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# Comparison of the best approximation of holomorphic functions from Hardy space

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### Abstract

We compare the best approximations of holomorphic functions in the Hardy space $H^1$ by algebraic polynomials and trigonometric polynomials. Particulary, we establish a class of functions $f \in H^1$ for which the best trigonometric approximation do not coincide with the best algebraic approximation.

**Keywords:** Best approximation, Hardy space, non-negative trigonometric polynomials.

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## 1. Introduction and main results

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and let $dm$ be a normalized Lebesgue measure on $\mathbb{T}$. The Hardy space $H^q$ for $1 \leqslant q \leqslant \infty$ is the class of holomorphic in the $\mathbb{D}$ functions $f$ satisfied $\|f\|_q < \infty$, where

$$\|f\|_q := \begin{cases} \sup\limits_{\rho\in(0,1)} \left(\int_{\mathbb{T}} |f(\rho t)|^q dm(t)\right)^{1/q}, & 1 \leqslant q < \infty, \\ \sup\limits_{z\in\mathbb{D}} |f(z)|, & q = \infty. \end{cases}$$

It is well-known that each function $f$ from $H^q$ has the nontangential limits $f(t)$, $t \in \mathbb{T}$, almost everywhere and

$$\|f\|_q = \begin{cases} \left(\int_{\mathbb{T}} |f|^q dm\right)^{1/q}, & 1 \leqslant q < \infty, \\ \operatorname*{ess\,sup}\limits_{t\in\mathbb{T}} |f(t)|, & q = \infty. \end{cases}$$

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The best polynomial approximation of $f \in H^q$ of order $n$, $n \in \mathbb{N}$, is the quantity

$$E_n(f)_q := \inf_{P \in \mathcal{P}_{n-1}} \|f - P\|_q, \quad n \in \mathbb{N},$$

where $\mathcal{P}_{n-1}$ is the set of all algebraic polynomials of degree at most $n-1$.

We will denote by $\mathcal{T}_{n-1}$ the set of all trigonometric polynomials of degree at most $n-1$ on the circle $\mathbb{T}$ with complex coefficients, that is a functions of the form $T(t) = \sum_{|k| \leqslant n-1} c_k t^k$, $t \in \mathbb{T}$.

The best trigonometric approximation of $f \in H^q$ of order $n$, $n \in \mathbb{N}$, is the quantity

$$\widetilde{E}_n(f)_q := \inf_{T \in \mathcal{T}_{n-1}} \|f - T\|_q, \quad n \in \mathbb{N}.$$

The polynomials $P^*$ and $T^*$ satisfied $\|f - P^*\|_q = E_n(f)_q$ and $\|f - T^*\|_q = \widetilde{E}_n(f)_q$ are called a best approximation to $f$ among the set $\mathcal{P}_{n-1}$ and $\mathcal{T}_{n-1}$ respectively in the metric $\|\cdot\|_q$.

Obviously, for any $1 \leqslant q \leqslant \infty$, one has

$$\widetilde{E}_1(f)_q = E_1(f)_q,$$

and

$$\widetilde{E}_n(f)_q \leqslant E_n(f)_q, \quad \forall\, n \in \mathbb{N} \setminus \{1\}. \tag{1.1}$$

Pekarskii [3] was the first to point out in print that there exists a function $f \in H^\infty$ such that

$$\widetilde{E}_n(f)_\infty < E_n(f)_\infty, \tag{1.2}$$

for a given natural $n > 1$.

Particularly, in [3] it was shown that for the sequence $\{f_{n,\rho}\}_{0<\rho<1}$ of functions

$$f_{n,\rho}(z) = z^n \frac{1 - \rho^{2(n+1)}}{1 - \rho^{n+1} z^{n+1}}, \quad n \in \mathbb{N} \setminus \{1\}, \tag{1.3}$$

we have

$$\widetilde{E}_n(f_{n,\rho})_\infty = 1 < E_n(f_{n,\rho})_\infty = \|f_{n,\rho}\|_\infty = 1 + \rho^{n+1} \to 2 \quad \text{as} \quad \rho \to 1.$$

In view of these results, it is natural to assume that function $f_{n,\rho}$ must satisfy the inequality

$$\widetilde{E}_n(f)_1 < E_n(f)_1. \tag{1.4}$$

As we will show later (see Proposition 2.4), this is indeed the case, but the method of [3] cannot be applied to proving this one.

