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Optimization of two-step block method with three hybrid points for solving third order initial value problems

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Abstract

An optimized two-step hybrid block method for the numerical solution of third-order initial value problems is presented. The method takes into regard three hybrid points which are selected suitably to optimize the local truncation errors of the main formulas for the block. The method is zero-stable and consistent with sixth algebraic order. Some numerical examples are debated to demonstrate the efficiency and the accuracy of the proposed method.

Keywords: Two-step hybrid block method, third-order initial value problems, stability, consistent.

2010 MSC: 34A38, 65C20, 93C30.

1. Introduction

We are interested in concerned the approximate solution of third order initial value problems of Ordinary Differential Equations (ODEs) in the next form:

$$y''' = f(x, y, y', y''), y(a) = y_0, y'(a) = y(a), y''(a) = y'(a).x \in [a, b]. \tag{1.1}$$

Eq. (1.1) often arises in several areas of engineering, biology, fluid flows, mechanics, electric circuits model, vibrations and other real-life problems. Direct method for solving Eq. (1.1) has been reported to be more active than the method of lowering to a system of first order ordinary differential equations [5, 16, 17]. Block methods were suggested firstly by Milne [14]. They have advantages over predictor-corrector method for being estimate-effectiveness, and time of execution, accuracy, and give better approximations [6, 11, 15, 19]. Hybrid method has the feature of curtailment the step number of a method and still remains zero stable. This method while retaining certain properties of the continuous linear multi-step method participate with the Runge-Kutta method the property of use data at another points other than the step points [4]. Many authors have used hybrid block method for solving third order initial value problem

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[1, 3, 8–10, 20] in this article we will present a two-step continuous hybrid block method with three intermediate points through interpolation and collocation that are obtained through the optimization of the local truncation errors of the major formulas for the solution and the derivative at the last point of the block. This paper is separated as follows. Section 2 explain the method, Section 3 establishes the dissection of the method which involves order six, Section 4 presents the numerical examples are included, that demonstrate the productivity of the new technique when it is contrasted with different strategies proposed in the scientist writing.

2. Derivation of the method

In this paper, we will approximate the solution $y(x)$ of Eq. (1.1) by a polynomial $p(x)$ at the grid points $a = x_0 < x_1 < \cdots < x_M = b$ of the interval [a,b], with constant step size h, where $h = x_{i+1} - x_i, i = 0, 1, \ldots, M-1$, by introduction of three intermediate points; $x_{m+r} = x_m + rh$, $x_{m+s} = x_m + sh$, and $x_{m+g} = x_m + gh$ with $0 < r, s, g < 2$.

Let us consider a power series approximate solution in the form

$$y(x) \simeq p(x) = \sum_{m=0}^{8} a_m x^m, \tag{2.1}$$

where the coefficients $a_m \in \mathbb{R}$, are real unknown to be determined. Therefore, from Eq. (2.1) we get

$$\begin{aligned} y'(x) &\simeq p'(x) = \sum_{m=1}^{8} m a_m x^{m-1}, \\ y''(x) &\simeq p''(x) = \sum_{m=2}^{8} m(m-1) a_m x^{m-2}, \\ y'''(x) &\simeq p'''(x) = \sum_{m=3}^{8} m(m-1)(m-2) a_m x^{m-3}. \end{aligned} \tag{2.2}$$

We impose that for approximating the solution in the two-step on $[x_m, x_{m+2}]$, we consider the polynomial in Eq. (2.1) applied to the points x_m and x_{m+r}, x_{m+1} and its third derivative in Eq. (2.2) applied to the points $x_m, x_{m+r}, x_{m+1}, x_{m+s}, x_{m+g}, x_{m+2}$. In this way we obtain a system of eight equations with eight unknowns $a_m, m = 0, 1, \ldots, 8$, given by

$$p(x_{m+i}) = y_{m+i}, \quad i = 0, r, 1$$

and

$$p'''(x_{m+i}) = f_{m+i}, \quad i = 0, g, 1, s, r, 2,$$

where as usual, the y_{m+j} and f_{m+j} are respectively approximations for the $y(x_{m+j})$ and $y'''(x_{m+j}) = f(x_{m+j}, y(x_{m+j}), y'(x_{m+j}), y''(x_{m+j}))$. This system of linear equations can be written in matrix form as

$$\begin{bmatrix} 1 & x_m & {x_m}^2 & x_m^3 & x_m^4 & x_m^5 & x_m^6 & x_m^7 & x_m^8 \\ 1 & x_{m+r} & x_{m+r}^2 & x_{m+r}^3 & x_{m+r}^4 & x_{m+r}^5 & x_{m+r}^6 & x_{m+r}^7 & x_{m+r}^8 \\ 1 & x_{m+1} & x_{m+1}^2 & x_{m+1}^3 & x_{m+1}^4 & x_{m+1}^5 & x_{m+1}^6 & x_{m+1}^7 & x_{m+1}^8 \\ 0 & 0 & 0 & 6 & 24x_m & 60x_m^2 & 120x_m^3 & 210x_m^4 & 336x_m^5 \\ 0 & 0 & 0 & 6 & 24x_{m+r} & 60x_{m+r}^2 & 120x_{m+r}^3 & 210x_{m+r}^4 & 336x_{m+r}^5 \\ 0 & 0 & 0 & 6 & 24x_{m+1} & 60x_{m+1}^2 & 120x_{m+1}^3 & 210x_{m+1}^4 & 336x_{m+1}^5 \\ 0 & 0 & 0 & 6 & 24x_{m+s} & 60x_{m+s}^2 & 120x_{m+s}^3 & 210x_{m+s}^4 & 336x_{m+s}^5 \\ 0 & 0 & 0 & 6 & 24x_{m+g} & 60x_{m+g}^2 & 120x_{m+g}^3 & 210x_{m+g}^4 & 336x_{m+g}^5 \\ 0 & 0 & 0 & 6 & 24x_{m+2} & 60x_{m+2}^2 & 120x_{m+2}^3 & 210x_{m+2}^4 & 336x_{m+2}^5 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} y_m \\ y_{m+r} \\ y_{m+1} \\ f_m \\ f_{m+r} \\ f_{m+1} \\ f_{m+s} \\ f_{m+g} \\ f_{m+2} \end{bmatrix}.$$

Solving the above system gives us the coefficients of the polynomial $a_m, m = 0, 1, \ldots, 8$.

By making the substitution $x = x_m + th$, the polynomial in Eq. (2.1) may be written in the form:

$$p(x_m + th) = \alpha_0 y_m + \alpha_r y_{m+r} + \alpha_1 y_{m+1} + h^3(\beta_0 f_m + \beta_r f_{m+r} + \beta_1 f_{m+1} + \beta_s f_{m+s} + \beta_g f_{m+g} + \beta_2 f_{m+2})$$

where

$$\begin{aligned}
\alpha_0 &= \frac{-(r-t)(t-1)}{r},\\
\alpha_r &= \frac{t(t-1)}{r(r-1)},\\
\alpha_1 &= \frac{t(r-t)}{r-1},\\
\beta_0 &= \frac{1}{(1680s(g-s)(r-s)(s^2-3s+2))}(t(r-t)(t-1)(22g+13r-9t-15r^2t^2+3r^2t^3\\
&\quad +3r^3t^2-48gr+22gt+13rt-48gr^2+36gr^3-6gr^4+22gt^2-34gt^3+8gt^4+13rt^2\\
&\quad +13r^2t-15rt^3-15r^3t+3rt^4+3r^4t+13r^2+13r^3-15r^4+3r^5-9t^2-9t^3+19t^4\\
&\quad -5t^5+36grt^2+36gr^2t-6grt^3-6gr^3t-6gr^2t^2-48grt-9))\\
\beta_r &= \frac{1}{1680r(g-r)(r-s)(r^2-3r+2)}(t(r-t)(t-1)(9r-22g-22s+9t-19r^2t^2+5r^2t^3\\
&\quad +5r^3t^2-22gr+70gs-22gt-22rs+9rt-22st-22gr^2+34gr^3-8gr^4-22gt^2\\
&\quad +34gt^3-8gt^4-22r^2s+34r^3s-8r^4s+9rt^2+9r^2t-19rt^3-19r^3t+5rt^4+5r^4t\\
&\quad -22st^2+34st^3-8st^4+9r^2+9r^3-19r^4+5r^5+9t^2+9t^3-19t^4+5t^5-70gr^2s\\
&\quad +14gr^3s+34grt^2+34gr^2t-8grt^3-8gr^3t-70gst^2+14gst^3+34rst^2+34r^2st\\
&\quad -8rst^3-8r^3st-8gr^2t^2-8r^2st^2+70grs-22grt+70gst-22rst-70grst+14grst^2+14gr^2st+9)),\\
\beta_1 &= \frac{1}{(1680(g-1)(r-1)(s-1))}(-t(r-t)(t-1)(20g+9r+20s-11t+9r^2t^2-3r^2t^3\\
&\quad -3r^3t^2-22gr-42gs+20gt-22rs+9rt+20st-22gr^2-22gr^3+6gr^4+20gt^2\\
&\quad +20gt^3-8gt^4-22r^2s-22r^3s+6r^4s+9rt^2+9r^2t+9rt^3+9r^3t-3rt^4-3r^4t+20st^2\\
&\quad +20st^3-8st^4+9r^2+9r^3+9r^4-3r^5-11t^2-11t^3-11t^4+5t^5+70gr^2s-14gr^3s\\
&\quad -22grt^2-22gr^2t+6grt^3+6gr^3t-42gst^2+14gst^3-22rst^2-22r^2st+6rst^3+6r^3st\\
&\quad +6gr^2t^2+6r^2st^2+70grs-22grt-42gst-22rst+70grst-14grst^2-14gr^2st-11)),\\
\beta_s &= \frac{1}{3360grs}(-(t(r-t)(t-1)(22g+13r+22s-9t-15r^2t^2+3r^2t^3+3r^3t^2-48gr-70gs\\
&\quad +22gt-48rs+13rt+22st-48gr^2+36gr^3-6gr^4+22gt^2-34gt^3+8gt^4-48r^2s\\
&\quad +36r^3s-6r^4s+13rt^2+13r^2t-15rt^3-15r^3t+3rt^4+3r^4t+22st^2-34st^3+8st^4+13r^2\\
&\quad +13r^3-15r^4+3r^5-9t^2-9t^3+19t^4-5t^5-112gr^2s+14gr^3s+36grt^2+36gr^2t\\
&\quad -6grt^3-6gr^3t+70gst^2-14gst^3+36rst^2+36r^2st-6rst^3-6r^3st-6gr^2t^2-6r^2st^2\\
&\quad +308grs-48grt-70gst-48rst-112grst+14grst^2+14gr^2st-9))),\\
\beta_g &= \frac{1}{1680g(g-r)(g-s)(g^2-3g+2)}(t(r-t)(t-1)(13r+22s-9t-15r^2t^2+3r^2t^3+3r^3t^2\\
&\quad -48rs+13rt+22st-48r^2s+36r^3s-6r^4s+13rt^2+13r^2t-15rt^3-15r^3t+3rt^4+3r^4t\\
&\quad +22st^2-34st^3+8st^4+13r^2+13r^3-15r^4+3r^5-9t^2-9t^3+19t^4-5t^5+36rst^2+36r^2st\\
&\quad -6rst^3-6r^3st-6r^2st^2-48rst-9)),
\end{aligned}$$

