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Contact CR-warped product submanifolds of a generalized Sasakian space form admitting a nearly Sasakian structure

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Abstract

This paper studies the contact CR-warped product submanifolds of a generalized Sasakian space form admitting a nearly Sasakian structure. Some Characterization of the existence of these warped product submanifolds are also obtained. We illustrate that the warping function is a harmonic function under certain conditions. Moreover, a sharp estimate for the squared norm of the second fundamental form is investigated, and the equality case is also discussed. The results obtained in this paper generalize the results that have appeared in [I. Hasegawa, I. Mihai, Geom. Dedicata, **102** (2003), 143–150], [I. Mihai, Geom. Dedicata, **109** (2004), 165–173], and [M. Atçeken, Hacet. J. Math. Stat., **44** (2015), 23–32].

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1. Introduction

It is well known that warped products of manifolds play an important role in differential geometry, the theory of relativity and mathematical physics. One of the most important examples of a warped product manifold is the excellent setting to the model space time near black holes or bodies with high gravitational fields [16]. For a recent survey of warped products on Riemannian manifolds, one can consult the reference [14].

The notion of the CR-warped product submanifolds was first introduced by Chen in [12]. Basically, Chen considered the warped product submanifolds of the types $N_\perp \times_\psi N_T$ and $N_T \times_\psi N_\perp$, where N_T and $N_\perp$ are the holomorphic and totally real submanifolds of a Kaehler manifold and showed that the first type of warped product does not exist. Chen obtained some basic results for the second type of warped product submanifolds. He also proved a sharp estimate for the squared norm of the second fundamental form in terms of the warping function. Hasegawa and Mihai [15] extended the results of Chen [12] in the setting of the Sasakian space forms and obtained a sharp inequality for the squared norm of the second fundamental form in terms of the warping function. A step forward was made by Mihai [21] who

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improved the same inequality for the contact CR-warped product submanifolds in the Sasakian space forms. In [22], Munteanu carried out a thorough study on the warped product contact CR-submanifolds of Sasakian space forms and obtained the various estimates for the squared norm of the second fundamental form. Moreover, Atceken in [4–6] studied the contact CR-warped product submanifolds of the cosymplectic, Kenmotsu, and Sasakian space forms. He also obtained the characterizing inequalities for the existence of these warped product submanifolds. In this line of research, many papers have appeared in the setting of the almost contact metric manifolds ([2, 3, 18, 19]). After reviewing the literature, the author of the present paper observed that the study of the contact CR-warped product submanifolds in the setting of the generalized Sasakian space forms admitting a nearly Sasakian structure is an interesting research problem which can be generalized by many existing results in this direction. Therefore, the author tries to fill this gap and investigate the contact CR-warped product submanifolds in the setting of the generalized Sasakian space forms.

The paper is organized so that Section 2 is introductory and contains a brief introduction to the almost contact metric manifolds and in particular for the nearly Sasakian manifolds. Moreover, in this section, the definition of a generalized Sasakian space form is given. In Section 3, the warped product of the Riemannian manifolds is discussed and some useful results are collected for later use. Furthermore, a characterizing inequality for the contact CR-warped product submanifolds of the generalized Sasakian space forms admitting a nearly Sasakian structure and some special cases are also discussed. In the last section, we prove a sharp estimate for the squared norm of the second fundamental form in terms of warping function, and the equality case is also discussed.

2. Preliminaries

A $(2n+1)$-dimensional C^∞-manifold $\bar{M}$ is said to have an almost contact structure if there exist on $\bar{M}$ a tensor field ϕ of the type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \oplus \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1. \tag{2.1}$$

There always exists a Riemannian metric g on an almost contact metric manifold $\bar{M}$ satisfying the following conditions

$$\eta(U) = g(U, \xi), \quad g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V)$$

for all $U, V \in T\bar{M}$.