Finally, let us pay attention that Pekarskii's example says nothing about the inequality (1.2) for an individual function for each natural $n$.

The aim of this note is to establish a class of functions $f$ satisfied (1.4) for a given $n$ as well as to construct an individual function $f$ for which (1.4) holds true for each natural $n$.

Let $\Pi_n$, $n \in \mathbb{N} \setminus \{1\}$, denote the set of all algebraic polynomials $f(z) = \sum_{k=0}^{3n-1} a_k z^k$ of degree at most $3n-1$ with complex coefficients satisfied $|a_n| \sum_{k=2n+1}^{3n-1} |a_k| > 0$ and such that

$$\min_{z \in \mathbb{T}} \operatorname{Re} \sum_{k=0}^{2n-1} \frac{a_{n+k}}{a_n} z^k \geqslant \frac{1}{2}.$$

Let $\widehat{f}_k = \frac{f^{(k)}(0)}{k!}$, $k = 0, 1, \cdots$, denote the Taylor coefficients of $f \in H^1$.

Our main results are the following two theorems.

**Theorem 1.1.** *Suppose* $n \in \mathbb{N}\setminus\{1\}$ *and* $f \in \Pi_n$. *Then*

$$\left|\widehat{f}_n\right| = \widetilde{E}_n(f)_1 < E_n(f)_1.$$

**Theorem 1.2.** *Suppose that* $0<\rho<1$ *and let* $f(z)=1/(1-\rho z)$. *Then for each natural* $n$,

$$\widetilde{E}_n(f)_1 = \frac{2}{\pi}\rho^n \mathbf{K}\left(\rho^{2n}\right), \tag{1.5}$$

*and*

$$E_n(f)_1 = \frac{2}{\pi}\rho^n \mathbf{K}\left(\rho^{n+1}\right), \tag{1.6}$$

*where*

$$\mathbf{K}(x) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}},$$

*is the complete elliptic integral of the first kind.*

**Corollary 1.3.** *Suppose that* $0<\rho<1$ *and let* $f(z)=1/(1-\rho z)$. *Then for each natural* $n>1$ *the inequality* (1.2) *holds true.*

## 2. Examples

In this section, our goal is to prove the following propositions that may be considered as examples to the main theorems.

Firstly, we give some examples of polynomials from $\Pi_n$ .

**Proposition 2.1.** *Let* $n,p\in\mathbb{N}$, $n\leqslant p\leqslant 2n-1$ *and let* $A_p(t)=\sum_{|k|\leqslant p} a_k t^k$, $B_{2n-p-1}(t)=\sum_{|k|\leqslant 2n-p-1} b_k t^k$, $|a_p b_{2n-p-2}|>0$ *be two trigonometric polynomials, such that* $A_p(t)B_{2n-p-1}(t)\geqslant 0$ *for all* $t\in\mathbb{T}$. *Then the function*

$$f(z) = z^n\left(\sum_{k=0}^{p} a_k z^k\right)\left(\sum_{|k|\leqslant 2n-p-1} b_k z^k\right), \tag{2.1}$$

*belongs to* $\Pi_n$ *and*

$$\widetilde{E}_n(f)_1 = \widehat{f}_n = a_0 b_0 + 2\,\mathrm{Re}\sum_{k=1}^{2n-p-1} a_k \bar{b}_k,$$

*where for* $p=2n-1$, $\sum_{k=1}^{0}=0$. *Moreover, the trigonometric polynomial*

$$T(t) = -t^n\left(\sum_{k=1}^{p} a_{-k}t^{-k}\right)B_{2n-p-1}(t),$$

*is the best approximation to* $f$ *among the set* $\mathcal{T}_{n-1}$ *in the metric* $\|\cdot\|_1$.

Indeed, a straightforward calculation shows that $T$ has the form $T(t)=\sum_{k=-(n-1)}^{3n-2p-1} c_k t^k$, $c_k\in\mathbb{C}$. Therefore, $T\in\mathcal{T}_{n-1}$, since $3n-2p-1\leqslant n-1$.