$$\beta_2 = \frac{1}{3360(g-2)(r-2)(s-2)}(t(r-t)(t-1)(6g+3r+6s-3t+3r^2t^2-3r^2t^3-3r^3t^2-8gr$$
$$-14gs+6gt-8rs+3rt+6st-8gr^2-8gr^3+6gr^4+6gt^2+6gt^3-8gt^4-8r^2s-8r^3s$$
$$+6r^4s+3rt^2+3r^2t+3rt^3+3r^3t-3rt^4-3r^4t+6st^2+6st^3-8st^4+3r^2+3r^3+3r^4$$
$$-3r^5-3t^2-3t^3-3t^4+5t^5+28gr^2s-14gr^3s-8grt^2-8gr^2t+6grt^3+6gr^3t-14gst^2$$
$$+14gst^3-8rst^2-8r^2st+6rst^3+6r^3st+6gr^2t^2+6r^2st^2+28grs-8grt-14gst-8rst$$
$$+28grst-14grst^2-14gr^2st-3)).$$

Now, by evaluating the approximation of the solution at the point x_{m+2}

$$y_{m+2} = -\frac{(r-2)}{r}y_m + \frac{2}{r(r-1)}y_{m+r} + \frac{(2r-4)}{(r-1)}y_{m+1} + \frac{h^3}{840s(g-s)(r-s)(s^2-3s+2)}((r-2)(10g$$
$$+19r-48gr+24gr^3-6gr^4+3r^2-5r^3-9r^4+3r^5+9))f_{m+s}$$
$$-\frac{h^3}{(840r(g-r)(r-s)(r-1))}(10g+9r+10s-6gr-42gs-6rs-14gr^2-18gr^3+8gr^4$$
$$-14r^2s-18r^3s+8r^4s+9r^2+9r^3+9r^4-5r^5+42gr^2s-14gr^3s+14grs+9)f_{m+r}$$
$$-\frac{h^3}{(840(g-1)(r-1)(s-1))}((r-2)(172g+87r+172s-106gr-182gs-106rs-42gr^2$$
$$-10gr^3+6gr^4-42r^2s-10r^3s+6r^4s+39r^2+15r^3+3r^4-3r^5+42gr^2s-14gr^3s+154grs \tag{2.3}$$
$$-181))f_{m+1} - \frac{h^3}{(1680grs)}((r-2)(10g+19r+10s-48gr-42gs-48rs+24gr^3-6gr^4$$
$$+24r^3s-6r^4s+3r^2-5r^3-9r^4+3r^5-84gr^2s+14gr^3s+140grs+9))f_m$$
$$+\frac{h^3}{(840g(g-r)(g-s)(g^2-3g+2))}((r-2)(19r+10s-48rs+24r^3s-6r^4s+3r^2-5r^3-9r^4$$
$$+3r^5+9))f_{m+g} - \frac{h^3}{(1680(g-2)(s-2))}((38g+3r+38s+8gr-14gs+8rs-4gr^3$$
$$-6gr^4-4r^3s-6r^4s+3r^2+3r^3+3r^4+3r^5+14gr^3s-28grs-67))f_{m+2}.$$

Evaluate the approximation of the first derivative at the point x_{m+2}:

$$hy'_{m+2} = -\frac{(r-3)}{r}y_m + \frac{3}{(r(r-1))}y_{m+r} + \frac{(r-4)}{(r-1)}y_{m+1} + \frac{h^3}{(1680s(g-s)(r-s)(s^2-3s+2))}((88g+61r$$
$$+234gr-144gr^2-144gr^3+108gr^4-18gr^5+39r^2+39r^3+39r^4-45r^5+9r^6-292))f_{m+s}$$
$$-\frac{h^3}{(1680r(g-r)(r-s)(r^2-3r+2))}((88g-27r+88s+66gr+168gs+66rs+66gr^2+66gr^3$$
$$-102gr^4+24gr^5+66r^2s+66r^3s-102r^4s+24r^5s-27r^2-27r^3-27r^4+57r^5$$
$$-15r^6-210gr^2s+210gr^3s-42gr^4s-210grs-292))f_{m+r}$$
$$+\frac{h^3}{(1680(g-1)(r-1)(s-1))}((1712g+1745r+1712s-1684gr-1624gs-1684rs+66gr^2$$
$$+66gr^3+66gr^4-18gr^5+66r^2s+66r^3s+66r^4s-18r^5s-27r^2-27r^3-27r^4-27r^5$$
$$+9r^6-210gr^2s-210gr^3s+42gr^4s+1918grs-2004))f_{m+1} \tag{2.4}$$
$$-\frac{h^3}{(3360grs)}((88g+61r+88s+234gr+168gs+234rs-144gr^2-144gr^3+108gr^4$$
$$-18gr^5-144r^2s-144r^3s+108r^4s-18r^5s+39r^2+39r^3+39r^4-45r^5+9r^6$$

$$+924gr^2s-336gr^3s+42gr^4s-938grs-292))f_m$$
$$+\frac{h^3}{(1680g(g-r)(g-s)(g^2-3g+2))}((61r+88s+234rs-144r^2s-144r^3s+108r^4s$$
$$-18r^5s+39r^2+39r^3+39r^4-45r^5+9r^6-292))f_{m+g}$$
$$+\frac{h^3}{(3360(g-2)(r-2)(s-2))}((920g+911r+920s-486gr-504gs-486rs-24gr^2-24gr^3$$
$$-24gr^4+18gr^5-24r^2s-24r^3s-24r^4s+18r^5s+9r^2+9r^3+9r^4+9r^5-9r^6$$
$$+84gr^2s+84gr^3s-42gr^4s+126grs-1548))f_{m+2}.$$

Evaluate the approximation of the second derivative at the point x_{m+2}

$$\begin{aligned}
h^2y''_{m+2}&=\frac{2}{r}y_m+\frac{2}{r(r-1)}y_{m+r}-\frac{2}{(r-1)}y_{m+1}+\frac{h^3}{(840s(g-s)(r-s)(s^2-3s+2))}((246g+237r\\
&-48gr-48gr^2-48gr^3+36gr^4-6gr^5+13r^2+13r^3+13r^4-15r^5+3r^6-457))f_{m+s}\\
&-\frac{h^3}{(840r(g-r)(r-s)(r^2-3r+2))}((246g-9r+246s+22gr-70gs+22rs+22gr^2\\
&+22gr^3-34gr^4+8gr^5+22r^2s+22r^3s-34r^4s+8r^5s-9r^2-9r^3-9r^4+19r^5-5r^6\\
&-70gr^2s+70gr^3s-14gr^4s-70grs-457))f_{m+r}\\
&+\frac{h^3}{(840(g-1)(r-1)(s-1))}(1324g+1335r+1324s-1098gr-1078gs-1098rs+22gr^2\\
&+22gr^3+22gr^4-6gr^5+22r^2s+22r^3s+22r^4s-6r^5s-9r^2-9r^3-9r^4-9r^5\\
&+3r^6-70gr^2s-70gr^3s+14gr^4s+1050grs-1781))f_{m+1}\\
&+\frac{h^3}{(1680grs)}(48gr-237r-246s-246g+70gs+48rs+48gr^2+48gr^3-36gr^4\\
&+6gr^5+48r^2s+48r^3s-36r^4s+6r^5s-13r^2-13r^3-13r^4+15r^5-3r^6-308gr^2s\\
&+112gr^3s-14gr^4s+252grs+457)f_m\\
&+\frac{h^3}{(840g(g-r)(g-s)(g^2-3g+2))}(237r+246s-48rs-48r^2s-48r^3s+36r^4s-6r^5s\\
&+13r^2+13r^3+13r^4-15r^5+3r^6-457)f_{m+g}\\
&+\frac{h^3}{(1680(g-2)(r-2)(s-2))}(2022g+2019r+2022s-1128gr-1134gs-1128rs-8gr^2\\
&-8gr^3-8gr^4+6gr^5-8r^2s-8r^3s-8r^4s+6r^5s+3r^2+3r^3+3r^4+3r^5-3r^6\\
&+28gr^2s+28gr^3s-14gr^4s+588grs-3587))f_{m+2}.
\end{aligned}\tag{2.5}$$

In order to determine appropriate values for r, s and g we choose to optimize the local truncation errors in the formulae for Eq. (2.3)-(2.5). This choice at y_{m+2}, y'_{m+2} and y''_{m+2}, the end of the block.

