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On positive travelling wave solutions for a general class of KdV-Burger type equation

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Abstract

In this paper, we establish the existence of positive traveling waves solutions for the third order differential equation $u_t + \alpha u_{xx} + \beta u_{xxx} + (f(x, u(x)))_x = 0$, where $t, x \in \mathbb{R}$, f is a non-negative continuous function with some properties. The result is a consequence of the characterization of the travelling wave solutions as fixed points of some functional, defined using the Green's function associated to the linear problem, and the Krasnosel'skii fixed point theorem on cone expansion and compression of norm type.

Keywords: Travelling wave solutions, Green function, Krasnosel'skii fixed point theorem.

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1. Introduction

Mathematical models defined by nonlinear evolution equations have been used for the description of different physical, biological, chemical, electrical, atmospheric phenomena, among others. In the case of these models, a natural question that arises is related with the existence of the simplest solutions for those models that are known as traveling waves. Those solutions in a certain sense correspond to stationary points because these solutions maintain its shape while traveling to a constant speed. As is well-known, these type of solutions appear naturally as a result of an effective cancellation of the nonlinear effects and the dispersive effects in the medium. In this paper, we are concerned with the existence of positive traveling waves solutions for the third order differential equations

$$u_t + \alpha u_{xx} + \beta u_{xxx} + (f(x, u(x)))_x = 0, \tag{1.1}$$

where $t, x \in \mathbb{R}$, f is a non-negative continuous function under some assumptions. We note that the nonlinear model (1.1) is related with the Burgers equation ($\alpha = \beta = 0$ and $f(u) = u^2$), the viscous Burgers equation ($\alpha \neq 0$, $\beta = 0$, and $f(u) = u^2$), the KdV equation ($\alpha = 0$, $\beta \neq 0$ and $f(u) = u^2$), the KdV-Burgers

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equation ($\alpha \neq 0$, $\beta \neq 0$, and $f(u) = u^2$), the modified KdV-Burgers equation ($\alpha \neq 0$, $\beta \neq 0$, and $f(u) = u^3$), and the quadratic-cubic KdV-Burgers equation ($\alpha \neq 0$, $\beta \neq 0$ and $f(u) = au^2 + bu^3$). Moreover, the nonlinear model (1.1) is related with the generalized KdV equation ($\alpha = 0$, $\beta \neq 0$, and $f(u) = u^{p+1}$) and the generalized p-Gardner equation ($\alpha = 0$, $\beta \neq 0$, and $f(u) = u^{p+1} + u^{2p+1}$).

One of the main techniques to establish the existence of solutions for the Cauchy problem or the existence of travelling waves associated with a differential equation is the well-know fixed point theorem in cones of Banach spaces due to Krasnosel'skii (see for instance [2, 4, 6, 9, 11–13, 16] and their references). In particular, Zima in [16] showed the existence of positive solutions for the following boundary value problem for second order differential equations

$$\begin{cases} x''(t) + k^2x(t) + g\,(t, x(t)) = 0, \\ x(0) = 0, \quad \lim\limits_{t\to\infty} x(t) = 0, \end{cases} \tag{1.2}$$

where $t \in [0, \infty)$, and g is continuous non-negative function. In this case, the existence result of positive solutions for the Cauchy problem associated with the equation (1.2) is obtained by characterizing solutions via the Green's function in an appropriate function spaces equipped with Bielecki's norm and by using the Krasnosel'skii fixed point theorem on cone expansion and compression of norm type (see [3, 7]). On the other hand, using a Krasnosel'skii fixed point theorem together with a compactness criterion due to Zima (see [16]), Torres in [14], established the existence of solutions for the nonlinear differential equation of type (1.1)

$$u''(x) + a(x)u(x) = b(x)f(u(x)),$$

where $a, b \in L^\infty(\mathbb{R})$ are non-negative almost everywhere and f is a given continuous function, under the assumption that

$$\lim_{|x|\to\infty} |u(x)| + |u'(x)| = 0.$$

The importance of this model lies in that under appropriate hypotheses, it models the propagation of electromagnetic waves through a medium formed by layers of dielectric material.