As it is easy to see that

$$t^{-n}\left(f(t)-T(t)\right) = A_p(t)B_{2n-p-1}(t)\geqslant 0,\quad t\in\mathbb{T},$$

then

$$\widetilde{E}_n(f)_1 \leqslant \|f-T\|_1 = \int_{\mathbb{T}} A_p B_{2n-p-1}\,dm = a_0 b_0 + 2\,\mathrm{Re}\sum_{k=1}^{2n-p-1} a_k \bar{b}_k.$$

Since $\left|\widehat{f}_n\right| \leqslant \widetilde{E}_n(f)_1$ and

$$\widehat{f}_n = \int_{\mathbb{T}} f(t)t^{-n}dm(t) = \int_{\mathbb{T}} (f(t) - T(t))\, t^{-n}dm(t) = \int_{\mathbb{T}} A_p B_{2n-p-1} dm \geqslant \widetilde{E}_n(f)_1,$$

we get $\widetilde{E}_n(f)_1 = \widehat{f}_n$. As it follows from Lemma 3.1 below,

$$\widetilde{E}_n(f)_1 = \left|\widehat{f}_n\right| \Longleftrightarrow \min_{t\in\mathbb{T}} \sum_{k=0}^{2n-1} \frac{\widehat{f}_{k+n}}{\widehat{f}_n} t^k \geqslant \frac{1}{2}.$$

Therefore $f \in \Pi_n$, because $\left|\widehat{f}_n\right| \sum_{k=n+1}^{3n-1} \left|\widehat{f}_k\right| \geqslant \left|\widehat{f}_n \widehat{f}_{3n-1}\right| = \left|\widehat{f}_n a_p b_{2n-p-1}\right| > 0$.

The computation of the value of best approximation $E_n(f)_1$ for function from $\Pi_n$ is a more complicated problem. In the next two propositions we would like to pay attention to how a sieve-method may be applied to computation of this one.

Let $r \in \mathbb{Z}_+$ and $s \in \mathbb{N}$. The linear operator $\mathcal{W}_{r,s}$ defined on $H^1$ by

$$\mathcal{W}_{r,s}(f)(z) = \sum_{j=0}^{\infty} \widehat{f}_{js+r} z^j, \quad z \in \mathbb{D},$$

is called the sieve operator.

The importance of sieve operator for the theory of approximation of holomorphic functions is recognized by the following.

**Proposition 2.2.** *Suppose that* $1 \leqslant q \leqslant \infty$, $r \in \mathbb{Z}_+$, $s \in \mathbb{N}$, $s \geqslant r+1$ *and let*

$$\mathcal{G}^q_{r,s} := \{g \in H^q : \mathcal{W}_{r,s}(g) = 0\}.$$

*Then for any function* $f \in H^p$,

$$\|\mathcal{W}_{r,s}(f)\|_q = \min_{g\in\mathcal{G}^q_{r,s}} \|f-g\|_q \tag{2.2}$$

$$\leqslant E_r(f)_q. \tag{2.3}$$

*The minimum in* (2.2) *is attained for the function* $g(z) = f(z) - z^r\mathcal{W}_{r,s}(f)(z^s)$. *The equality sign in* (2.3) *is attained for function of the form* $f(z) = \sum_{j=0}^{\infty} a_j z^{js+r}$.

*Proof.* Indeed, for any $g \in \mathcal{G}^q_{r,s}$,

$$\mathcal{W}_{r,s}(f) = \mathcal{W}_{r,s}(f-g).$$

Therefore,

$$\|\mathcal{W}_{r,s}(f)\|_q \leqslant \|\mathcal{W}_{r,s}\|_q \|f-g\|_q,$$

where $\|\mathcal{W}_{r,s}\|_q$ is the norm of operator $\mathcal{W}_{r,s}$ on $H^q$ space.

It was shown in [5] that

$$\|\mathcal{W}_{r,s}\|_\infty = 1 \Leftrightarrow s \geqslant r+1 \Rightarrow \|\mathcal{W}_{r,s}\|_q = 1.$$

Thus, for any $g \in \mathcal{G}^q_{r,s}$,

$$\|\mathcal{W}_{r,s}(f)\|_q \leqslant \|f-g\|_q.$$

It is clear that the best we can minimize the right side in the above inequality is to choose

$$g(z) = f(z) - z^r\mathcal{W}_{r,s}(f)(z^s).$$

For proving (2.3) it suffices to note that $\mathcal{P}_{r-1} \subset \mathcal{G}^q_{r,s}$. □