$$\begin{aligned}
\mathcal{L}(y(x_{m+2};h))&=\frac{y^{(9)}(x_m)h^9}{1814400}(r-2)(27g-18r+27s+57gr+30gs+57rs+9gr^2-15gr^3\\
&-27gr^4+9gr^5+9r^2s-15r^3s-27r^4s+9r^5s-2r^2+6r^3+10r^4+12r^5\\
&-5r^6+72gr^3s-18gr^4s-144grs-77)\\
&+\frac{y^{(10)}(x_m)h^{10}}{4233600}(r-2)(3g^2r^5-6g^2r^4s-9g^2r^4+24g^2r^3s-5g^2r^3+3g^2r^2-48g^2rs\\
&+19g^2r+10g^2s+9g^2+3gr^6-3gr^5s-6gr^4s^2-3gr^4s-32gr^4+24gr^3s^2+67gr^3s
\end{aligned}$$

$$
\begin{aligned}
&- 12gr^3 - 45gr^2s + 3r^5s^2 + r^5 - 9r^4s^2 - 32r^4s - 21r^4 - 5r^3s^2 - 12r^3s - 65r^3 + 3r^2s^2\\
&+ 28r^2s - 153r^2 + 19rs^2 + 66rs - 338r + 9s^2 + 27s - 717),\\
\mathcal{L}(y'(x_{m+2}; h)) = {}& \frac{y^{(9)}(x_m)h^9}{3628800}(183gr - 837r - 876s - 876g + 264gs + 183rs + 117gr^2 + 117gr^3\\
&+ 117gr^4 - 135gr^5 + 27gr^6 + 117r^2s + 117r^3s + 117r^4s - 135r^5s + 27r^6s - 42r^2\\
&- 42r^3 - 42r^4 - 42r^5 + 66r^6 - 15r^7 - 432gr^2s - 432gr^3s + 324gr^4s - 54gr^5s\\
&+ 702grs + 1588)\\
&+ \frac{y^{(10)}(x_m)h^{10}}{25401600}(27g^2r^6 - 54g^2r^5s - 135g^2r^5 + 324g^2r^4s + 117g^2r^4 - 432g^2r^3s\\
&+ 117g^2r^3 - 432g^2r^2s + 117g^2r^2 + 702g^2rs + 183g^2r + 264g^2s - 876g^2 + 27gr^7\\
&- 27gr^6s - 54gr^6 - 54gr^5s^2 + 27gr^5s - 288gr^5 + 324gr^4s^2 + 657gr^4s\\
&+ 468gr^4 - 432gr^3s^2 - 1611gr^3s + 468gr^3 - 432gr^2s^2 - 477gr^2s + 534gr^2\\
&+ 702grs^2 + 2553grs - 327gr + 264gs^2 - 84gs - 2628g - 45r^8 + 27r^7s + 36r^7\\
&+ 27r^6s^2 - 54r^6s + 117r^6 - 135r^5s^2 - 288r^5s - 207r^5 + 117r^4s^2 + 468r^4s\\
&- 207r^4 + 117r^3s^2 + 468r^3s - 207r^3 + 117r^2s^2 + 534r^2s - 1002r^2 + 183rs^2\\
&- 327rs - 2511r - 876s^2 - 2628s + 7324),\\
\mathcal{L}(y''(x_{m+2}; h)) = {}& -\frac{y^{(9)}(x_m)h^9}{1814400}(1371g + 1358r + 1371s - 711gr - 738gs - 711rs - 39gr^2 - 39gr^3\\
&- 39gr^4 + 45gr^5 - 9gr^6 - 39r^2s - 39r^3s - 39r^4s + 45r^5s - 9r^6s + 14r^2 + 14r^3\\
&+ 14r^4 + 14r^5 - 22r^6 + 5r^7 + 144gr^2s + 144gr^3s - 108gr^4s + 18gr^5s + 144grs - 2317)\\
&+ \frac{y^{(10)}(x_m)h^{10}}{4233600}(3g^2r^6 - 6g^2r^5s - 15g^2r^5 + 36g^2r^4s + 13g^2r^4 - 48g^2r^3s + 13g^2r^3\\
&- 48g^2r^2s + 13g^2r^2 - 48g^2rs + 237g^2r + 246g^2s - 457g^2 + 3gr^7 - 3gr^6s - 6gr^6\\
&- 6gr^5s^2 + 3gr^5s - 32gr^5 + 36gr^4s^2 + 73gr^4s + 52gr^4 - 48gr^3s^2 - 179gr^3s\\
&+ 52gr^3 - 48gr^2s^2 - 179gr^2s + 276gr^2 - 48grs^2 + 339grs + 254gr + 246gs^2\\
&+ 281gs - 1371g - 5r^8 + 3r^7s + 4r^7 + 3r^6s^2 - 6r^6s + 13r^6 - 15r^5s^2 - 32r^5s - 23r^5\\
&+ 13r^4s^2 + 52r^4s - 23r^4 + 13r^3s^2 + 52r^3s - 23r^3 + 13r^2s^2 + 276r^2s - 471r^2\\
&+ 237rs^2 + 254rs - 1358r - 457s^2 - 1371s + 3597).
\end{aligned}
$$

To determine the values of r, s and g, equating to zero the coefficients of h^9 in the formulas of local truncation errors above, we obtain the system

$$
\begin{aligned}
&(r-2)(27g - 18r + 27s + 57gr + 30gs + 57rs + 9gr^2 - 15gr^3 - 27gr^4 + 9gr^5 + 9r^2s - 15r^3s\\
&\quad - 27r^4s + 9r^5s - 2r^2 + 6r^3 + 10r^4 + 12r^5 - 5r^6 + 72gr^3s - 18gr^4s - 144grs - 77) = 0,\\
&183gr - 837r - 876s - 876g + 264gs + 183rs + 117gr^2 + 117gr^3 + 117gr^4 - 135gr^5 + 27gr^6\\
&\quad + 117r^2s + 117r^3s + 117r^4s - 135r^5s + 27r^6s - 42r^2 - 42r^3 - 42r^4 - 42r^5 + 66r^6 - 15r^7\\
&\quad - 432gr^2s - 432gr^3s + 324gr^4s - 54gr^5s + 702grs + 1588 = 0,\\
&1371g + 1358r + 1371s - 711gr - 738gs - 711rs - 39gr^2 - 39gr^3 - 39gr^4 + 45gr^5 - 9gr^6\\
&\quad - 39r^2s - 39r^3s - 39r^4s + 45r^5s - 9r^6s + 14r^2 + 14r^3 + 14r^4 + 14r^5 - 22r^6 + 5r^7\\
&\quad + 144gr^2s + 144gr^3s - 108gr^4s + 18gr^5s + 144grs - 23177 = 0,
\end{aligned}
$$

whose solution is

$$r=\frac{-36870+2\sqrt{619998117}}{30867},\qquad s=\frac{40897+\sqrt{4332680249}}{69790},\qquad g=\frac{18627-\sqrt{50225239}}{6206}.$$

Substituting the above values of r, s and g in the local truncation errors gives:

$$\begin{aligned}
\mathcal{L}(y(x_m;h)&\simeq\frac{5h^{10}y^{(10)}}{44694}+O(h^{11}),\\
\mathcal{L}(y'(x_m;h)&\simeq\frac{-19h^{10}y^{(10)}}{418018}+O(h^{11}),\\
\mathcal{L}(y''(x_m;h)&\simeq\frac{-18h^{10}y^{(10)}}{596411}+O(h^{11}).
\end{aligned}$$

To get a two-step hybrid block method for solving Eq. (1.1), we evaluate $p(x)$ at the points x_{m+g}, x_{m+s}, x_{m+r} and $p'(x), p''(x)$ at the points $x_m, x_{m+g}, x_{m+s}, x_{m+r}, x_{m+1}$. We will get the following system:

$$\begin{aligned}
hy'_m=&-\frac{(r+1)}{r}y_m-\frac{1}{(r(r-1))}y_{m+r}+\frac{r}{(r-1)}y_{m+1}-\frac{h^3}{(1680s(g-s)(r-s)(s^2-3s+2))}\\
&\times(r(22g+13r-48gr-48gr^2+36gr^3-6gr^4+13r^2+13r^3-15r^4+3r^5-9))f_{m+s}\\
&+\frac{h^3}{(1680(g-r)(r-s)(r^2-3r+2))}(22g-9r+22s+22gr-70gs+22rs+22gr^2\\
&-34gr^3+8gr^4+22r^2s-34r^3s+8r^4s-9r^2-9r^3+19r^4-5r^5+70gr^2s-14gr^3s-70grs\\
&-9)f_{m+r}+\frac{h^3}{(1680(g-1)(r-1)(s-1))}(r(20g+9r+20s-22gr-42gs-22rs-22gr^2-22gr^3\\
&+6gr^4-22r^2s-22r^3s+6r^4s+9r^2+9r^3+9r^4-3r^5+70gr^2s-14gr^3s+70grs-11))f_{m+1}\\
&+\frac{h^3}{(3360gs)}(22g+13r+22s-48gr-70gs-48rs-48gr^2+36gr^3-6gr^4\\
&-48r^2s+36r^3s-6r^4s+13r^2+13r^3-15r^4+3r^5-112gr^2s+14gr^3s+308grs-9)f_m\\
&-\frac{h^3}{(1680g(g-r)(g-s)(g^2-3g+2))}(r(13r+22s-48rs-48r^2s+36r^3s-6r^4s+13r^2\\
&+13r^3-15r^4+3r^5-9))f_{m+g}\\
&-\frac{h^3}{(3360(g-2)(r-2)(s-2))}(r(6g+3r+6s-8gr-14gs-8rs-8gr^2-8gr^3+6gr^4\\
&-8r^2s-8r^3s+6r^4s+3r^2+3r^3+3r^4-3r^5+28gr^2s-14gr^3s+28grs-3))f_{m+2},
\end{aligned}$$

$$\begin{aligned}
h^2y''_m=&\frac{2}{r}y_m+\frac{2}{(r(r-1))}y_{m+r}-\frac{2}{(r-1)}y_{m+1}+\frac{h^3}{(840s(g-s)(r-s)(s^2-3s+2))}((22g+13r\\
&-48gr-48gr^2-48gr^3+36gr^4-6gr^5+13r^2+13r^3+13r^4-15r^5+3r^6-9))f_{m+s}\\
&-\frac{h^3}{(840r(g-r)(r-s)(r^2-3r+2))}((22g-9r+22s+22gr-70gs+22rs+22gr^2\\
&+22gr^3-34gr^4+8gr^5+22r^2s+22r^3s-34r^4s+8r^5s-9r^2-9r^3-9r^4+19r^5-5r^6\\
&-70gr^2s+70gr^3s-14gr^4s-70grs-9))f_{m+r}\\
&-\frac{h^3}{(840(g-1)(r-1)(s-1))}(20g+9r+20s-22gr-42gs-22rs-22gr^2-22gr^3\\
&-22gr^4+6gr^5-22r^2s-22r^3s-22r^4s+6r^5s+9r^2+9r^3+9r^4+9r^5-3r^6+70gr^2s
\end{aligned}$$