An almost contact structure (ϕ, ξ, η) is said to be normal if the almost complex structure J on the product manifold $\bar{M} \times R$ is given by

$$J(U, f\frac{d}{dt}) = (\phi U - f\xi, \eta(U)\frac{d}{dt}),$$

where f is a C^∞-function on $\bar{M} \times R$ has no torsion that is J is integrable and the condition for normality in terms of ϕ, ξ and η is $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on $\bar{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. Finally, the *fundamental 2-form* Φ is defined by $\Phi(U, V) = g(U, \phi V)$.

An almost contact metric manifold $(\bar{M}, \phi, \xi, \eta, g)$ is said to be a Sasakian manifold [10], if the following condition holds

$$(\bar{\nabla}_U \phi)V = -g(U, V)\xi + \eta(V)U,$$

from the above equation one can conclude that

$$\bar{\nabla}_U \xi = \phi U$$

for all $U, V \in T\bar{M}$.

An almost contact metric manifold is said to be nearly Sasakian manifold [10], if

$$(\bar{\nabla}_U \phi)V + (\bar{\nabla}_V \phi)U = -2g(U, V)\xi + \eta(V)U + \eta(U)V \tag{2.2}$$

for all $U, V \in T\bar{M}$. The equation (2.2) is equivalent to the following equation

$$(\bar{\nabla}_U \phi)U = -g(U, U)\xi + \eta(U)U \tag{2.3}$$

for each $U \in T\bar{M}$.

In [1], Blair et al. introduced the notion of generalized Sasakian space form as that an almost contact metric manifold $(\bar{M}, \phi, \xi, \eta, g)$ whose curvature tensor $\bar{R}$ satisfies

$$\begin{aligned}\bar{R}(U, V)W = f_1\{g(V, W)U - g(U, W)V\} + f_2\{g(U, \phi W)\phi V - g(V, \phi W)\phi U + 2g(U, \phi V)\phi W\}\\ + f_3\{\eta(U)\eta(W)V - \eta(V)\eta(W)U + g(U, W)\eta(V)\xi - g(V, W)\eta(U)\xi\}\end{aligned} \tag{2.4}$$

for all vector fields U, V, W and certain differentiable functions f_1, f_2, f_3 on $\bar{M}$. A generalized Sasakian space form with functions f_1, f_2, f_3 is denoted by $\bar{M}(f_1, f_2, f_3)$. If $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$, then $\bar{M}(f_1, f_2, f_3)$ is a Sasakian space form $\bar{M}(c)$ [8]. If $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$, then $\bar{M}(f_1, f_2, f_3)$ is a Kenmotsu space form $\bar{M}(c)$ [17] and if $f_1 = f_2 = f_3 = \frac{c}{4}$, then $\bar{M}(f_1, f_2, f_3)$ is a cosymplectic space form $\bar{M}(c)$ [1].

Let M be a submanifold of an almost contact metric manifold $\bar{M}$ with induced metric g and if ∇ and $\nabla^{\perp}$ are the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively, then the Gauss and Weingarten formulae are given by the following equations

$$\bar{\nabla}_U V = \nabla_U V + h(U, V), \tag{2.5}$$

$$\bar{\nabla}_U N = -A_N U + \nabla^{\perp}_U N \tag{2.6}$$

for each $U, V \in TM$ and $N \in T^{\perp}M$, where h and A_N are the second fundamental form and the shape operator respectively, for the immersion of M in $\bar{M}$, they are associated as

$$g(h(U, V), N) = g(A_N U, V), \tag{2.7}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as on M.

The mean curvature vector H of M is given by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),$$

where n is the dimension of M and $\{e_1, e_2, \dots, e_n\}$ is a local orthonormal frame of the vector fields on M. The squared norm of the second fundamental form is defined as

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be a totally geodesic submanifold, if $h(U, V) = 0$, for each $U, V \in TM$ and totally umbilical submanifold, if $h(U, V) = g(U, V)H$.