**Proposition 2.3.** *Suppose that* $n \in \mathbb{N} \setminus \{1\}$ *and let* $f(z) = z^n + \frac{1}{2}z^{2n+1}$. *Then* $f \in \Pi_n$,

$$\widetilde{E}_n(f)_1 = 1,$$

*and*

$$E_n(f)_1 = \|f\|_1 = \frac{3}{\pi}\mathbf{E}\left(\frac{2\sqrt{2}}{3}\right),$$

*where*

$$\mathbf{E}(x) = \int_0^{\frac{\pi}{2}} \sqrt{1 - x^2 \sin^2\theta}\, d\theta,$$

*is a complete elliptic integral of the second kind.*

Clearly, $f$ has the form (2.1), in which $p = n+1$, $a_0 = b_0 = 1$, $a_{n+1} = \frac{1}{2}$ , and $a_k = a_{n-1} = b_k = 0$ for $k = 1, 2, \cdots, n-2$. Therefore, by Proposition 2.1, $f \in \Pi_n$ and $\widetilde{E}_n(f)_1 = 1$.

On the other hand, $\mathcal{W}_{n,n+1}(f)(z) = 1 + \frac{1}{2}z$. Thus, by Proposition 2.2

$$\begin{aligned} E_n(f)_1 &= \|f\|_1 \\ &= \int_{\mathbb{T}} \left|1 + \frac{t}{2}\right| dm(t) \\ &= \frac{3}{4\pi}\int_{-\pi}^{\pi} \sqrt{1 - \frac{8}{9}\sin^2\frac{\theta}{2}}\, d\theta \\ &= \frac{3}{\pi}\mathbf{E}\left(\frac{2\sqrt{2}}{3}\right) \approx 1.01925. \end{aligned}$$

**Proposition 2.4.** *Suppose that* $n \in \mathbb{N} \setminus \{1\}$, $0 < \rho < 1$ *and let* $f_{n,\rho}$ *be the function defined in* (1.3). *Then*

$$\widetilde{E}_n\left(f_{n,\rho}\right)_1 \leqslant 1,$$

*and*

$$E_n\left(f_{n,\rho}\right)_1 = \|f_{n,\rho}\|_1 = \frac{2}{\pi}\left(1 - \rho^{2(n+1)}\right)\mathbf{K}\left(\rho^{n+1}\right).$$

*Therefore, for the* $\rho_* \approx (0.139793)^{\frac{1}{n+1}}$ *that maximizes the function* $\rho \mapsto \left(1 - \rho^{2(n+1)}\right)\mathbf{K}\left(\rho^{n+1}\right)$ *on* $[0, 1]$, *we get* (1.4).

Indeed,

$$\widetilde{E}_n\left(f_{n,\rho}\right)_1 \leqslant \widetilde{E}_n\left(f_{n,\rho}\right)_\infty = 1.$$

On the other hand,

$$z^n \mathcal{W}_{n,n+1}\left(f_{n,\rho}\right)\left(z^{n+1}\right) = f_{n,\rho}(z).$$

Therefore, $E_n\left(f_{n,\rho}\right)_1 = \|f_{n,\rho}\|_1$.

## 3. Auxiliary lemmas

The proof of Theorem 1.1 is based on the following assertions that are also of some independent interest.

**Lemma 3.1.** *Suppose that* $n \in \mathbb{N}$, $f \in H^1$ *and* $\left|\widehat{f}_n\right| > 0$. *Then equality* $\widetilde{E}_n(f)_1 = \left|\widehat{f}_n\right|$ *holds true if and only if*

$$f(z) = \sum_{k=0}^{3n-1} \widehat{f}_k z^k, \quad \left|\widehat{f}_n\right| > 0,$$

*and*

$$\min_{t\in\mathbb{T}} \operatorname{Re} \sum_{k=0}^{2n-1} \frac{\widehat{f}_{n+k}}{\widehat{f}_n} t^k \geqslant \frac{1}{2}.$$

*Moreover, the trigonometric polynomial*

$$T^*(t) = \sum_{k=0}^{n-1} \left(\widehat{f}_k - \overline{\widehat{f}_{2n-k}} e^{i2\arg \widehat{f}_n}\right) t^k - e^{i2\arg \widehat{f}_n} \sum_{k=1}^{n-1} \overline{\widehat{f}_{k+2n+1}} t^{-k},$$

*is the unique best approximation to* $f$ *among the set* $\mathcal{T}_{n-1}$ *in the metric* $\|\cdot\|_1$.