$$
\begin{aligned}
&+70gr^3s-14gr^4s+70grs-11)f_{m+1}\\
&-\frac{h^3}{1680grs}(22g+13r+22s-48gr-70gs-48rs-48gr^2-48gr^3+36gr^4-6gr^5-48r^2s\\
&-48r^3s+36r^4s-6r^5s+13r^2+13r^3+13r^4-15r^5+3r^6+308gr^2s-112gr^3s+14gr^4s\\
&+308grs-9)f_m\\
&+\frac{h^3}{(840g(g-r)(g-s)(g^2-3g+2))}(13r+22s-48rs-48r^2s-48r^3s+36r^4s-6r^5s\\
&+13r^2+13r^3+13r^4-15r^5+3r^6-9)f_{m+g}\\
&+\frac{h^3}{(1680(g-2)(r-2)(s-2))}(6g+3r+6s-8gr-14gs-8rs-8gr^2-8gr^3-8gr^4\\
&+6gr^5-8r^2s-8r^3s-8r^4s+6r^5s+3r^2+3r^3+3r^4+3r^5-3r^6+28gr^2s+28gr^3s\\
&-14gr^4s+28grs-3)f_{m+2},\\
hy'_{m+r}=&\frac{r-1}{r}y_m+\frac{(2r-1)}{(r(r-1))}y_{m+r}-\frac{r}{(r-1)}y_{m+1}-\frac{h^3}{(1680s(g-s)(r-s)(s^2-3s+2))}\\
&\times(r(r-1)(22g+4r-26gr-74gr^2+74gr^3-16gr^4+17r^2+30r^3-41r^4+10r^5-9))f_{m+s}\\
&+\frac{h^3}{(1680(g-r)(r-s)(r-2))}(22g-18r+22s+44gr-70gs+44rs+66gr^2\\
&-136gr^3+40gr^4+66r^2s-136r^3s+40r^4s-27r^2-36r^3+95r^4-30r^5+210gr^2s\\
&-56gr^3s-140grs-9)f_{m+r}\\
&-\frac{h^3}{(1680(g-1)(s-1))}(r(2r-20g-20s+2gr+42gs+2rs+24gr^2+46gr^3-16gr^4\\
&+24r^2s+46r^3s-16r^4s-7r^2-16r^3-25r^4+10r^5-98gr^2s+28gr^3s-28grs+11))f_{m+1}\\
&+\frac{h^3}{(3360gs)}((r-1)(22g+4r+22s-26gr-70gs-26rs-74gr^2+74gr^3-16gr^4\\
&-74r^2s+74r^3s-16r^4s+17r^2+30r^3-41r^4+10r^5-154gr^2s+28gr^3s+238grs-9))f_m\\
&-\frac{h^3}{(1680g(g-r)(g-s)(g^2-3g+2))}(r(r-1)(4r+22s-26rs-74r^2s+74r^3s)\\
&-16r^4s+17r^2+30r^3-41r^4+10r^5-9))f_{m+g}\\
&+\frac{h^3}{(3360(g-2)(r-2)(s-2))}(r(r-1)(2gr-6s-6g+14gs+2rs+10gr^2+18gr^3\\
&-16gr^4+10r^2s+18r^3s-16r^4s-3r^2-6r^3-9r^4+10r^5-42gr^2s+28gr^3s-14grs+3))f_{m+2},\\
h^2y''_{m+r}=&\frac{2}{r}y_m+\frac{2}{(r(r-1))}y_{m+r}-\frac{2}{(r-1)}y_{n+1}+\frac{h^3}{(840s(g-s)(r-s)(s^2-3s+2))}(22g+13r-48gr\\
&-48gr^2+232gr^3-174gr^4+36gr^5+13r^2+13r^3-127r^4+111r^5-25r^6-9)f_{m+s}\\
&-\frac{h^3}{(840r(g-r)(r-s)(r^2-3r+2))}(22g-9r+22s+22gr-70gs+22rs+22gr^2-538gr^3\\
&+596gr^4-160gr^5+22r^2s-538r^3s+596r^4s-160r^5s-9r^2-9r^3+411r^4-485r^5\\
&+135r^6+770gr^2s-770gr^3s+196gr^4s-70grs-9)f_{m+r}\\
&-\frac{h^3}{(840(g-1)(r-1)(s-1))}(20g+9r+20s-22gr-42gs-22rs-22gr^2-22gr^3+118gr^4\\
&-36gr^5-22r^2s-22r^3s+118r^4s-36r^5s+9r^2+9r^3+9r^4-75r^5+25r^6+70gr^2s
\end{aligned}
$$

$$
\begin{aligned}
&- 210gr^3s + 56gr^4s + 70grs - 11)f_{m+1} \\
&- \frac{h^3}{1680grs}(22g + 13r + 22s - 48gr - 70gs - 48rs - 48gr^2 + 232gr^3 - 174gr^4 + 36gr^5 \\
&- 48r^2s + 232r^3s - 174r^4s + 36r^5s + 13r^2 + 13r^3 - 127r^4 + 111r^5 - 25r^6 - 532gr^2s \\
&+ 308gr^3s - 56gr^4s + 308grs - 9)f_m \\
&+ \frac{h^3}{(840g(g-r)(g-s)(g^2-3g+2))}(13r + 22s - 48rs - 48r^2s + 232r^3s - 174r^4s + 36r^5s \\
&+ 13r^2 + 13r^3 - 127r^4 + 111r^5 - 25r^6 - 9)f_{m+g} \\
&+ \frac{h^3}{(1680(g-2)(r-2)(s-2))}(6g + 3r + 6s - 8gr - 14gs - 8rs - 8gr^2 - 8gr^3 + 62gr^4 - 36gr^5 \\
&- 8r^2s - 8r^3s + 62r^4s - 36r^5s + 3r^2 + 3r^3 + 3r^4 - 39r^5 + 25r^6 + 28gr^2s - 112gr^3s \\
&+ 56gr^4s + 28grs - 3)f_{m+2},
\end{aligned}
$$

$$
\begin{aligned}
hy'_{m+1} = &-\frac{(r-1)}{r}y_m + \frac{1}{(r(r-1))}y_{m+r} + \frac{(r-2)}{(r-1)}y_{m+1} + \frac{h^3}{1680s(g-s)(r-s)(s^2-3s+2)} \\
&\times ((r-1)(40g + 27r - 66gr - 18gr^2 + 30gr^3 - 6gr^4 + 14r^2 + r^3 - 12r^4 + 3r^5 - 22))f_{m+s} \\
&+ \frac{h^3}{1680r(g-r)(r-s)(r-2)}(13r - 40g - 40s - 18gr + 84gs - 18rs + 4gr^2 + 26gr^3 \\
&- 8gr^4 + 4r^2s + 26r^3s - 8r^4s + 4r^2 - 5r^3 - 14r^4 + 5r^5 - 56gr^2s + 14gr^3s + 14grs + 22)f_{m+r} \\
&- \frac{h^3}{1680(g-1)(s-1)}(72g + 33r + 72s - 60gr - 112gs - 60rs - 38gr^2 - 16gr^3 + 6gr^4 \\
&- 38r^2s - 16r^3s + 6r^4s + 24r^2 + 15r^3 + 6r^4 - 3r^5 + 56gr^2s - 14gr^3s + 126grs - 50)f_{m+1} \\
&- \frac{h^3}{3360grs}((r-1)(40g + 27r + 40s - 66gr - 84gs - 66rs - 18gr^2 + 30gr^3 - 6gr^4 \\
&- 18r^2s + 30r^3s - 6r^4s + 14r^2 + r^3 - 12r^4 + 3r^5 - 98gr^2s + 14gr^3s + 210grs - 22))f_m \\
&+ \frac{h^3}{(1680g(g-r)(g-s)(g^2-3g+2))}((r-1)(27r + 40s - 66rs - 18r^2s + 30r^3s - 6r^4s \\
&+ 14r^2 + r^3 - 12r^4 + 3r^5 - 22))f_{m+g} \\
&- \frac{h^3}{(3360(g-2)(r-2)(s-2))}((r-1)(18gr - 9r - 16s - 16g + 28gs + 18rs + 10gr^2 \\
&+ 2gr^3 - 6gr^4 + 10r^2s + 2r^3s - 6r^4s - 6r^2 - 3r^3 + 3r^5 - 14gr^2s + 14gr^3s - 42grs + 10))f_{m+2},
\end{aligned}
$$

$$
\begin{aligned}
h^2y''_{m+1} = &\frac{2}{r}y_m + \frac{2}{r(r-1)}y_{m+r} - \frac{2}{(r-1)}y_{m+1} + \frac{h^3}{(840s(g-s)(r-s)(s^2-3s+2))}(162gr \\
&- 85r - 76g - 48gr^2 - 48gr^3 + 36gr^4 - 6gr^5 + 13r^2 + 13r^3 + 13r^4 - 15r^5 + 3r^6 + 47)f_{m+s} \\
&+ \frac{h^3}{840r(g-r)(r-s)(r^2-3r+2)}(76g + 9r + 76s - 22gr - 140gs - 22rs - 22gr^2 \\
&- 22gr^3 + 34gr^4 - 8gr^5 - 22r^2s - 22r^3s + 34r^4s - 8r^5s + 9r^2 + 9r^3 + 9r^4 - 19r^5 \\
&+ 5r^6 + 70gr^2s - 70gr^3s + 14gr^4s + 70grs - 47)f_{m+r} \\
&+ \frac{h^3}{840(g-1)(r-1)(s-1)}(232g + 243r + 232s - 328gr - 308gs - 328rs + 22gr^2 \\
&+ 22gr^3 + 22gr^4 - 6gr^5 + 22r^2s + 22r^3s + 22r^4s - 6r^5s - 9r^2 - 9r^3 - 9r^4 - 9r^5 + 3r^6
\end{aligned}
$$

$$
\begin{aligned}
&-70gr^2s-70gr^3s+14gr^4s+490grs-185)f_{m+1}\\
&+\frac{h^3}{1680grs}(76g+85r+76s-162gr-140gs-162rs+48gr^2+48gr^3-36gr^4+6gr^5\\
&+48r^2s+48r^3s-36r^4s+6r^5s-13r^2-13r^3-13r^4+15r^5-3r^6-308gr^2s+112gr^3s\\
&-14gr^4s+392grs-47)f_m\\
&+\frac{h^3}{(840g(g-r)(g-s)(g^2-3g+2))}(162rs-76s-85r-48r^2s-48r^3s+36r^4s-6r^5s\\
&+13r^2+13r^3+13r^4-15r^5+3r^6+47)f_{m+g}\\
&-\frac{h^3}{(1680(g-2)(r-2)(s-2))}(36g+39r+36s-62gr-56gs-62rs+8gr^2+8gr^3\\
&+8gr^4-6gr^5+8r^2s+8r^3s+8r^4s-6r^5s-3r^2-3r^3-3r^4-3r^5+3r^6-28gr^2s\\
&-28gr^3s+14gr^4s+112grs-25)f_{m+2},
\end{aligned}
$$