For any $U \in TM$, we write

$$\phi U = PU + FU, \tag{2.8}$$

where PU and FU are the tangential and normal components of ϕU, respectively.

Similarly, for $N \in T^{\perp}M$, we can write

$$\phi N = tN + fN, \tag{2.9}$$

where tN and fN are the tangential and normal components of ϕN, respectively.

The covariant differentiation of the tensors ϕ, P, and F are defined as

$$(\bar{\nabla}_U \phi)V = \bar{\nabla}_U \phi V - \phi\bar{\nabla}_U V, \tag{2.10}$$

$$(\bar{\nabla}_U P)V = \nabla_U PV - P\nabla_U V, \tag{2.11}$$
$$(\bar{\nabla}_U F)V = \nabla^{\perp}_U FV - F\nabla_U V, \tag{2.12}$$

respectively. Furthermore, for any $U, V \in TM$, the tangential and normal parts of $(\bar{\nabla}_U \phi)V$ are denoted by $\mathcal{P}_U V$ and $\mathcal{Q}_U V$, i.e.,

$$(\bar{\nabla}_U \phi)V = \mathcal{P}_U V + \mathcal{Q}_U V. \tag{2.13}$$

By using the equations (2.5)-(2.13), we may obtain that

$$\begin{aligned} \mathcal{P}_U V &= (\bar{\nabla}_U P)V - A_{FV}U - th(U, V), \\ \mathcal{Q}_U V &= (\bar{\nabla}_U F)V + h(U, TV) - fh(U, V). \end{aligned} \tag{2.14}$$

On a submanifold M of a nearly Sasakian manifold by (2.2) and (2.13)

$$(a)\ \mathcal{P}_U V + \mathcal{P}_V U = -2g(U, V)\xi + \eta(U)V + \eta(V)U, \quad (b)\ \mathcal{Q}_U V + \mathcal{Q}_V U = 0 \tag{2.15}$$

for any $U, V \in TM$.

An m-dimensional Riemannian submanifold M of an almost contact metric manifold $\bar{M}$, where ξ is tangent to M, is called a *contact CR-submanifold*, if it admits an invariant distribution D whose orthogonal complementary distribution $D^{\perp}$ is anti invariant, the tangent bundle of the submanifold M can be written as

$$TM = D \oplus D^{\perp} \oplus \langle \xi \rangle,$$

where $\phi D \subseteq D$ and $\phi D^{\perp} \subseteq T^{\perp}M$. $\langle \xi \rangle$ denotes the 1-dimensional distribution which is spanned by the structure vector field ξ.

If μ is the invariant subspace of the normal bundle $T^{\perp}M$, then in the case of contact CR-submanifold, the normal bundle $T^{\perp}M$ can be decomposed as follows:

$$T^{\perp}M = \mu \oplus \phi D^{\perp}. \tag{2.16}$$

A contact CR-submanifold M is called contact CR-product, if the distribution D and $D^{\perp}$ are parallel on M.

As a generalization of the product manifolds and in particular of the contact CR-product submanifolds, one can consider the warped product of manifolds which is defined as follows.

Let (B, g_B) and (F, g_F) be the two Riemannian manifolds with the Riemannian metrics g_B and g_F respectively and ψ be a positive differentiable function on B and $\pi : B \times F \to B$, $\eta : B \times F \to F$. The projection maps given by $\pi(p, q) = p$ and $\eta(p, q) = q$ for every $(p, q) \in B \times F$. The warped product $M = B \times_{\psi} F$ is the manifold $B \times F$ ([7]), equipped with the Riemannian structure such that

$$g(U, U_2) = g_B(\pi_* U, \pi_* U_2) + (\psi \circ \pi)^2 g_F(\pi_* U, \pi_* U_2)$$

for all $U, U_2 \in TM$, where $*$ denotes the tangent map. The function ψ is called the warping function of the warped product manifold. If the warping function is constant then the warped product manifold M is said to be trivial.

