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Dass and Gupta's fixed point theorem in $\mathcal{F}$-metric spaces

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Abstract

The purpose of this article is to define Dass and Gupta's contraction in the context of $\mathcal{F}$-metric spaces and obtain some new fixed point theorems to elaborate, generalize and synthesize several known results in the literature including Jleli and Samet [M. Jleli, B. Samet, J. Fixed Point Theory Appl., **20** (2018), 20 pages] and Dass and Gupta [B. K. Dass, S. Gupta, Indian J. Pure Appl. Math., **6** (1975), 1455–1458]. Also we have provided a non trivial example to validate our main result.

Keywords: $\mathcal{F}$-metric space, fixed point, rational contraction.

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1. Introduction and preliminaries

In the field of fixed point theory, to find the solution of fixed point problems, the contractive conditions on underlying functions play a significance role. The fundamental result in metric fixed point theory is Banach contraction principle [2] which was first introduced by the great Polish Mathematician Stephan Banach in this way.

Theorem 1.1 ([2])**.** *Let (X, d) be a complete metric space, and let $T : X \to X$ be a self mapping. If there exists $\mu \in [0, 1)$ such that*

$$d(T(x), T(y)) \leqslant \mu d(x, y),$$

for all $x, y \in X$, then T has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$, the sequence $\{x_n\} \subset X$ defined by

$$x_{n+1} = T(x_n), \quad n \in \mathbb{N},$$

is $\mathcal{F}$-convergent to x^*.

Over the years, due to its importance and applications in different fields of science, several authors generalized and elaborated the well known Banach Contraction Principle by introducing rational contractions in complete metric spaces. One of these attempts is due to Jaggi [5] in this way.

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Theorem 1.2. *Assume that* (X, d) *be a complete metric space and let* $T : X \to X$ *be a self mapping. If there exist* $\mu_1, \mu_2 \in [0, 1)$ *with* $\mu_1 + \mu_2 < 1$ *such that*

$$d(T(x), T(y)) \leqslant \mu_1 d(x, y) + \mu_2 \frac{d(x, T(x))d(y, T(y))}{d(x, y)}$$

for all $x, y \in X$ *with* $x \neq y$, *then* T *has a unique fixed point* $x^* \in X$.

On the other hand, in [7] Khan proved the following fixed point theorem.

Theorem 1.3. *Assume that the pair* (X, d) *be a complete metric space and* T *be a self-mapping on* X *that satisfies the following contractive condition*

$$d(Tx, Ty) \leqslant \mu \frac{d(x, Tx)d(x, Ty) + d(y, Tx)d(y, Ty)}{d(x, Ty) + d(y, Tx)}, \tag{1.1}$$

for all $x, y \in X$ *and for some* $\mu \in [0, 1)$. *Then* T *has a unique fixed point in* X.

Remark 1.4. In the inequality (1.1) if the denominator vanishes, then $x = Ty$ and $y = Tx$ and consequently also the numerator vanishes. Moreover, we have $d(Tx, Ty) = d(y, x)$ and so the contractive condition is not well defined.

In 1975, Dass and Gupta gave the following fixed point theorem of new rational contraction to generalize the Banach contraction principle.

Theorem 1.5 ([3])**.** *Let* (X, d) *be a complete metric space, and let* $T : X \to X$ *be a self mapping. If there exist* $\mu_1, \mu_2 \in [0, 1)$ *with* $\mu_1 + \mu_2 < 1$ *such that*

$$d(T(x), T(y)) \leqslant \mu_1 d(x, y) + \mu_2 \frac{[1 + d(x, T(x))]d(y, T(y))}{1 + d(x, y)},$$

then T *has a unique fixed point* $x^* \in X$.

Very recently, Jleli and Samet [6] established an interesting generalization of a metric space in the following way.

Let $\mathcal{F}$ be the set of continuous functions $f : (0, +\infty) \to \mathbb{R}$ satisfying the following conditions:

($\mathcal{F}_1$) f is non-decreasing, i.e., $0 < s < t \Longrightarrow f(s) \leqslant f(t)$;

($\mathcal{F}_2$) for every sequence $\{t_n\} \subseteq R^+$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} f(\alpha_n) = -\infty$.