$$
\begin{aligned}
y_{m+s}=&-\frac{(r-s)(s-1)}{r}y_m+\frac{s(s-1)}{r(r-1)}y_{m+r}+\frac{s(r-s)}{(r-1)}y_{m+1}+\frac{h^3}{(1680(g-s)(s-2))}\\
&\times(22g+13r-9s-15r^2s^2+3r^2s^3+3r^3s^2-48gr+22gs+13rs-48gr^2+36gr^3\\
&-6gr^4+22gs^2-34gs^3+8gs^4+13rs^2+13r^2s-15rs^3-15r^3s+3rs^4+3r^4s+13r^2\\
&+13r^3-15r^4+3r^5-9s^2-9s^3+19s^4-5s^5+36grs^2+36gr^2s-6grs^3-6gr^3s\\
&-6gr^2s^2-48grs-9)f_{m+s}\\
&-\frac{h^3}{1680r(g-r)(r^2-3r+2)}(s(s-1)(22g-9r+13s-15r^2s^2+3r^2s^3+3r^3s^2+22gr\\
&-48gs+13rs+22gr^2-34gr^3+8gr^4-48gs^2+36gs^3-6gs^4+13rs^2+13r^2s-15rs^3\\
&-15r^3s+3rs^4+3r^4s-9r^2-9r^3+19r^4-5r^5+13s^2+13s^3-15s^4+3s^5+36grs^2\\
&+36gr^2s-6grs^3-6gr^3s-6gr^2s^2-48grs-9))f_{m+r}\\
&-\frac{h^3}{(1680(g-1)(r-1))}(s(r-s)(20g+9r+9s-13r^2s^2+3r^2s^3+3r^3s^2-22gr-22gs\\
&-13rs-22gr^2-22gr^3+6gr^4-22gs^2-22gs^3+6gs^4-13rs^2-13r^2s-13rs^3-13r^3s\\
&+3rs^4+3r^4s+9r^2+9r^3+9r^4-3r^5+9s^2+9s^3+9s^4-3s^5+48grs^2+48gr^2s\\
&-8grs^3-8gr^3s-8gr^2s^2+48grs-11))f_{m+1}\\
&-\frac{h^3}{3360gr(s^2-3s+2)}(r-s)(s-1)^2(s-2)(22g+13r+13s+21r^2s^2-3r^2s^3-3r^3s^2\\
&-48gr-48gs-35rs-48gr^2+36gr^3-6gr^4-48gs^2+36gs^3-6gs^4-35rs^2-35r^2s\\
&+21rs^3+21r^3s-3rs^4-3r^4s+13r^2+13r^3-15r^4+3r^5+13s^2+13s^3-15s^4+3s^5\\
&-76grs^2-76gr^2s+8grs^3+8gr^3s+8gr^2s^2+260grs-9)f_m\\
&-\frac{h^3}{(1680g(g-r)(g-s)(g^2-3g+2))}(s(s-1)(-3r^6+6r^5s+15r^5-36r^4s-13r^4+48r^3s\\
&-13r^3+48r^2s-13r^2-6rs^5+36rs^4-48rs^3-48rs^2+9r+3s^6-15s^5+13s^4+13s^3\\
&+13s^2-9s))f_{m+g}\\
&+\frac{h^3}{(3360(g-2)(r-2)(s-2))}(s(r-s)(s-1)(6g+3r+3s-5r^2s^2+3r^2s^3+3r^3s^2-8gr\\
&-8gs-5rs-8gr^2-8gr^3+6gr^4-8gs^2-8gs^3+6gs^4-5rs^2-5r^2s-5rs^3-5r^3s
\end{aligned}
$$

$$+3rs^4+3r^4s+3r^2+3r^3+3r^4-3r^5+3s^2+3s^3+3s^4-3s^5+20grs^2+20gr^2s-8grs^3$$
$$-8gr^3s-8gr^2s^2+20grs-3))f_{m+2},$$

$$\begin{aligned}
hy'_{m+s} &= -\frac{(r-2s+1)}{r}y_m+\frac{(2s-1)}{r(r-1)}y_{m+r}+\frac{(r-2s)}{(r-1)}y_{m+1}+\frac{h^3}{1680s(g-s)(r-s)(s^2-3s+2)}\\
&\times(9r-18s-22gr+44gs+26rs+48gr^2+48gr^3-36gr^4+6gr^5-280gs^4+252gs^5\\
&-56gs^6+26r^2s+26r^3s-13r^3-13r^4+15r^5-3r^6+168s^5-168s^6+40s^7-96gr^2s\\
&+560grs^3-96gr^3s-420grs^4+72gr^4s+84grs^5-12gr^5s-96grs)f_{m+s}\\
&-\frac{h^3}{(1680r(g-r)(r-s)(r^2-3r+2))}(9r-18s+44r^2s^2+44r^3s^2-68r^4s^2+16r^5s^2\\
&-22gr+44gs-40rs-22gr^2-22gr^3+34gr^4-8gr^5-140gs^2+280gs^4-168gs^5\\
&+28gs^6+44rs^2-40r^2s-40r^3s+16r^4s+30r^5s-10r^6s+9r^2+9r^3+9r^4-19r^5\\
&+5r^6+44s^2-112s^5+84s^6-16s^7-140grs^2+114gr^2s-26gr^3s-54gr^4s+16gr^5s\\
&-140gr^2s^2+140gr^3s^2-28gr^4s^2+114grs)f_{m+r}\\
&+\frac{h^3}{1680(g-1)(r-1)(s-1)}(22s-11r+44r^2s^2+44r^3s^2+44r^4s^2-12r^5s^2+20gr\\
&-40gs+2rs-22gr^2-22gr^3-22gr^4+6gr^5+84gs^2-112gs^5+28gs^6+44rs^2-40r^2s\\
&-40r^3s-40r^4s-112rs^5-12r^5s+28rs^6+6r^6s+9r^2+9r^3+9r^4+9r^5-3r^6-40s^2\\
&+56s^6-16s^7-140grs^2+114gr^2s+114gr^3s+280grs^4+30gr^4s-56grs^5-12gr^5s\\
&-140gr^2s^2-140gr^3s^2+28gr^4s^2+2grs)f_{m+1}\\
&+\frac{h^3}{(3360grs)}(18s-9r+96r^2s^2+96r^3s^2-72r^4s^2+12r^5s^2+22gr-44gs-4rs-48gr^2\\
&-48gr^3+36gr^4-6gr^5+140gs^2-280gs^4+168gs^5-28gs^6+96rs^2-74r^2s-74r^3s\\
&-280rs^4+10r^4s+168rs^5+24r^5s-28rs^6-6r^6s+13r^2+13r^3+13r^4-15r^5+3r^6\\
&-44s^2+112s^5-84s^6+16s^7-616grs^2+404gr^2s+1120grs^3-16gr^3s-420grs^4\\
&-58gr^4s+56grs^5+12gr^5s-616gr^2s^2+224gr^3s^2-28gr^4s^2+26grs)f_m\\
&-\frac{h^3}{(1680g(g-r)(g-s)(g^2-3g+2))}(-6r^6s+3r^6+12r^5s^2+24r^5s-15r^5-72r^4s^2\\
&+10r^4s+13r^4+96r^3s^2-74r^3s+13r^3+96r^2s^2-74r^2s+13r^2-28rs^6+168rs^5\\
&-280rs^4+96rs^2-4rs-9r+16s^7-84s^6+112s^5-44s^2+18s)f_{m+g}\\
&-\frac{h^3}{(3360(g-2)(r-2)(s-2))}(6s-3r+16r^2s^2+16r^3s^2+16r^4s^2-12r^5s^2+6gr\\
&-12gs-8gr^2-8gr^3-8gr^4+6gr^5+28gs^2-56gs^5+28gs^6+16rs^2-14r^2s-14r^3s\\
&-14r^4s-56rs^5+28rs^6+6r^6s+3r^2+3r^3+3r^4+3r^5-3r^6-12s^2+28s^6-16s^7\\
&-56grs^2+44gr^2s+44gr^3s+140grs^4+2gr^4s-56grs^5-12gr^5s-56gr^2s^2-56gr^3s^2\\
&+28gr^4s^2+2grs)f_{m+2},
\end{aligned}$$

$$\begin{aligned}
h^2y''_{m+s} &= \frac{2}{r}y_m+\frac{2}{r(r-1))}y_{m+r}-\frac{2}{(r-1))}y_{m+1}+\frac{h^3}{(840s(g-s)(r-s)(s^2-3s+2))}(22g\\
&+13r-48gr-48gr^2-48gr^3+36gr^4-6gr^5-560gs^3+630gs^4-168gs^5-560rs^3\\
&+630rs^4-168rs^5+13r^2+13r^3+13r^4-15r^5+3r^6+420s^4-504s^5+140s^6
\end{aligned}$$

$$
\begin{aligned}
&+840grs^2-840grs^3+210grs^4-9)f_{m+s}\\
&-\frac{h^3}{(840r(g-r)(r-s)(r^2-3r+2))}(22g-9r+22s+22gr-70gs+22rs+22gr^2\\
&+22gr^3-34gr^4+8gr^5+280gs^3-210gs^4+42gs^5+22r^2s+22r^3s-34r^4s+8r^5s\\
&-9r^2-9r^3-9r^4+19r^5-5r^6-140s^4+126s^5-28s^6-70gr^2s+70gr^3s-14gr^4s\\
&-70grs-9)f_{m+r}\\
&+\frac{h^3}{1680(g-2)(r-2)(s-2)}(6g+3r+6s-8gr-14gs-8rs-8gr^2-8gr^3-8gr^4\\
&+6gr^5+70gs^4-42gs^5-8r^2s-8r^3s+70rs^4-8r^4s-42rs^5+6r^5s+3r^2+3r^3\\
&+3r^4+3r^5-3r^6-42s^5+28s^6+28gr^2s-140grs^3+28gr^3s+70grs^4-14gr^4s\\
&+28grs-3)f_{m+2}\\
&-\frac{h^3}{(840(g-1)(r-1)(s-1))}(20g+9r+20s-22gr-42gs-22rs-22gr^2-22gr^3\\
&-22gr^4+6gr^5+140gs^4-42gs^5-22r^2s-22r^3s+9r^2+9r^3+9r^4+9r^5-3r^6-84s^5\\
&+28s^6+70gr^2s-280grs^3+70gr^3s+70grs^4-14gr^4s+70grs-11)f_{m+1}\\
&-\frac{h^3}{1680grs}(22g+13r+22s-48gr-70gs-48rs-48gr^2-48gr^3+36gr^4-6gr^5\\
&+280gs^3-210gs^4+42gs^5-48r^2s+280rs^3-48r^3s-210rs^4+36r^4s+42rs^5-6r^5s\\
&+13r^2+13r^3+13r^4-15r^5+3r^6-140s^4+126s^5-28s^6-840grs^2+308gr^2s\\
&+420grs^3-112gr^3s-70grs^4+14gr^4s+308grs-9)f_m\\
&+\frac{h^3}{840g(g-r)(g-s)(g^2-3g+2)}(3r^6-6r^5s-15r^5+36r^4s+13r^4-48r^3s+13r^3\\
&-48r^2s+13r^2+42rs^5-210rs^4+280rs^3-48rs+13r-28s^6+126s^5-140s^4+22s-9)f_{m+g}\\
&+\frac{h^3}{1680(g-2)(r-2)(s-2)}(6g+3r+6s-8gr-14gs-8rs-8gr^2-8gr^3-8gr^4\\
&+6gr^5+70gs^4-42gs^5-8r^2s-8r^3s+70rs^4-8r^4s-42rs^5+6r^5s+3r^2+3r^3+3r^4\\
&+3r^5-3r^6-42s^5+28s^6+28gr^2s-140grs^3+28gr^3s+70grs^4-14gr^4s+28grs-3)f_{m+2},
\end{aligned}
$$

