| \leqslant \|x_n - u_n^m\| + \|u_n^m - T_m^n(x_n)\|,$$

we obtain that

$$\lim_{n\to\infty} \|x_n - T_m^n(x_n)\| = 0 \quad \text{for all } m \geqslant 1.$$

Similarly, since

$$\|x^* - T_m^n(x_n)\| \leqslant \|x^* - x_n\| + \|x_n - T_m^n(x_n)\|,$$

we have

$$\lim_{n\to\infty} \|x^* - T_m^n(x_n)\| = 0 \ \text{ for all } m \geqslant 1.$$

Moreover, we also have

$$\|x^* - T_m^{n+1}(x_n)\| \leqslant \|x^* - T_m^n(x_n)\| + \|T_m^n(x_n) - T_m^{n+1}(x_n)\|.$$

Since $\{T_m\}_{m=1}^{\infty}$ is uniformly asymptotically regular, we obtain that

$$\lim_{n\to\infty} \|x^* - T_m^{n+1}(x_n)\| = 0 \ \text{ for all } m \geqslant 1.$$

This implies that $T_m T_m^n(x_n) \to x^*$ (as $n \to \infty$), $\forall m \geqslant 1$.

By assumption we know that T_m is closed, it follows from Definition 2.11 (c), we obtain $T_m x^* = x^*$. Therefore for each $m \geqslant 1$, we have $x^* \in \bigcap_{m=1}^{\infty} F(T_m)$. Since f is uniformly Frêchet differentiable and by Lemma 2.1, we obtain that ∇f is uniformly continuous on bounded sets. It follows that

$$\lim_{n\to\infty} \|\nabla f(u_n^m) - \nabla f(T_m^n(x_n))\| = 0 \ \text{ for all } m \geqslant 1.$$

Since $u_n^m = \mathrm{Res}_{g,\varphi}^f(T_m^n(x_n))$, we have

$$g(u_n^m, y) + \varphi(y) - \varphi(u_n^m) + \langle \nabla f(u_n^m) - \nabla f(T_m^n(x_n)), y - u_n^m \rangle \geqslant 0 \ \text{ for all } y \in C.$$

From (C2), we obtain that

$$\varphi(y) - \varphi(u_n^m) + \langle \nabla f(u_n^m) - \nabla f(T_m^n(x_n)), y - u_n^m \rangle \geqslant -g(u_n^m, y) \geqslant g(y, u_n^m)$$

for all $y \in C$ and for all $m \geqslant 1$. For any $y \in C$ and $t \in (0,1]$, we let $y_t = ty + (1-t)x^* \in C$. This implies

$$\varphi(y_t) - \varphi(u_n^m) - \langle \nabla f(u_n^m) - \nabla f(T_m^n(x_n)), y_t - u_n^m \rangle \geqslant g(y_t, u_n^m).$$

Using (C4), this yields

$$g(y_t, x^*) + \varphi(x^*) - \varphi(y_t) \leqslant 0.$$

It follows that

$$\begin{aligned}
0 &= g(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&= g(y_t, ty + (1-t)x^*) + \varphi(ty + (1-t)x^*) - \varphi(y_t) \\
&\leqslant tg(y_t, y) + (1-t)g(y_t, x^*) + t\varphi(y) + (1-t)\varphi(x^*) - t\varphi(y_t) - (1-t)\varphi(y_t) \\
&= t\,[g(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\,[g(y_t, x^*) + \varphi(x^*) - \varphi(y_t)] \\
&\leqslant t\,[g(y_t, y) + \varphi(y) - \varphi(y_t)]\,.
\end{aligned}$$

Therefore

$$g(y_t, y) + \varphi(y) - \varphi(y_t) \geqslant 0.$$

From (C3), we have

$$\begin{aligned}
0 \leqslant \limsup_{t\to 0^+} (g(y_t, y) + \varphi(y) - \varphi(y_t)) &= \limsup_{t\to 0^+} (g(ty + (1-t)x^*, y) + \varphi(y) - \varphi(y_t)) \\
&\leqslant g(x^*, y) + \varphi(y) - \varphi(x^*).
\end{aligned}$$

This implies that $x^* \in \mathrm{MEP}(g, \varphi)$. Hence $x^* \in \Omega := \bigcap_{m=1}^{\infty} F(T_m) \cap \mathrm{MEP}(g, \varphi)$.

Step 5: We shall show that $\{x_n\}$ converges strongly to $\bar{x} = \mathrm{proj}^f_\Omega u$. By Lemmas 2.16 and 2.18, we have $\bigcap_{m=1}^{\infty} F(T_m) \cap MEP(g,\varphi)$ is a nonempty closed convex subset of E. It follows that $\mathrm{proj}^f_\Omega u$ is well-defined. Since $\mathrm{proj}^f_\Omega u \subset C_n \subset D_n$ and $x_{n+1} = \mathrm{proj}^f_{D_n} u$, we have

$$D_f(x_{n+1}, u) \leqslant D_f(\mathrm{proj}^f_\Omega u, u).$$

By Lemma 2.19, we have $x_n \to \mathrm{proj}^f_\Omega u$ (as $n \to \infty$). This implies that $\{x_n\}$ strongly converges to $\bar{x} = \mathrm{proj}^f_\Omega u$. This completes the proof. □