Definition 1.6 ([6])**.** Assume that X be a nonempty set, and $D : X \times X \to [0, +\infty)$ be a given mapping. Suppose that there exists $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$ such that

(D_1) $(x, y) \in X \times X$, $D(x, y) = 0 \Longleftrightarrow x = y$;

(D_2) $D(x, y) = D(y, x)$, for all $(x, y) \in X \times X$;

(D_3) for every $(x, y) \in X \times X$, for every $N \in \mathbb{N}$, $N \geqslant 2$, and for every $(u_i)_{i=1}^{N} \subset X$ with $(u_1, u_N) = (x, y)$ we have

$$D(x, y) > 0 \Rightarrow f(D(x, y)) \leqslant f(\sum_{i=1}^{N-1} D(x_i, x_{i+1})) + \alpha.$$

Then D is said to be an $\mathcal{F}$-metric on X, and the pair (X, D) is said to be an $\mathcal{F}$-metric space.

Remark 1.7. They showed that any metric space is an $\mathcal{F}$-metric space but the converse is not true in general, which validate that the concept of $\mathcal{F}$-metric space is more general than the standard metric concept.

Example 1.8 ([6]). The set of real numbers $\mathbb{R}$ is an $\mathcal{F}$-metric Space if we define D by

$$D(x,y) = \begin{cases} (x-y)^2 & \text{if } (x,y) \in [0,3] \times [0,3], \\ |x-y| & \text{if } (x,y) \notin [0,3] \times [0,3], \end{cases}$$

with $f(t) = \ln(t)$ and $\alpha = \ln(3)$.

Definition 1.9 ([6]). Assume that (X, D) be an $\mathcal{F}$-metric space.

(i) Let $\{x_n\}$ be a sequence in X. We call the sequence $\{x_n\}$ is $\mathcal{F}$-convergent to $x \in X$, if $\{x_n\}$ is convergent to x with respect to the $\mathcal{F}$-metric D.

(ii) A sequence $\{x_n\}$ is called $\mathcal{F}$-Cauchy, if

$$\lim_{n,m\to\infty} D(x_n, x_m) = 0.$$

(iii) If every $\mathcal{F}$-Cauchy sequence in X is $\mathcal{F}$-convergent to a certain element in X, then the pair (X, D) is called $\mathcal{F}$-complete.

Theorem 1.10 ([6]). *Let the pair (X, D) be an $\mathcal{F}$-metric space and $T : X \to X$ be a given mapping. Assume that the following conditions are satisfied*

(i) *(X, D) is $\mathcal{F}$-complete;*

(ii) *there exists $k \in (0, 1)$ such that*

$$D(T(x), T(y)) \leqslant kD(x, y).$$

Then T has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$, the sequence $\{x_n\} \subset X$ defined by

$$x_{n+1} = T(x_n), \quad n \in \mathbb{N},$$

is $\mathcal{F}$-convergent to x^*.

Afterward, Hussain et al. [4] considered the concept of β-ψ-contraction in the setting of $\mathcal{F}$-metric spaces and established the fixed point theorem given below.

Theorem 1.11 ([4]). *Let the pair (X, D) be an $\mathcal{F}$-metric space and $T : X \to X$ be β-admissible mapping. Assume that the conditions given below are satisfied*

(i) *(X, D) is $\mathcal{F}$-complete;*

(ii) *there exist two functions $\beta : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that*

$$\beta(x, y)D(T(x), T(y)) \leqslant \psi(M(x, y)),$$

where

$$M(x, y) = \max\{D(x, y), D(x, Tx), D(y, Ty)\},$$

for $x, y \in X$;

(iii) *there exists $x_0 \in X$ such that $\beta(x_0, T(x_0))$.*

Then T has a unique fixed point $x^* \in X$.

Later on [1] Alnaser et al. defined relation theoretic contractions and extended Jleli et al. [6] results and some known results of literature.

In the present paper, first we define Dass and Gupta's contraction in the context of $\mathcal{F}$-metric spaces and then prove some new fixed point theorems. In this way, we extend, generalize and elaborate several known results of the literature. We also support our main result by providing a non trivial example.

2. Main results

In this section, we first define Dass and Gupta's contraction in the context of $\mathcal{F}$-metric space and then establish a new fixed point theorem.