$$
\begin{aligned}
y_{m+g}={}&\frac{(g-r)(g-1)}{r}y_m+\frac{g(g-1)}{r(r-1)}y_{m+r}-\frac{g(g-r)}{(r-1)}y_{m+1}\\
&+\frac{h^3}{(1680s(g-s)(r-s)(s^2-3s+2))}(g(g-1)(-3g^6+6g^5r+15g^5-36g^4r-13g^4\\
&+48g^3r-13g^3+48g^2r-13g^2-6gr^5+36gr^4-48gr^3-48gr^2+9g+3r^6-15r^5+13r^4\\
&+13r^3+13r^2-9r))f_{m+s}\\
&+\frac{h^3}{(1680r(r-s)(r^2-3r+2))}(g(g-1)(13g-9r+22s+13gr-48gs+22rs+13gr^2\\
&+13g^2r-15gr^3-15g^3r+3gr^4+3g^4r-48g^2s+36g^3s-6g^4s+22r^2s-34r^3s+8r^4s\\
&+13g^2+13g^3-15g^4+3g^5-9r^2-9r^3+19r^4-5r^5-15g^2r^2+3g^2r^3+3g^3r^2+36gr^2s\\
&+36g^2rs-6gr^3s-6g^3rs-6g^2r^2s-48grs-9))f_{m+r}\\
&+\frac{h^3}{(1680(r-1)(s-1))}(g(g-r)(9g+9r+20s-13gr-22gs-22rs-13gr^2-13g^2r\\
&-13gr^3-13g^3r+3gr^4+3g^4r-22g^2s-22g^3s+6g^4s-22r^2s-22r^3s+6r^4s+9g^2+9g^3
\end{aligned}
$$

$$
\begin{aligned}
&+9g^4-3g^5+9r^2+9r^3+9r^4-3r^5-13g^2r^2+3g^2r^3+3g^3r^2+48gr^2s+48g^2rs-8gr^3s\\
&-8g^3rs-8g^2r^2s+48grs-11))f_{m+1}\\
&+\frac{h^3}{(3360rs(g^2-3g+2))}((g-r)(g-1)^2(g-2)(13g+13r+22s-35gr-48gs-48rs\\
&-35gr^2-35g^2r+21gr^3+21g^3r-3gr^4-3g^4r-48g^2s+36g^3s-6g^4s-48r^2s+36r^3s\\
&-6r^4s+13g^2+13g^3-15g^4+3g^5+13r^2+13r^3-15r^4+3r^5+21g^2r^2-3g^2r^3-3g^3r^2\\
&-76gr^2s-76g^2rs+8gr^3s+8g^3rs+8g^2r^2s+260grs-9))f_m\\
&-\frac{h^3}{(1680(g-s)(g-2))}((13r-9g+22s+13gr+22gs-48rs+13gr^2+13g^2r-15gr^3\\
&-15g^3r+3gr^4+3g^4r+22g^2s-34g^3s+8g^4s-48r^2s+36r^3s-6r^4s-9g^2-9g^3+19g^4\\
&-5g^5+13r^2+13r^3-15r^4+3r^5-15g^2r^2+3g^2r^3+3g^3r^2+36gr^2s+36g^2rs-6gr^3s\\
&-6g^3rs-6g^2r^2s-48grs-9))f_{m+g}\\
&-\frac{h^3}{(3360(g-2)(r-2)(s-2))}(g(g-r)(g-1)(3g+3r+6s-5gr-8gs-8rs-5gr^2\\
&-5g^2r-5gr^3-5g^3r+3gr^4+3g^4r-8g^2s-8g^3s+6g^4s-8r^2s-8r^3s+6r^4s+3g^2+3g^3\\
&+3g^4-3g^5+3r^2+3r^3+3r^4-3r^5-5g^2r^2+3g^2r^3+3g^3r^2+20gr^2s+20g^2rs-8gr^3s\\
&-8g^3rs-8g^2r^2s+20grs-3))f_{m+2},
\end{aligned}
$$

$$
\begin{aligned}
hy'_{m+g}=&-\frac{(r-2g+1)}{r}y_m+\frac{(2g-1)}{(r(r-1))}y_{m+r}-\frac{(2g-r)}{(r-1)}y_{m+1}\\
&-\frac{h^3}{(1680s(g-s)(r-s)(s^2-3s+2))}(16g^7-28g^6r-84g^6+168g^5r+112g^5-280g^4r\\
&+12g^2r^5-72g^2r^4+96g^2r^3+96g^2r^2+96g^2r-44g^2-6gr^6+24gr^5+10gr^4-74gr^3\\
&-74gr^2-4gr+18g+3r^6-15r^5+13r^4+13r^3+13r^2-9r)f_{m+s}\\
&-\frac{h^3}{(1680r(g-r)(r-s)(r^2-3r+2))}(9r-18g-40gr+44gs-22rs-40gr^2+44g^2r\\
&-40gr^3+16gr^4+30gr^5-10gr^6-140g^2s+280g^4s-168g^5s+28g^6s-22r^2s-22r^3s\\
&+34r^4s-8r^5s+44g^2-112g^5+84g^6-16g^7+9r^2+9r^3+9r^4-19r^5+5r^6+44g^2r^2\\
&+44g^2r^3-68g^2r^4+16g^2r^5+114gr^2s-140g^2rs-26gr^3s-54gr^4s+16gr^5s-140g^2r^2s\\
&+140g^2r^3s-28g^2r^4s+114grs)f_{m+r}\\
&+\frac{h^3}{(1680(g-1)(r-1)(s-1))}(22g-11r+2gr-40gs+20rs-40gr^2+44g^2r-40gr^3\\
&-40gr^4-12gr^5-112g^5r+6gr^6+28g^6r+84g^2s-112g^5s+28g^6s-22r^2s-22r^3s\\
&-22r^4s+6r^5s-40g^2+56g^6-16g^7+9r^2+9r^3+9r^4+9r^5-3r^6+44g^2r^2+44g^2r^3\\
&+44g^2r^4-12g^2r^5+114gr^2s-140g^2rs+114gr^3s+30gr^4s+280g^4rs-12gr^5s-56g^5rs\\
&-140g^2r^2s-140g^2r^3s+28g^2r^4s+2grs)f_{m+1}\\
&+\frac{h^3}{(3360grs)}(18g-9r-4gr-44gs+22rs-74gr^2+96g^2r-74gr^3+10gr^4-280g^4r\\
&+24gr^5+168g^5r-6gr^6-28g^6r+140g^2s-280g^4s+168g^5s-28g^6s-48r^2s-48r^3s\\
&+36r^4s-6r^5s-44g^2+112g^5-84g^6+16g^7+13r^2+13r^3+13r^4-15r^5+3r^6+96g^2r^2\\
&+96g^2r^3-72g^2r^4+12g^2r^5+404gr^2s-616g^2rs-16gr^3s+1120g^3rs-58gr^4s-420g^4rs
\end{aligned}
$$

$$
\begin{aligned}
&+12gr^5s+56g^5rs-616g^2r^2s+224g^2r^3s-28g^2r^4s+26grs)f_m\\
&-\frac{h^3}{(1680g(g-r)(g-s)(g^2-3g+2))}(18g-9r-26gr-44gs+22rs-26gr^2-26gr^3\\
&-26gr^4+280g^4r+30gr^5-252g^5r-6gr^6+56g^6r+280g^4s-252g^5s+56g^6s-48r^2s\\
&-48r^3s+36r^4s-6r^5s-168g^5+168g^6-40g^7+13r^2+13r^3+13r^4-15r^5+3r^6\\
&+96gr^2s+96gr^3s-560g^3rs-72gr^4s+420g^4rs+12gr^5s-84g^5rs+96grs)f_{m+g}\\
&-\frac{h^3}{(3360(g-2)(r-2)(s-2))}(6g-3r-12gs+6rs-14gr^2+16g^2r-14gr^3-14gr^4\\
&-56g^5r+6gr^6+28g^6r+28g^2s-56g^5s+28g^6s-8r^2s-8r^3s-8r^4s+6r^5s-12g^2\\
&+28g^6-16g^7+3r^2+3r^3+3r^4+3r^5-3r^6+16g^2r^2+16g^2r^3+16g^2r^4-12g^2r^5+44gr^2s\\
&-56g^2rs+44gr^3s+2gr^4s+140g^4rs-12gr^5s-56g^5rs-56g^2r^2s-56g^2r^3s+28g^2r^4s\\
&+2grs)f_{m+2},
\end{aligned}
$$