Let U be a vector field on B and let V be a vector field on F. Then from [7, Lemma 7.3], we have

$$\nabla_U V = \nabla_V U = (\frac{U\psi}{\psi})V, \tag{2.17}$$

where ∇ is the Levi-Civita connection on M. From (2.17) for the warped product $M = B \times_{\psi} F$, it is easy to conclude that

$$\nabla_U V = \nabla_V U = (U \ln \psi)V \tag{2.18}$$

for any $U \in TB$ and $V \in TF$. $\nabla\psi$ is the gradient of ψ and is defined as

$$g(\nabla\psi, U) = U\psi \tag{2.19}$$

for all $U \in TM$.

Let M be a m-dimensional Riemannian manifold with the Riemannian metric g and let $\{e_1, \dots, e_m\}$ be an orthogonal basis of TM. As a consequence of (2.18), we have

$$\|\nabla\psi\|^2 = \sum_{i=1}^{m} (e_i(\psi))^2.$$

The Laplacian of ψ is defined by

$$\Delta\psi = \sum_{i=1}^{m} \{(\nabla_{e_i} e_i)\psi - e_i e_i \psi\}. \tag{2.20}$$

Now, we state the Hopf's Lemma.

Lemma 2.1 (Hopf's Lemma [13])**.** *Let M be a n-dimensional compact Riemannian manifold. If ψ is differentiable function on M such that $\Delta\psi \geqslant 0$ everywhere on M (or $\Delta\psi \leqslant 0$ everywhere on M), then ψ is a constant function.*

3. Results and discussion

In this section, we consider the contact CR-warped product submanifolds of the type $N_T \times_\psi N_\perp$ of the nearly Sasakian manifolds $\bar{M}$, where N_T and $N_\perp$ are the invariant and anti-invariant submanifolds respectively of $\bar{M}$, Throughout, this section, we consider ξ tangent to N_T.

Now we obtain the following results, which will be used to prove the main results.

Lemma 3.1. *Let $M = N_T \times_\psi N_\perp$ be a contact CR-warped product submanifold of a nearly Sasakian manifold $\bar{M}$. Then*

(i) $\xi \ln\psi = 0$;
(ii) $g(h(U, W), \phi W) = (-\phi U \ln\psi + \eta(U))\|W\|^2$;
(iii) $g(h(\phi U, W), \phi W) = U \ln\psi \|W\|^2$

for any $U \in TN_T$ and $W \in TN_\perp$.

Proof. The equations (2.15) (a), (2.5), and (2.18) give $\xi \ln\psi = 0$. Hence, part (i) is proved.

By the Gauss formula,

$$g(h(U, W), \phi W) = g(\bar{\nabla}_W U, \phi W) = -g(\bar{\nabla}_W \phi W, U).$$

The above equation alongwith (2.10), (2.3), and (2.14) takes the following form

$$g(h(U, W), \phi W) = (\eta(U) - \phi U \ln\psi)\|W\|^2.$$

The above equation is the required result. Now replacing U by ϕU and using (2.1) and the part (i), we get the part (iii). □

Lemma 3.2. *Let $M = N_T \times_\psi N_\perp$ be a contact CR-warped product submanifold of a nearly Sasakian manifold $\bar{M}$. Then*

$$g(h(\phi U, W), \phi h(U, W)) = \|h_\mu(U, W)\|^2 - g(\phi h(U, W), \mathcal{Q}_U W)$$

for any $U \in TN_T$ and $W \in TN_\perp$.