*Proof.* By definition

$$\begin{aligned}\widetilde{E}_n(f)_1 &= \inf_{a_k\in\mathbb{C}} \int_{\mathbb{T}} \left| f(t) - \sum_{|k|\leqslant n-1} a_k t^k \right| dm(t)\\ &= \inf_{a_k\in\mathbb{C}} \int_{\mathbb{T}} \left| t^{n-1} f(t) - \sum_{k=0}^{2(n-1)} a_{k-n+1} t^k \right| dm(t) = E_{2n-1}(g)_1,\end{aligned}$$

where $g(t) = t^{n-1} f(t)$.

Therefore, taking into account that $\widehat{f}_n = \widehat{g}_{2n-1}$, we get the equivalence

$$\widetilde{E}_n(f)_1 = \left|\widehat{f}_n\right| \Longleftrightarrow E_{2n-1}(g)_1 = |\widehat{g}_{2n-1}|.$$

In addition, the trigonometric polynomial $\sum_{|k|\leqslant n-1} a_k t^k$ is the best approximation to $f$ among the set $\mathcal{T}_{n-1}$ if and only if the algebraic polynomial $\sum_{k=0}^{2(n-1)} a_{k-n+1} t^k$ is the best approximation to $g$ among the set $\mathcal{P}_{2(n-1)}$.

The remainder of this proof follows immediately from the next lemma proved in [1]. □

**Lemma 3.2.** *Suppose that* $n \in \mathbb{N}$, $f \in H^1$ *and* $\left|\widehat{f}_n\right| > 0$. *The equality* $E_n(f)_1 = \left|\widehat{f}_n\right|$ *holds true if and only if*

$$f(z) = \sum_{k=0}^{2n} \widehat{f}_k z^k,$$

*and*

$$\min_{t\in\mathbb{T}} \operatorname{Re} \sum_{k=0}^{n} \frac{\widehat{f}_{n+k}}{\widehat{f}_n} t^k \geqslant \frac{1}{2}.$$

*Moreover, the polynomial*

$$P^*(z) = \sum_{k=0}^{n-1} \left(\widehat{f}_k - \overline{\widehat{f}_{2n-k}} e^{i2\arg \widehat{f}_n}\right) z^k,$$

*is the unique best approximation to* $f$ *among the set* $\mathcal{P}_{n-1}$ *in the metric* $\|\cdot\|_1$.

**Lemma 3.3.** *Suppose that* $0 \leqslant \rho < 1$ *and* $p \in \mathbb{N}$. *Then*

$$\int_{\mathbb{T}} \frac{|1-\rho^p t^p|}{|1-\rho t|^2} dm(t) = \frac{2}{\pi} \frac{1-\rho^{2p}}{1-\rho^2} \mathbf{K}(\rho^p),$$

*where* $\mathbf{K}$ *is the complete elliptic integral of the first kind.*

This assertion essentially is contained in [6]. For convenience, we present here its proof.

*Proof.* From the expansion

$$\frac{1}{\sqrt{1-x}} = 1 + \sum_{k=1}^{\infty} \alpha_k x^k, \quad \forall\, x \in \mathbb{D},$$

where

$$\alpha_k := \frac{(2k-1)!!}{(2k)!!}, \quad k \in \mathbb{N},$$

we get

$$\begin{aligned}\frac{\sqrt{1-x^p}}{1-x} &= \frac{1-x^p}{1-x}\frac{1}{\sqrt{1-x^p}}\\ &= \frac{1-x^p}{1-x}\left(1+\sum_{k=1}^{\infty}\alpha_k x^{pk}\right)\\ &= \frac{1-x^p}{1-x} + \sum_{k=1}^{\infty}\alpha_k \frac{1-x^p}{1-x}x^{pk}\\ &= \sum_{\nu=0}^{p-1} x^{\nu} + \sum_{k=1}^{\infty}\alpha_k \sum_{\nu=pk}^{p(k+1)-1} x^{\nu}, \quad \forall\, x \in \mathbb{D}.\end{aligned}$$