Definition 2.1. Assume that (X, D) be an $\mathcal{F}$-metric space. A self mapping T on X is said to be Dass and Gupta's contraction, if there exist $\mu_1, \mu_2 \in (0, 1)$ with $\mu_1 + \mu_2 < 1$ such that

$$D(T(x), T(y)) \leqslant \mu_1 D(x, y) + \mu_2 \frac{[1 + D(x, T(x))]D(y, T(y))}{1 + D(x, y)},$$

for $(x, y) \in X \times X$.

Theorem 2.2. *Assume that (X, D) be an $\mathcal{F}$-metric space, and let $T : X \to X$ be a Dass and Gupta's contraction. If (X, D) is $\mathcal{F}$-complete, then T has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$, the sequence $\{x_n\} \subset X$ defined by*

$$x_{n+1} = T(x_n), \quad n \in \mathbb{N}, \tag{2.1}$$

is $\mathcal{F}$-convergent to x^*.

Proof. First, we observe that T has at most one fixed point. In fact, if $(u, v) \in X \times X$ are two fixed points of T with $u \neq v$, i.e.

$$D(u, v) > 0, \qquad T(u) = u, \qquad T(v) = v.$$

As $T : X \to X$ is Dass and Gupta's contraction, so

$$\begin{aligned}
D(u, v) = D(T(u), T(v)) &\leqslant \mu_1 D(u, v) + \mu_2 \frac{[1 + D(u, T(u))]D(v, T(v))}{1 + D(u, v)} \\
&= \mu_1 D(u, v) + \mu_2 \frac{[1 + D(u, u)]D(v, v)}{1 + D(u, v)} \\
&= \mu_1 D(u, v) \\
&< D(u, v),
\end{aligned}$$

which is a contradiction.

Next, let $(f, \alpha) \in F \times [0, +\infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Longrightarrow f(t) < f(\delta) - \alpha. \tag{2.2}$$

Let $x_0 \in X$ be an arbitrary element. Let the sequence $\{x_n\} \subset X$ be defined by (2.1). As $T : X \to X$ is Dass and Gupta's contraction, we have

$$\begin{aligned}
D(x_n, x_{n+1}) = D(T(x_{n-1}), T(x_n)) &\leqslant \mu_1 D(x_{n-1}, x_n) \\
&\quad + \mu_2 \frac{[1 + D(x_{n-1}, T(x_{n-1}))]D(x_n, T(x_n))}{1 + D(x_{n-1}, x_n)} \\
&= \mu_1 D(x_{n-1}, x_n) \\
&\quad + \mu_2 \frac{[1 + D(x_{n-1}, x_n)]D(x_n, x_{n+1})}{1 + D(x_{n-1}, x_n)} \\
&= \mu_1 D(x_{n-1}, x_n) + \mu D(x_n, x_{n+1}) \\
&\leqslant \frac{\mu_1}{1 - \mu_2} D(x_{n-1}, x_n),
\end{aligned}$$

which further yields that

$$D(x_n, x_{n+1}) \leqslant \frac{\mu_1}{1 - \mu_2} D(x_{n-1}, x_n). \tag{2.3}$$

Similarly

$$\begin{aligned} D(x_{n-1}, x_n) = D(T(x_{n-2}), T(x_{n-1})) &\leqslant \mu_1 D(x_{n-2}, x_{n-1}) \\ &\quad + \mu_2 \frac{[1 + D(x_{n-2}, T(x_{n-2}))] D(x_{n-1}, T(x_{n-1}))}{1 + D(x_{n-2}, x_{n-1})} \\ &= \mu_1 D(x_{n-2}, x_{n-1}) \\ &\quad + \mu_2 \frac{[1 + D(x_{n-2}, x_{n-1})] D(x_{n-1}, x_n)}{1 + D(x_{n-2}, x_{n-1})} \\ &\leqslant \mu_1 D(x_{n-2}, x_{n-1}) + \mu_2 D(x_{n-1}, x_n), \end{aligned}$$

which further yields that

$$D(x_{n-1}, x_n) \leqslant \frac{\mu_1}{1-\mu_2} D(x_{n-2}, x_{n-1}). \tag{2.4}$$