Proof. By (2.5) and (2.10), we have

$$h(\phi U, W) = (\bar{\nabla}_W \phi)U + \phi\nabla_W U + \phi h(U, W) - \nabla_W \phi U.$$

In the view of (2.13) and (2.16), the above equation reduces to

$$h(\phi U, W) = \mathcal{P}_W U + \mathcal{Q}_W U + U \ln\psi\phi W + \phi h(U, W) - \phi U \ln\psi W.$$

On comparing the normal parts, we get

$$h(\phi U, W) = \mathcal{Q}_W U + U \ln\psi\phi W + \phi h_\mu(U, W),$$

or

$$g(h(\phi U, W), \phi h(U, W)) = g(\mathcal{Q}_W U, \phi h(U, W)) + \|h_\mu(U, W)\|^2.$$

By using (2.15) (b), we obtain

$$g(h(\phi U, W), \phi h(U, W)) = \|h_\mu(U, W)\|^2 - g(\phi h(U, W), \mathcal{Q}_U W).$$

□

3.1. The existence theorem

Now, we prove the following characterization for existence of the contact CR-warped product submanifolds.

Theorem 3.3. *Let $M = N_T \times_\psi N_\perp$ be a contact CR-warped product submanifold of a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting a nearly Sasakian structure such that N_T is compact. Then M is a contact CR-product submanifold, if either one of the following inequality holds*

(i) $\sum_{i=1}^{2p}\sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2 \geqslant 2.p.q.f_2 + \sum_{i=1}^{2p}\sum_{j=1}^{q} \|\mathcal{Q}_{e_i} e^j\|^2$;

(ii) $\sum_{i=1}^{2p}\sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2 \leqslant 2.p.q.f_2$,

where h_μ denotes the component of h in μ, $2p+1$ and q are the dimensions of N_T and $N_\perp$, respectively.

Proof. For any unit vector fields $U \in TN_T - \langle\xi\rangle$ and $W \in TN_\perp$. Then from (2.4) we have

$$R(U, \phi U, W, \phi W) = -2.f_2.g(U, U)\ g(W, W). \tag{3.1}$$

On the other hand by Codazzi equation

$$\begin{aligned}\bar{R}(U, \phi U, W, \phi W) = {} & g(\nabla^\perp_U h(\phi U, W), \phi W) - g(h(\nabla_U \phi U, W), \phi W)\\ & - g(h(\phi U, \nabla_U W), \phi W) - g(\nabla^\perp_{\phi U} h(U, W), \phi W)\\ & + g(h(\nabla_{\phi U} U, W), \phi W) + g(h(U, \nabla_{\phi U} W), \phi W).\end{aligned} \tag{3.2}$$

By using the part (iii) of Lemma 3.1, (2.10), (2.5), and (2.13), we get

$$\begin{aligned}g(\nabla^\perp_U h(\phi U, V), \phi W) &= U g(h(\phi U, W), \phi W) - g(h(\phi U, W), \bar{\nabla}_U \phi W)\\ &= U(U \ln\psi g(W, W)) - g(h(\phi U, W), (\bar{\nabla}_U \phi)W + \phi\bar{\nabla}_U W).\end{aligned}$$

On further simplification the above equation yields

$$\begin{aligned}g(\nabla^\perp_U h(\phi U, W), \phi W) = {} & U^2 \ln\psi g(W, W) + 2(U \ln\psi)^2 g(W, W) - g(h(\phi U, W), \mathcal{Q}_U W)\\ & - g(h(\phi U, W), \phi h(U, W)) - U \ln\psi g(h(\phi U, W), \phi W).\end{aligned}$$

By using the 3.2, we have

$$g(\nabla^{\perp}_{U}h(\phi U,W),\phi W)=U^2\ln\psi g(W,W)+(U\ln\psi)^2g(W,W)-\|h_\mu(U,W)\|^2 \\ -g(\phi h(U,W)-h(\phi U,W),\mathcal{Q}_U W).$$

Further, using (2.5), (2.13), (2.15) (b), and (2.18) in the last term of the above equation, we get

$$g(\nabla^{\perp}_{U}h(\phi U,W),\phi W)=U^2\ln\psi g(W,W)+(U\ln\psi)^2g(W,W)-\|h_\mu(U,W)\|^2+\|\mathcal{Q}_U W\|^2. \tag{3.3}$$