Therefore, by Parseval's identity,

$$\begin{aligned}\int_{\mathbb{T}} \frac{|1-\rho^p t^p|}{|1-\rho t|^2} dm(t) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\sqrt{1-\rho^p e^{ip\theta}}}{1-\rho e^{i\theta}} \frac{\sqrt{1-\rho^p e^{-ip\theta}}}{1-\rho e^{-i\theta}} d\theta\\ &= \sum_{\nu=0}^{p-1} \rho^{2\nu} + \sum_{k=1}^{\infty}\alpha_k^2 \sum_{\nu=pk}^{p(k+1)-1} \rho^{2\nu}\\ &= \frac{1-\rho^{2p}}{1-\rho^2}\left(1+\sum_{k=1}^{\infty}\alpha_k^2\rho^{2pk}\right)\\ &= \frac{1-\rho^{2p}}{1-\rho^2}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1-\rho^p e^{-i\theta}|}d\theta\\ &= \frac{2}{\pi}\frac{1-\rho^{2p}}{1-\rho^2}\mathbf{K}(\rho^p).\end{aligned}$$

□

## 4. Proofs of main results

*Proof of Theorem 1.1.* It is ease to see that functions from the set $\Pi_n$ satisfies conditions of Lemma 3.1. Therefore, $\left|\widehat{f}_n\right| = \widetilde{E}_n(f)_1$. On the other hand, any function from $\Pi_n$ does not satisfy conditions of Lemma 3.2. Thus, for any $f \in \Pi_n$, $E_n(f)_1 \neq \left|\widehat{f}_n\right|$. The result follows, since (1.1). □

*Proof of Theorem 1.2.* First for all we note that as was shown by Alper [2] (see also some generalizations in [4])

$$E_p(f)_1 = \rho^p \frac{1-\rho^2}{1-\rho^{2(p+1)}} \int_{\mathbb{T}} \frac{\left|1-\rho^{p+1}t^{p+1}\right|}{|1-\rho t|^2} dm(t).$$

From this formula, by Lemma 3.3 we get the equality (1.6).

Let us prove (1.5).
We have

$$\begin{aligned}
\widetilde{E}_n(f)_1 &= \min_{a_k\in\mathbb{C}} \int_{\mathbb{T}} \left| \frac{1}{1-\rho t} - \sum_{|k|\leqslant n-1} a_k t^k \right| dm(t) \\
&= \min_{a_k\in\mathbb{C}} \frac{1}{\rho^{n-1}} \int_{\mathbb{T}} \left| \frac{\rho^{n-1}t^{n-1}}{1-\rho t} - \sum_{k=0}^{2(n-1)} \rho^{n-1} a_{k-n+1} t^k \right| dm(t) \\
&= \frac{1}{\rho^{n-1}} \min_{a_k\in\mathbb{C}} \int_{\mathbb{T}} \left| \frac{1}{1-\rho t} - \sum_{k=0}^{n-2} \rho^k t^k - \sum_{k=0}^{2(n-1)} \rho^{n-1} a_{k-n+1} t^k \right| dm(t) \\
&= \frac{1}{\rho^{n-1}} \min_{b_k\in\mathbb{C}} \int_{\mathbb{T}} \left| \frac{1}{1-\rho t} - \sum_{k=0}^{2(n-1)} b_k t^k \right| dm(t) \\
&= \frac{1}{\rho^{n-1}} E_{2n-1}(f)_1.
\end{aligned}$$

Therefore by (1.6) we get the equality (1.5). □

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## References

[1] F. G. Abdullayev, G. A. Abdullayev, V. V. Savchuk, *Best approximation of holomorphic functions from Hardy space in terms of Taylor coefficient*, to appear in Filomat. 3

[2] S. Y. Al'per, *On the Best Mean First-degree Approximation of Analytic Functions on Circle (Russian)*, Dokl. Akad. Nauk S.S.S.R., **153** (1963), 503–506. 4

[3] A. A. Pekarskii, *Comparison of the Best Uniform Approximations of Analytic Functions in the Disk and on Its Boundary (Russian)*, translated from Tr. Mat. Inst. Steklova, **255** (2006), 227–232. 1, 1, 1

[4] V. V. Savchuk, *Best Approximation of Cauchy–Szegö Kernel in the Mean on Circle (Ukrainian)*, Ukr. Mat. Zh., **70** (2018), 708–714. 4

[5] V. V. Savchuk, S. O. Chaichenko, *Addendum to a theorem of F. Wiener about sieve (Ukrainian)*, Praci Instytutu Matematyky NAN Ukrainy, **12** (2015), 262–272. 2

[6] V. V. Savchuk, M. V. Savchuk, S. O. Chaichenko, *Approximation of Analytic Functions byde Valle Poussin sums (Ukrainian)*, Matematychni Studii, **34** (2010), 207–219. 3