Let $\lambda = \frac{\mu_1}{1-\mu_2} < 1$. Then from (2.3) and (2.4) and continuing the process, we get

$$D(x_n, x_{n+1}) \leqslant \lambda^n D(x_0, x_1), \qquad n \in N.$$

This implies that

$$\sum_{i=n}^{m-1} D(x_i, x_{i+1}) \leqslant \frac{\lambda^n}{1-\lambda} D(x_0, x_1), \quad m > n.$$

Since

$$\lim_{n\to\infty} \frac{\lambda^n}{1-\lambda} D(x_0, x_1) = 0,$$

so there exists some $N \in \mathbb{N}$ such that

$$0 < \frac{\lambda^n}{1-\lambda} D(x_0, x_1) < \delta, \quad n \geqslant N.$$

Hence, by (2.2) and $(\mathcal{F}_2)$, we have

$$f(\sum_{i=n}^{m-1} D(x_i, x_{i+1})) \leqslant f(\frac{\lambda^n}{1-\lambda} D(x_0, x_1)) < f(\epsilon) - \alpha, \tag{2.5}$$

for $m > n \geqslant N$. Using (D_3) and (2.5), we obtain $D(x_n, x_m) > 0$, $m > n \geqslant N$ implies

$$f(D(x_n, x_m)) \leqslant f(\sum_{i=n}^{m-1} D(x_i, x_{i+1})) + \alpha < f(\epsilon),$$

which indicates by $(\mathcal{F}_1)$ that $D(x_n, x_m) < \epsilon$, $m > n \geqslant N$. This shows that $\{x_n\}$ is $\mathcal{F}$-Cauchy. Since the pair (X, D) is $\mathcal{F}$-complete, there exists $x^* \in X$ such that the sequence $\{x_n\}$ is $\mathcal{F}$-convergent to x^*, i.e.,

$$\lim_{n\to\infty} D(x_n, x^*) = 0. \tag{2.6}$$

We shall show that x^* is a fixed point of T. We declare by contradiction that $D(T(x^*), x^*) > 0$. By (D_3), we get

$$f(D(T(x^*), x^*)) \leqslant f(D(T(x^*), T(x_n)) + D(T(x_n), x^*)) + \alpha, \quad n \in N.$$

As $T : X \to X$ is Dass and Gupta's contraction. By $(\mathcal{F}_1)$, we get

$$f(D(T(x^*), x^*)) \leqslant f(\mu_1 D(x^*, x_n) + \mu_2 \frac{[1 + D(x^*, T(x^*))] D(x_n, T(x_n))}{1 + D(x^*, x_n)}$$

$$+ D(x_{n+1}, x*)) + \alpha,$$

for $n \in N$. On the other hand, using ($\mathcal{F}_2$) and (2.6), we have

$$\lim_{n\to\infty} f(\mu_1 D(x^*, x_n) + \mu_2 \frac{[1 + D(x^*, T(x^*))]D(x_n, T(x_n))}{1 + D(x^*, x_n)} + D(x_{n+1}, x*)) + \alpha = -\infty.$$

This is a contradiction. Hence, we have $D(T(x^*), x^*) = 0$, i.e., $T(x^*) = x^*$. Therefore it concludes that $x^* \in X$ is the unique fixed point of T. □

Example 2.3. Let $X = [0, 3)$ endowed with $\mathcal{F}$-complete $\mathcal{F}$-metric D given by

$$D(x, y) = \begin{cases} e^{|x-y|}, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Take $f(t) = \frac{-1}{t}$ and $\alpha = 1$. Define $T : X \to X$ by

$$T(x) = \begin{cases} \frac{1}{2}(x + 1), & \text{if } 0 \leqslant x \leqslant 1, \\ 0, & \text{otherwise.} \end{cases}$$

Case I: If $x \in [0, 1]$. Then $D(x, y) = e^{|x-y|}$ and $D(T(x), T(y)) = e^{\frac{1}{2}|x-y|}$. Now for $\mu_1 = \frac{1}{2}$ and $\mu_2 = \frac{1}{3}$, we have

$$e^{\frac{1}{2}|x-y|} \leqslant \frac{1}{2} e^{|x-y|} + \frac{1}{3} \frac{[1 + e^{|x-\frac{1}{2}|}]e^{|y-\frac{1}{2}|}}{1 + e^{|x-y|}},$$

which shows that

$$D(T(x), T(y)) \leqslant \mu_1 D(x, y) + \mu_2 \frac{[1 + D(x, T(x))]D(y, T(y))}{1 + D(x, y)}.$$

Hence T is Dass and Gupta's contraction.