Similarly, we can calculate

$$-g(\nabla^{\perp}_{\phi U}h(U,W),\phi W)=(\phi U)^2\ln\psi g(W,W)+(\phi U\ln\psi)^2g(W,W)-\|h_\mu(\phi U,W)\|^2+\|\mathcal{Q}_{\phi U}W\|^2. \tag{3.4}$$

From the part (iii) of the 3.1, we have

$$g(A_{\phi W}W,\phi U)=U\ln\psi,$$

replacing U by $\nabla_U U$

$$g(A_{\phi W}W,\phi\nabla_U U)=\nabla_U U\ln\psi.$$

By using the Gauss formula in the last equation, we get

$$g(A_{\phi W}W,\phi(\bar{\nabla}_U U-h(U,U))=\nabla_U U\ln\psi. \tag{3.5}$$

By use of (2.5), (2.10), (2.3), and (2.18), it is easy to see that $h(U,U)\in\mu$, applying this fact in (3.5), then we get

$$g(A_{\phi W}W,\bar{\nabla}_U\phi U-(\bar{\nabla}_U\phi)U)=\nabla_U U\ln\psi.$$

In view of (2.3) and (2.5) the above equation reduces to

$$g(A_{\phi W}W,\nabla_U\phi U)+g(U,U)g(A_{\phi W}W,\xi)=\nabla_U U\ln\psi.$$

In view of (2.7) and (2.15) (b), the second term of the last equation becomes zero and we have

$$g(h(\nabla_U\phi U,W),\phi W)=\nabla_U U\ln\psi g(W,W). \tag{3.6}$$

Similarly,

$$g(h(\nabla_{\phi U}U,W),\phi W)=\nabla_{\phi U}\phi U\ln\psi g(W,W). \tag{3.7}$$

By use of (2.18) and the part (iii) of the 3.1, it is easy to see the following

$$g(h(\phi U,\nabla_U W),\phi W)=(U\ln\psi)^2g(W,W) \tag{3.8}$$

and

$$g(h(U,\nabla_{\phi U}W),\phi W)=-(\phi U\ln\psi)^2g(W,W). \tag{3.9}$$

Substituting (3.3), (3.4), (3.6), (3.7), (3.8), and (3.9) in (3.2), we find

$$R(U,\phi U,W,\phi W)=U^2\ln\psi g(W,W)+(\phi U)^2\ln\psi g(W,W)-\nabla_U U\ln\psi g(W,W) \\ -\nabla_{\phi U}\phi U g(W,W)-\|h_\mu(U,W)\|^2-\|h_\mu(\phi U,W)\|^2+\|\mathcal{Q}_U W\|^2+\|\mathcal{Q}_{\phi U}W\|^2. \tag{3.10}$$

Let $\{e_0=\xi,e_1,e_2,\dots,e_p,e_{p+1}=\phi e_1,e_{p+2}=\phi e_2,\dots,e_{2p}=\phi e_p,e^1,e^2,\dots,e^q\}$ be an orthonormal frame of TM such that the set $\{e_1,e_2,\dots,e_p,\phi e_1,\phi e_2,\dots,\phi e_p\}$ is tangent to TN_T and $\{e^1,e^2,\dots,e^q\}$ is tangent to $TN_\perp$. Using (3.1) and (2.20) in (3.10) and summing over $i=1,2,\dots,p$ and $j=1,2,\dots,q$, we get

$$q\Delta\ln\psi=2.p.q.f_2-\sum_{i=1}^{2p}\sum_{j=1}^{q}\|h_\mu(e_i,e^j)\|^2+\sum_{i=1}^{2p}\sum_{j=1}^{q}\|\mathcal{Q}_{e_i}e^j\|^2. \tag{3.11}$$

From the Hopf's Lemma and (3.11), if

$$\sum_{i=1}^{2p}\sum_{j=1}^{q}\|h_{\mu}(e_i,e^j)\|^2 \geqslant 2.p.q.f_2+\sum_{i=1}^{2p}\|\mathcal{Q}_{e_i}e^j\|^2,$$

or

$$\sum_{i=1}^{2p}\sum_{j=1}^{q}\|h_{\mu}(e_i,e^j)\|^2 \leqslant 2.p.q.f_2,$$

then the warping function ψ is constant on M, i.e., M is simply a contact CR-product submanifold, which proves the theorem completely.