$$
\begin{aligned}
h^2y''_{m+g}=&\frac{2}{r}y_m+\frac{2}{(r(r-1))}y_{m+r}-\frac{2}{(r-1)}y_{m+1}+\frac{h^3}{(840s(g-s)(r-s)(s^2-3s+2))}\\
&\times(-28g^6+42g^5r+126g^5-210g^4r-140g^4+280g^3r-6gr^5+36gr^4-48gr^3\\
&-48gr^2-48gr+22g+3r^6-15r^5+13r^4+13r^3+13r^2+13r-9)f_{m+s}\\
&-\frac{h^3}{(840r(g-r)(r-s)(r^2-3r+2))}(22g-9r+22s+22gr-70gs+22rs+22gr^2\\
&+22gr^3-34gr^4+8gr^5+280g^3s-210g^4s+42g^5s+22r^2s+22r^3s-34r^4s+8r^5s\\
&-140g^4+126g^5-28g^6-9r^2-9r^3-9r^4+19r^5-5r^6-70gr^2s+70gr^3s-14gr^4s\\
&-70grs-9)f_{m+r}\\
&-\frac{h^3}{(840(g-1)(r-1)(s-1))}(20g+9r+20s-22gr-42gs-22rs-22gr^2-22gr^3\\
&-22gr^4+140g^4r+6gr^5-42g^5r+140g^4s-42g^5s-22r^2s-22r^3s-22r^4s+6r^5s\\
&-84g^5+28g^6+9r^2+9r^3+9r^4+9r^5-3r^6+70gr^2s+70gr^3s-280g^3rs-14gr^4s\\
&+70g^4rs+70grs-11)f_{m+1}\\
&-\frac{h^3}{(1680grs)}(22g+13r+22s-48gr-70gs-48rs-48gr^2-48gr^3+280g^3r\\
&+36gr^4-210g^4r-6gr^5+42g^5r+280g^3s-210g^4s+42g^5s-48r^2s-48r^3s+36r^4s\\
&-6r^5s-140g^4+126g^5-28g^6+13r^2+13r^3+13r^4-15r^5+3r^6+308gr^2s\\
&-840g^2rs-112gr^3s+420g^3rs+14gr^4s-70g^4rs+308grs-9)f_m\\
&+\frac{h^3}{(840g(g-r)(g-s)(g^2-3g+2))}(13r+22s-48rs-560g^3r+630g^4r-168g^5r\\
&-560g^3s+630g^4s-168g^5s-48r^2s-48r^3s+36r^4s-6r^5s+420g^4-504g^5+140g^6\\
&+13r^2+13r^3+13r^4-15r^5+3r^6+840g^2rs-840g^3rs+210g^4rs-9)f_{m+g}\\
&+\frac{h^3}{(1680(g-2)(r-2)(s-2))}(6g+3r+6s-8gr-14gs-8rs-8gr^2-8gr^3-8gr^4\\
&+70g^4r+6gr^5-42g^5r+70g^4s-42g^5s-8r^2s-8r^3s-8r^4s+6r^5s-42g^5+28g^6\\
&+3r^2+3r^3+3r^4+3r^5-3r^6+28gr^2s+28gr^3s-140g^3rs-14gr^4s+70g^4rs\\
&+28grs-3)f_{m+2}.
\end{aligned}
\tag{2.6}
$$

3. Characteristics of the method

3.1. *Order of the method*

The hybrid block method in Eq. (2.3)-(2.6) may be rewritten in the form

$$AY_m = hBY'_m + h^2CY''_m + h^3VF_m, \tag{3.1}$$

where A, B, C, V are matrices of coefficients of dimensions 15×6, and

$$\begin{aligned} Y_m &= (y_m, y_{m+r}, y_{m+1}, y_{m+s}, y_{m+g}, y_{m+2})^T, \\ Y'_m &= (y'_m, y'_{m+r}, y'_{m+1}, y'_{m+s}, y'_{m+g}, y'_{m+2})^T, \\ Y''_m &= (y''_m, y''_{m+r}, y''_{m+1}, y''_{m+s}, y''_{m+g}, y''_{m+2})^T, \\ F_m &= (f_m, f_{m+r}, f_{m+1}, f_{m+s}, f_{m+g}, f_{m+2})^T. \end{aligned}$$

If $z(x)$ is a sufficiently differentiable function, we consider the linear difference operator $\mathcal{L}$ associated with the two-step implicit block hybrid method in Eq. (2.3)-(2.6), that gives

$$\mathcal{L}[z(x_m); h] = \sum_j \kappa_j z(x_m + jh) - h\tau_j z'(x_m + jh) - h^2\gamma_j z''(x_m + jh) - h^3\xi_j z'''(x_m + jh),$$

where $j = 0, r, 1, s, g, 2$, and $\kappa_j, \tau_j, \gamma_j$, and ξ_j are respectively the vector columns of the matrices A, B, C, and V.

Expanding $z(x_m + jh), z'(x_m + jh), z''(x_m + jh)$ and $z'''(x_m + jh)$ in Taylor series about x_m we get

$$\mathcal{L}[z(x_m); h] = C_0 y(x_m) + C_1 hy'(x_m) + C_2 h^2 y''(x_m) + \cdots + C_q h^q y^{(q)}(x_m) + \cdots$$

with $C_0 = C_1 = C_2 = \cdots = C_{p+2} = 0$ and c_{p+3} is called the error constant. Then we have that $C_0 = C_1 = C_2 = \cdots = C_8 = 0$,

$$\begin{aligned} C_9 = (&2.893851875718754 \times 10^{-9}, 5.785124635488442 \times 10^{-7}, 3.856749756992272 \times 10^{-7}, \\ &9.969132305965792 \times 10^{-6}, -7.045458790935663 \times 10^{-5}, 2.584688610051117 \times 10^{-7}, \\ &-3.333336924590299 \times 10^{-6}, 3.280861089595141 \times 10^{-7}, 3.463123072405624 \times 10^{-6}, \\ &-4.805579774815597 \times 10^{-8}, -1.541337830791623 \times 10^{-6}, 1.237632710240815 \times 10^{-5}, \\ &2.680234138648716 \times 10^{-7}, 7.071939846161973 \times 10^{-7}, -1.544224526826144 \times 10^{-6})^T \end{aligned}$$

indicating that the proposed method has at least order $p = 6$.

3.2. *Zero Stability*

As $h \to 0$ in Eq. (3.1), the method can be written as a vector form $A_0Y_m - A_1Y_{m-1} = 0$, where

$$Y_m = (y_{m+2}, y_{m+1}, y_{m+r}, y_{m+s}, y_{m+g})^T, Y_{m-1} = (y_m, y_{m-1}, y_{m+r-1}, y_{m+s-1}, y_{m+g-1})^T,$$

$$A_0 = \begin{bmatrix} 1 & 0.326948973328471 & -1.251387885055001 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -0.587710559540554 & -0.506293359904019 & 1 & 0 \\ 0 & -0.702090651924726 & 0.152305983263900 & 0 & 1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.075561088273469 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -0.094003919444572 & 0 & 0 & 0 & 0 & 0 \\ 0.450215331339174 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first characteristic of polynomial $\rho(z) = \det[A_0z - A_1]$, $(z-1)z^4 = 0$, is that the roots of polynomials are $z_1 = z_2 = z_3 = z_4 = 0$, $z_5 = 1$. Hence the block method is zero-stable.

3.3. Consistency

The block method has order $p = 6 > 1$, in case $p \geqslant 1$ this feature is a sufficient condition for the associated block method to be consistent [12].

3.4. Convergence

We can establish the convergence of the two-step with three points hybrid block method if and only if it is consistent and zero stable [7].

3.5. Region of absolute stability

The stability advantages are generally analyzed by considering the linear test equation introduced by [13]

$$y'''(x) = -\lambda^3 y(x) \ \text{ with } \lambda > 0.$$

Zero-stability depends just on the method but linear stability (in general for finite h) depends on the problem also. We will locate the region in which the numerical method reproduces the behavior of the true solutions.

Let us depict the procedure to get such a region [18]. Our method has fifteen equations in which there are twelve different terms of derivatives and three intermediate values, eradicated these terms from the system of equations, and get a recurrence equation

$$\begin{aligned}
&(9.36481 \times 10^{269} - 1.9316 \times 10^{268} z^3 + 1.12471 \times 10^{266} z^6 - 1.04835 \times 10^{264} z^9 \\
&\quad + 2.38044 \times 10^{262} z^{12}) y_{m+2} + (-3.74592 \times 10^{270} + 5.14288 \times 10^{269} z^3 - 1.38863 \times 10^{268} z^6 \\
&\quad + 1.42593 \times 10^{266} z^9 - 5.78221 \times 10^{263} z^{12}) y_{m+1} + (2.80944 \times 10^{270} - 1.87296 \times 10^{270} z \\
&\quad + 1.29348 \times 10^{269} z^3 + 7.41597 \times 10^{267} z^4 - 4.30619 \times 10^{267} z^6 + 8.27706 \times 10^{266} z^7 \\
&\quad + 3.98136 \times 10^{264} z^9 - 2.92902 \times 10^{264} z^{10} + 3.35152 \times 10^{262} z^{12}) y_m = 0,
\end{aligned}$$

where $z = \lambda h$. We study the extent boundedness of their solutions through its characteristic equation to define the stability region. The roots of the characteristic equation

$$\begin{aligned}
&(9.36481 \times 10^{269} - 1.9316 \times 10^{268} z^3 + 1.12471 \times 10^{266} z^6 - 1.04835 \times 10^{264} z^9 \\
&\quad + 2.38044 \times 10^{262} z^{12}) x^2 + (-3.74592 \times 10^{270} + 5.14288 \times 10^{269} z^3 - 1.38863 \times 10^{268} z^6 \\
&\quad + 1.42593 \times 10^{266} z^9 - 5.78221 \times 10^{263} z^{12}) x + 2.80944 \times 10^{270} - 1.87296 \times 10^{270} z \\
&\quad + 1.29348 \times 10^{269} z^3 + 7.41597 \times 10^{267} z^4 - 4.30619 \times 10^{267} z^6 + 8.27706 \times 10^{266} z^7 \\
&\quad + 3.98136 \times 10^{264} z^9 - 2.92902 \times 10^{264} z^{10} + 3.35152 \times 10^{262} z^{12} = 0
\end{aligned}$$

are

$$\begin{aligned}
x_1 = &(5.3575 \times 10^{23} - 7.35547 \times 10^{22} z^3 + 1.98605 \times 10^{21} z^6 - 2.0394 \times 10^{19} z^9 \\
&+ 8.26985 \times 10^{16} z^{12} - 1.41421 \sqrt{}(3.58786 \times 10^{46} + 7.17571 \times 10^{46} z - 4.21424 \times 10^{46} z^3 \\
&- 1.76419 \times 10^{45} z^4 + 4.02344 \times 10^{45} z^6 - 1.72328 \times 10^{43} z^7 - 1.61039 \times 10^{44} z^9 \\
&+ 6.51845 \times 10^{41} z^{10} + 3.54106 \times 10^{42} z^{12} - 3.98104 \times 10^{39} z^{13} - 4.68888 \times 10^{40} z^{15} \\
&+ 4.17544 \times 10^{37} z^{16} + 3.76411 \times 10^{38} z^{18} - 9.31689 \times 10^{35} z^{19} - 1.68899 \times +2.85245 \times 10^{33} z^{22} \\
&+ 3.38688 \times 10^{33} z^{24} (2.67875 \times 10^{23} - 5.52524 \times 10^{21} z^3 + 3.21717 \times 10^{19} z^6 - 2.99875 \times 10^{17} z^9 \\
&+ 6.80913 \times 10^{15} z^{12}))), \\
x_2 = &(5.3575 \times 10^{23} - 7.35547 \times 10^{22} z^3 + 1.98605 \times 10^{21} z^6 - 2.0394 \times 10^{19} z^9
\end{aligned}$$

$$+8.26985\times10^{16}z^{12}+1.41421\surd(3.58786\times10^{46}+7.17571\times10^{46}z-4.21424\times10^{46}z^{3}$$
$$-1.76419\times10^{45}z^{4}+4.02344\times10^{45}z^{6}-1.72328\times10^{43}z^{7}-1.61039\times10^{44}z^{9}+6.51845\times10^{41}z^{10}$$
$$+3.54106\times10^{42}z^{12}-3.98104\times10^{39}z^{13}-4.68888\times10^{40}z^{15}+4.17544\times10^{37}z^{16}$$
$$+3.76411\times10^{38}z^{18}-9.31689\times10^{35}z^{19}-1.68899\times10^{36}z^{21}+2.85245\times10^{33}z^{22}+3.38688\times10^{33}z^{24}$$
$$\times(2.67875\times10^{23}-5.52524\times10^{21}z^{3}+3.21717\times10^{19}z^{6}-2.99875\times10^{17}z^{9}+6.80913\times10^{15}z^{12}))).$$

The roots of the characteristic equation must be less than 1, for the method to be stable. Figure 1 has shown the stability region for the method considered in this paper.