Remark 3.2. According to the proof of Theorem 3.1, we can keep track of the iteration (3.1), we can choose $x_1 = u \in C$. By Lemma 2.16 (1), we obtain that, for each $m \geqslant 1$,

$$\mathrm{Res}^f_{g,\varphi}(T^1_m(x_1)) = \{z \in C : g(z,y) + \varphi(y) - \varphi(z) + \langle \nabla f(z) - \nabla f(T^1_m(x_1)), y - z\rangle \geqslant 0, \quad \forall y \in C\}$$

is single-valued. Iteration (3.1) implies that u^m_1 is contained in $\mathrm{Res}^f_{g,\varphi}(T^1_m(x_1))$. Therefore, $u^m_1 \in C$ and $g(u^m_1, y) + \varphi(y) - \varphi(u^m_1) + \langle \nabla f(u^m_1) - \nabla f(T^1_m(x_1)), y - u^m_1\rangle \geqslant 0$ for all $y \in C$. In step 1 of the proof of Theorem 3.1, we obtain that C_1 is a nonempty closed convex set. By the construction of D_1, we have D_1 is also a nonempty closed convex set. Therefore we can find $x_2 \in C$ and repeat of this process. It follows that $\{x_n\}$ is well-defined.

Remark 3.3. Theorem 3.1 can be applied to solving the common solutions of the fixed point problems and equilibrium problems and obtain the strong convergence theorems.

As a consequence of Theorem 3.1, if we set $T_m = T$ for all $m \geqslant 1$ being a Bregman totally quasi-asymptotically nonexpansive mapping and $\varphi : C \to \mathbb{R}$ being a zero mapping, we immediately obtain the following corollary.

Corollary 3.4 ([17]). *Let K be a bounded subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ be a strong coercive Legendre function which is bounded, uniformly Frêchet differentiable and totally convex on $K \subset E$. Let $C \subset E$ be a nonempty closed and convex subset of* $\mathrm{int}(\mathrm{dom}(f))$, $T : C \to C$ *be a closed Bregman totally asymptotically quasi-nonexpansive mapping with real sequences $\{\lambda_n\}, \{\mu_n\}$ in $\mathbb{R}^+$ and a strictly increasing continuous function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lambda_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and $\delta(0) = 0$. Let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions* (C1)-(C4). *Assume that T is uniformly asymptotically regular, $\Omega := F(T) \cap EP(g) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = u \in C, \text{chosen arbitrarily}, \\ u_n : g(u_n, y) + \langle \nabla f(u_n) - \nabla f(T^n x_n), y - u_n\rangle \geqslant 0, \quad \forall y \in C, \\ C_n = \{z \in C : D_f(z, u_n) \leqslant D_f(z, x_n) + \xi_n\}, \\ D_n = \bigcap_{i=1}^{n} C_i, \\ x_{n+1} = \mathrm{proj}^f_{D_n} u, \end{cases} \tag{3.2}$$

where $\xi_n = \lambda_n \sup \delta(D_f(\nu, x_n)) + \mu_n$ and $\mathrm{proj}^f_{D_n}$ is the Bregman projection of E onto D_n.

If $\Omega := F(T) \cap EP(g)$ is bounded, then the sequence $\{x_n\}$ converges strongly to $\bar{x} = \mathrm{proj}^f_\Omega u$.

Recall the generalization duality mapping $J_p : E \to E^*$ defined by $J_p(x) := \{x^* \in E^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2, \ \forall x \in E\}$.

Remark 3.5. Theorem 3.1 can be applied to the generalization duality mapping $J_p : E \to E^*$ from E onto the dual space E^*.

If in Theorem 3.1, we suppose that E is a uniformly smooth and uniformly convex Banach space and $f(x) = \frac{1}{p}\|x\|^p \ (1 < p < \infty)$, we obtain that $\nabla f = J_p$, where J_p is the generalization duality mapping from E onto E^*. Thus we obtain the following corollary.

Corollary 3.6. *Let E be a uniformly smooth and uniformly convex Banach space and $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$. Let C be a nonempty closed convex subset of $\text{int}(\text{dom}(f))$, $\{T_m : C \to C\}_{m=1}^{\infty}$ be a countable family of closed and uniformly Bregman totally quasi-asymptotically nonexpansive mappings with real sequences $\{\lambda_n\}, \{\mu_n\}$ in $\mathbb{R}^+$ and a strictly increasing continuous function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lambda_n \to 0, \mu_n \to 0$ (as $n \to \infty$) and $\delta(0) = 0$. Let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (C1)-(C4) and $\varphi : C \to \mathbb{R}$ be a proper lower semicontinuous and convex function.*

Assume that $\{T_m\}_{m=1}^{\infty}$ is uniformly asymptotically regular and $\Omega := \bigcap_{m=1}^{\infty} F(T_m) \cap MEP(g, \varphi) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 = u \in C, \text{chosen arbitrarily}, \\ u_n^m : g(u_n^m, y) + \varphi(y) - \varphi(u_n^m) + \langle J_p(u_n^m) - J_p(T_m^n x_n), y - u_n^m \rangle \geqslant 0, \quad \forall y \in C, \quad \forall m \geqslant 1, \\ C_n = \{z \in C : \sup_{m \geqslant 1} D_f(z, u_n^m) \leqslant D_f(z, x_n) + \xi_n\}, \\ D_n = \bigcap_{i=1}^{n} C_i, \\ x_{n+1} = \text{proj}_{D_n}^f u, \end{cases} \tag{3.3}$$

where $\xi_n = \lambda_n \sup \delta(D_f(v, x_n)) + \mu_n$ and $\text{proj}_{D_n}^f$ is the Bregman projection of E onto D_n.

If $\Omega := \bigcap_{m=1}^{\infty} F(T_m) \cap MEP(g, \varphi)$ is bounded, then the sequence $\{x_n\}$ converges strongly to $\overline{x} = \text{proj}_{\Omega}^f u$.

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