Under a certain condition, the equation (3.11) becomes the Laplace equation on the Riemannian manifold, then we can see the nature of the warping function ψ in the following proposition. □

Proposition 3.4. *Let $M = N_T \times_{\psi} N_{\perp}$ be a contact CR-warped product submanifold of a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting a nearly Sasakian structure such that N_T is compact. Then the function $\ln\psi$ is a Harmonic function, if and only if*

$$\sum_{i=1}^{2p}\sum_{j=1}^{q}\|h_{\mu}(e_i,e^j)\|^2 = 2.p.q.f_2+\sum_{i=1}^{2p}\sum_{j=1}^{q}\|\mathcal{Q}_{e_i}e^j\|^2.$$

Moreover, if the ambient manifold $\bar{M}$ is a Sasakian space form, then from the equation (3.7) of [21] and the Hopf's Lemma we derive the following.

Theorem 3.5. *Let $M = N_T \times_{\psi} N_{\perp}$ be a contact CR-warped product submanifold of a Sasakian space form $\bar{M}(c)$, such that N_T is compact. Then M is a contact CR-product submanifold, if either the inequality*

$$\sum_{i=1}^{2p}\sum_{j=1}^{q}\|h_{\mu}(e_i,e^j)\|^2 \geqslant \frac{c+3}{2}p.q,$$

or

$$\sum_{i=1}^{2p}\sum_{j=1}^{q}\|h_{\mu}(e_i,e^j)\|^2 \leqslant \frac{c+3}{2}p.q$$

holds, where h_{μ} is the component of h in μ, and $2p+1$ and q are the dimensions of N_T and $N_{\perp}$.

Corollary 3.6. *Let $M = N_T \times_{\psi} N_{\perp}$ be a compact contact CR-warped product submanifolds of a Sasakian space form $\bar{M}(c)$. Then M is a contact CR-product submanifold, if and only if*

$$\sum_{i=1}^{2p}\sum_{j=1}^{q}\|h_{\mu}(e_i,e^j)\|^2 = \frac{c+3}{2}p.q.$$

3.2. Another inequality

In the present subsection, we generalize the inequality proved in the 3.3 of the reference [21].

Theorem 3.7. *Let $\bar{M}(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian space form admitting a nearly Sasakian structure and $M = N_T \times_{\psi} N_{\perp}$ be an m-dimensional contact CR-warped product submanifold, such that N_T is a $(2p+1)$-dimensional invariant submanifold tangent to ξ and $N_{\perp}$ be a q-dimensional anti-invariant submanifold of $\bar{M}(f_1, f_2, f_3)$. Then*

(i) *the squared norm of the second fundamental form* h *satisfies*

$$\|h\|^2 \geqslant q[\|\nabla \ln \psi\|^2 - \Delta \ln \psi + 1] + 2.p.q.f_2 + \|\mathcal{Q}_D D^\perp\|^2, \tag{3.12}$$

where Δ *denotes the Laplace operator on* N_T;

(ii) *the equality sign of* (3.12) *holds identically, if and only if we have*

(a) N_T *is a totally geodesic invariant submanifold of* $\bar{M}(f_1, f_2, f_3)$, *hence* N_T *is a generalized Sasakian space form admitting a nearly Sasakian structure;*

(b) $N_\perp$ *is a totally umbilical anti-invariant submanifold of* $\bar{M}(f_1, f_2, f_3)$.