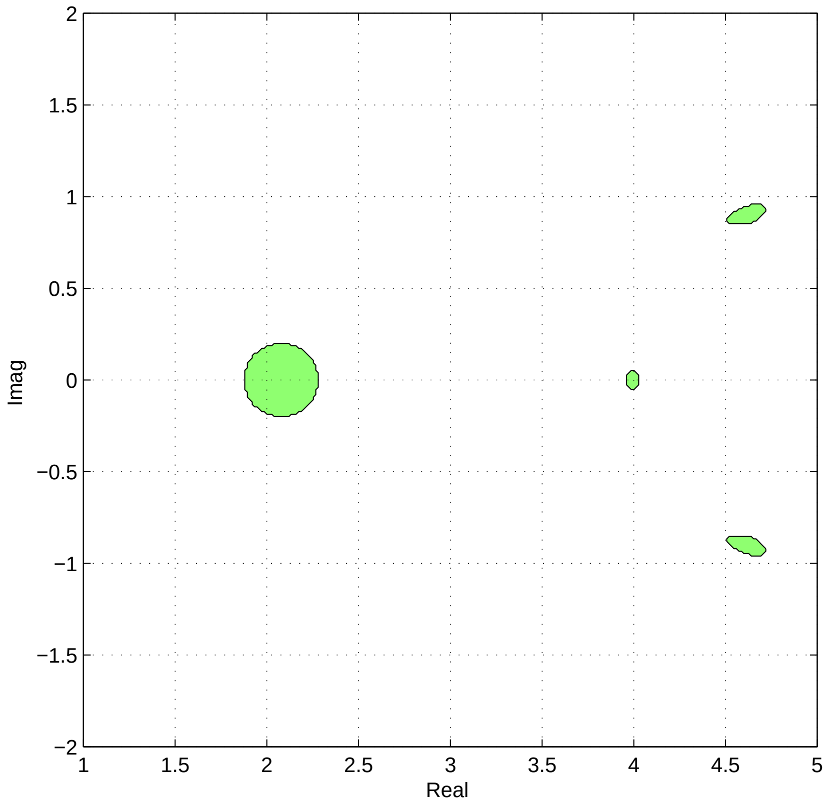

Figure 1: Region of absolute stability.

4. Numerical Examples

Example 4.1. Consider a special third order initial value problem

$$y'''=3\sin x,\quad y(0)=1, y'(0)=0, y''(0)=2,\quad x\in[0,1].$$

Exact solution is $y(x)=3\cos x+\frac{x^2}{2}-2$.

Example 4.1 stated above was solved in [3, 16]. Our new method was also applied to solve the same problem. The results are shown in Table 1.

Example 4.2. Consider a linear third order initial value problem

$$y'''+y'=0,\quad y(0)=0,\ y'(0)=1, y''(0)=2,\quad x\in[0,1].$$

Exact solution is $y(x)=2(1-\cos x)+\sin x$.

Example 4.2 above was considered by [3, 9]. We applied our new method to solve the same problem and the results are demonstrated in Table 2.

Example 4.3. Consider a linear third order initial value problem

$$y''' + 2y'' - 9y - 18y = 18x^2 18x + 22, \;\; y(0) = -2, \; y'(0) = -8, \; y''(0) = -12, \;\; x \in [0, 1].$$

Exact solution is $y(x) = 2e^{3x} + e^{2x} + x^2 - 1$.

Example 4.3 stated above was solved in [1, 2]. Our new method was also applied to solve the same problem. The results are shown in Table 3.

Example 4.4. Consider the nonlinear Blassius equation in fluid dynamics given by

$$2y''' + yy'' = 0, \;\; y(0) = 0, \; y'(0) = 0, \; y''(0) = 1, \;\; x \in [0, 1].$$

The exact solution does not exist.

Example 4.4 above was considered by [3]. We applied our new method to solve the same problem and the results are shown in Table 4.

Table 1: Comparison between the absolute errors in our method and the methods in [3, 16] .

x	Error in our method $h = 0.1$	Error in [3] $h = 0.1$	Error in our method $h = 0.01$	Error in [16] $h = 0.01$
0.1	4.1078×10^{-15}	2.5934×10^{-12}	4.4409×10^{-16}	9.9920×10^{-16}
0.2	1.6875×10^{-14}	1.1857×10^{-11}	1.2212×10^{-15}	7.6605×10^{-15}
0.3	5.0848×10^{-14}	2.6224×10^{-11}	2.4425×10^{-15}	2.2870×10^{-14}
0.4	1.1779×10^{-13}	4.7034×10^{-11}	3.7748×10^{-15}	5.9063×10^{-14}
0.5	2.4081×10^{-13}	7.2700×10^{-11}	5.5511×10^{-15}	1.1535×10^{-13}
0.6	4.3709×10^{-13}	1.0437×10^{-10}	8.4377×10^{-15}	1.9828×10^{-13}
0.7	7.3708×10^{-13}	1.4049×10^{-10}	1.1324×10^{-14}	3.1274×10^{-13}
0.8	1.1662×10^{-12}	1.8197×10^{-10}	1.4544×10^{-14}	4.6355×10^{-13}
0.9	1.7587×10^{-12}	2.2736×10^{-10}	1.8985×10^{-14}	6.5425×10^{-13}
1	2.5466×10^{-12}	2.7729×10^{-10}	2.3870×10^{-14}	8.8852×10^{-13}

Table 2: Comparison between the absolute errors in our method and the methods in [3, 9].

x	Error in our method $h = 0.1$	Error in [3] $h = 0.1$	Error in our method $h = 0.05$	Error in [9] $h = 0.05$
0.1	1.9345×10^{-14}	1.6613×10^{-12}	1.1796×10^{-16}	2.065×10^{-14}
0.2	7.1831×10^{-14}	7.5411×10^{-12}	2.2204×10^{-16}	1.950×10^{-11}
0.3	1.8218×10^{-13}	1.3843×10^{-9}	8.3267×10^{-17}	8.094×10^{-11}
0.4	3.6803×10^{-13}	4.5006×10^{-9}	8.0491×10^{-16}	1.964×10^{-10}
0.5	6.5725×10^{-13}	1.0520×10^{-8}	9.9920×10^{-16}	3.702×10^{-10}
0.6	1.0689×10^{-12}	1.9715×10^{-8}	2.1649×10^{-15}	-
0.7	1.6320×10^{-12}	3.2968×10^{-8}	3.4972×10^{-15}	-
0.8	2.3652×10^{-12}	5.0419×10^{-8}	5.9952×10^{-15}	-
0.9	3.2969×10^{-12}	7.2608×10^{-8}	8.5487×10^{-15}	-
1	4.4442×10^{-12}	9.9511×10^{-8}	1.0991×10^{-14}	-

Table 3: Comparison between the absolute errors in our method and the methods in [1, 2].

x	Error in our method $h = 0.1$	Error in [1] $h = 0.1$	Error in our method $h = 0.05$	Error in [2] $h = 0.05$
0.1	7.9564×10^{-10}	1.3408×10^{-3}	1.3953×10^{-12}	1.5405×10^{-9}
0.2	2.8239×10^{-9}	-	5.0511×10^{-12}	9.8455×10^{-9}
0.3	7.5227×10^{-9}	-	1.2736×10^{-11}	2.3652×10^{-8}
0.4	1.5802×10^{-8}	-	2.5549×10^{-11}	4.3273×10^{-8}
0.5	3.0550×10^{-8}	-	4.5870×10^{-11}	3.9018×10^{-8}
0.6	5.3869×10^{-8}	-	7.5249×10^{-11}	6.9700×10^{-8}
0.7	9.1670×10^{-8}	-	1.1703×10^{-10}	5.2032×10^{-8}
0.8	1.4869×10^{-7}	-	1.7348×10^{-10}	1.3522×10^{-7}
0.9	2.3696×10^{-7}	-	2.4939×10^{-10}	4.7403×10^{-7}
1	3.6683×10^{-7}	-	3.4815×10^{-10}	1.0693×10^{-6}

Table 4: Comparison between the errors in our method and the methods in [3]. Δ_1 denotes the numerical solution at $h = 0.1$ and Δ_2 denotes the numerical solution at $h = 0.01$.

x	Error in our method at $\Delta_2 - \Delta_1$	Error in [3] at $\Delta_2 - \Delta_1$
0.1	3.9647×10^{-15}	5.600×10^{-14}
0.2	1.4450×10^{-14}	1.3280×10^{-13}
0.3	6.5551×10^{-14}	9.7880×10^{-12}
0.4	1.8039×10^{-13}	1.1509×10^{-10}
0.5	4.4837×10^{-13}	7.9960×10^{-10}
0.6	9.2922×10^{-13}	2.4275×10^{-10}
0.7	1.7660×10^{-12}	6.3073×10^{-9}
0.8	3.0489×10^{-12}	1.3204×10^{-8}
0.9	4.9734×10^{-12}	2.5376×10^{-8}
1	7.6655×10^{-12}	4.3280×10^{-8}

5. Conclusion

We have progressed a two-step with three hybrid points method in this paper. The good convergent and stability properties of our method make it more attractive for the numerical solution of initial value problems of third order ordinary differential equations. The presented method is zero stable of sixth algebraic order, consistent and convergence. The numerical results evince its effectiveness and precision compared with other methods evidenced in the literature.

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