Case II: If $x \notin [0, 1]$. Then trivially T is Dass and Gupta's Contraction. Thus all the hypothesis of Theorem 2.2 are satisfied. So we conclude that T has a unique fixed point which is $x^* = 1$.

Remark 2.4. It can be easily observed that the result of Jleli and Samet cannot be applied to Example 2.3, as

$$D(T(x), T(y)) > kD(x, y),$$

for all $x, y \in X$.

Corollary 2.5 ([6])**.** *Assume that the pair* (X, D) *be an* $\mathcal{F}$*-metric space, and let* $T : X \to X$ *be a given mapping. Suppose that the conditions given below are satisfied*

(i) (X, D) *is* $\mathcal{F}$*-complete;*

(ii) *there exists* $\mu_1 \in (0, 1)$ *such that*

$$D(T(x), T(y)) \leqslant \mu_1 D(x, y), \quad (x, y) \in X \times X.$$

Then T *has a unique fixed point* $x^* \in X$*. Moreover, for any* $x_0 \in X$*, the sequence* $\{x_n\} \subset X$ *defined by*

$$x_{n+1} = T(x_n), \quad n \in \mathbb{N},$$

is $\mathcal{F}$*-convergent to* x^**.*

Proof. Taking $\mu_2 = 0$ in Theorem 2.2. □

Corollary 2.6. *Assume that the pair* (X, D) *be an* $\mathcal{F}$*-metric space, and let* $T : X \to X$ *be a given mapping. Suppose that the conditions given below are satisfied*

(i) (X, D) *is* $\mathcal{F}$*-complete;*

(ii) *there exists* $\mu_2 \in (0, 1)$ *such that*

$$D(T(x), T(y)) \leqslant \mu_2 \frac{[1 + D(x, T(x))]D(y, T(y))}{1 + D(x, y)},$$

for all $(x, y) \in X \times X$.

Then T *has a unique fixed point* $x^* \in X$*. Moreover, for any* $x_0 \in X$*, the sequence* $\{x_n\} \subset X$ *defined by*

$$x_{n+1} = T(x_n), \quad n \in \mathbb{N},$$

is $\mathcal{F}$*-convergent to* x^*.

Proof. Setting $\mu_1 = 0$ in Theorem 2.2. □

Now, we consider some special cases, when our result deduces several well-known fixed point theorems of the existing literature.

Corollary 2.7. *By Remark 1.7, we can have Dass and Gupta'fixed point theorem 1.5 from our main result 2.2.*

Remark 2.8. By setting $\mu_1 = \mu$ and $\mu_2 = 0$ in above Theorem, we can obtain the classical Banach contraction principle 1.1.

If we take $\mu_1 = 0$ in Corollary 2.7, then we have the following result in complete metric space.

Corollary 2.9. *Assume that* (X, d) *be a complete metric space, and let* $T : X \to X$ *be a self mapping. If there exists* $\mu_2 \in (0, 1)$ *such that*

$$d(T(x), T(y)) \leqslant \mu_2 \frac{d(x, T(x))d(y, T(y))}{1 + d(x, y)},$$

for all $x, y \in X$*, then* T *has a unique fixed point* $x^* \in X$*. Moreover, for any* $x_0 \in X$*, the sequence* $\{x_n\} \subset X$ *defined by*

$$x_{n+1} = T(x_n), \quad n \in \mathbb{N},$$

is $\mathcal{F}$*-convergent to* x^*.

3. Conclusions

Recently, Jleli and Samet [6] introduced the concept of $\mathcal{F}$-metric space and proved a new version of well known Banach contraction principle in the setting of this $\mathcal{F}$-metric spaces. In this article we defined Dass and Gupta's contraction and establish some fixed point theorems in the context of $\mathcal{F}$-metric spaces. Our results generalized and modified several known results in the literature.

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