Proof. For any $U \in TN_T$ and $W \in TN_\perp$, from (2.8) and the part (ii) of the 3.1, we have

$$g(h(\xi, W), \phi W) = 1$$

and

$$g(h(\phi U, W), \phi W) = U \ln \psi \|W\|^2.$$

From the above two equations one can get

$$\sum_{i=0}^{2p} \sum_{j=1}^{q} \|h_{\phi D^\perp}(e_i, e^j)\|^2 = q[\|\nabla \ln \psi\|^2 + 1]. \tag{3.13}$$

Again from the equation (3.11), we have

$$\sum_{i=1}^{2p} \sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2 = 2.p.q.f_2 - q\Delta \ln \psi + \sum_{i=1}^{2p} \sum_{j=1}^{q} \|\mathcal{Q}_{e^j} e_i\|^2. \tag{3.14}$$

We use the following notation

$$\sum_{i=1}^{2p} \sum_{j=1}^{q} \|\mathcal{Q}_{e_i} e^j\|^2 = \|\mathcal{Q}_D D^\perp\|^2.$$

Substituting above notation in (3.14) and combining it with (3.13), we obtain the inequality (3.12).

Let h'' be the second fundamental form of $N_\perp$ in M. Then, we have

$$g(h''(W, W'), U) = g(\nabla_W W', U) = -U \ln \psi g(W, W').$$

By using (2.19), we get

$$h''(W, W') = -g(W, W') \nabla \ln \psi. \tag{3.15}$$

If the equality sign of (3.12) holds identically, then we obtain

$$h(D, D) = 0, \qquad h(D^\perp, D^\perp) = 0. \tag{3.16}$$

The first condition of (3.16) implies that N_T is totally geodesic in M. On the other hand, one has

$$g(h(U, \phi V), \phi W) = g(\bar{\nabla}_U \phi V, \phi W) = -g(\phi V, (\bar{\nabla}_U \phi) W). \tag{3.17}$$

By use of (2.10) and (2.5) we get the following equation

$$g(\phi V, (\bar{\nabla}_W \phi) U) = g(\phi V, \nabla_W \phi U) - g(V, \nabla_W U),$$

in view of (2.18), the above equation reduced to

$$g(\phi V, (\bar{\nabla}_W \phi) U) = 0. \tag{3.18}$$

From (3.17), (3.18), and (2.15) (a) we have

$$g(h(U, \phi V), \phi W) = -g(\phi V, (\bar{\nabla}_U \phi) W + (\bar{\nabla}_W \phi) U) = 0. \tag{3.19}$$

From (3.19), it is evident that N_T is totally geodesic in $\bar{M}(f_1, f_2, f_3)$ and hence is a generalized Sasakian space form admitting a nearly Sasakian structure.

The second condition of (3.16) and (3.15) imply that $N_\perp$ is totally umbilical in $\bar{M}(f_1, f_2, f_3)$. □

In the last, we have the following corollary which can be deduced from inequality (3.12).

Corollary 3.8. *Let* $M = N_T \times_\psi N_\perp$ *be a contact CR-warped product submanifold of a Sasakian space form* $\bar{M}(c)$*, such that the dimensions of* N_T *and* $N_\perp$ *are* $2p+1$ *and* q*, respectively, then the squared norm of the second fundamental form satisfies*

$$\|h\|^2 \geqslant q[\|\nabla \ln\psi\|^2 - \Delta \ln\psi + 1] + \frac{c-1}{2}p.q,$$

where Δ *is the Laplace operator on* N_T.

4. Conclusion

In this paper, by using Hopf's Lemma, we have obtained the characterizing inequalities for the existence of the contact CR-warped product submanifolds in the setting of a generalized Sasakian space form, and simultaneously, some special cases are also discussed. The methodology and the techniques used in this paper are new and not available in the literature yet. Moreover, we also worked out an inequality for the squared norm of the second fundamental form in terms of the warping function and the slant angle.

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