In this paper, we establish the existence of positive traveling waves solutions for the third order differential equation (1.1), following the approach used by Zima in [16], where $t \in \mathbb{R}$, and f is a non-negative continuous function having the form

$$f(t, u) \leqslant b(t) + a(t)|u|^\sigma,$$

where $\sigma \geqslant 1$ and a, b are continuous non-negative functions with some properties. The main result is a consequence of the characterization of the traveling wave solutions as fixed points of some functional, defined using the Green's function associated to the linear problem, and the Krasnosel'skii fixed point theorem on cone expansion and compression of norm type, as done for Zima in [16] in the case of the existence of solutions for the Cauchy Problem for the equation (1.2).

It is important to point out that in the case $\alpha = 0$ and $f(u) = au^p + bu^q$ with $p, q \geqslant 1$, the dispersive model (1.1) has explicit solutions. Although, we will not establish the existence of such explicit solutions using the fixed point argument, we are able to show somehow that those solutions can be captured in the setting considered in our model, as we will show in this paper.

The paper is organized as follows. In Section 2 some preliminary results are collected, for the convenience of the reader, we include here the definitions of a cone and completely continuous operator and we state a theorem due to Krasnosel'skii concerning non-trivial fixed points of completely continuous operator in a Banach space. We also include the Green's function associated with the traveling wave equation for (1.1). Finally, in this section, we include some results related with the Banach spaces with Bielecki's type norm. Section 3 contains the main result about the existence of positive traveling waves. Section 4 contains some illustrative examples where it is possible to apply our main result. Section 5 presents some examples in order to illustrate that the explicit solution of well-known dispersive models can be captured in our model.

2. Preliminary results

For the convenience of the reader, we include some basic definitions and the explicit Green's function associated with the linear model.

Definition 2.1. A nonempty subset K of a Banach space E is called a cone, if K is convex, closed, and

1. $\alpha x \in K$ for all $x \in K$ and $\alpha \geqslant 0$;
2. $x, -x \in K$ implies $x = 0$.

Definition 2.2. An operator $F : E \to E$ is completely continuous, if F is continuous and maps bounded sets into precompact sets.

Next, we state a clever theorem due to Krasnosel'skii related with the existence of non-trivial fixed points of completely continuous operator in a Banach space (see [10]). The Krasnosel'skii fixed point theorems represents a good tool to establish the existence of non-trivial solutions of nonlinear problems for integral equations, ordinary differential equations, and partial differential equations. The Krasnosel'skii fixed point results for positive operators are used to prove not only the existence of solutions, but also to localize solutions in an annulus or other domains of this type, avoiding having trivial fixed points. This result has been extensively employed in the study of boundary value problems with separated boundary conditions (see for instance [2, 4, 6, 9] and their references), as well as for the periodic problem [11–13].

Theorem 2.3 ([10])**.** *Let E be a Banach space and $K \subset E$ be a cone in E. For $i = 1, 2$, let Ω_i be two bounded open sets in E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $F : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that one of the following conditions is satisfied:*

1. $\|Fx\| \leqslant \|x\|$ *for* $x \in K \cap \partial\Omega_1$ *and* $\|Fx\| \geqslant \|x\|$ *for* $x \in K \cap \partial\Omega_2$;
2. $\|Fx\| \geqslant \|x\|$ *for* $x \in K \cap \partial\Omega_1$ *and* $\|Fx\| \leqslant \|x\|$ *for* $x \in K \cap \partial\Omega_2$.

Then F has at least one fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2.1. *Green's function*

Here we discuss some properties of the Green's function associated with the traveling wave equation for (1.1). First note that if $u = u(x - ct)$ for $c > 0$ is a traveling wave solution for (1.1), then we have that

$$-cu_x + \alpha u_{xx} + \beta u_{xxx} + (f(x, u(x)))_x = 0,$$

where the function f satisfies that

$$\lim_{(|x|,u)\to(\infty,0)} f(x, u) = 0.$$

If we assume that

$$\lim_{|x|\to\infty} (|u(x)| + |u'(x)| + |u''(x)|) = 0,$$

then integrating we have that

$$-cu + \alpha u_x + \beta u_{xx} + f(x, u(x)) = 0. \tag{2.1}$$

If we set $\Delta = \alpha^2 + 4\beta c$, then the roots of the characteristic polynomial are given by

$$m_1 = \frac{-\alpha + \sqrt{\Delta}}{2\beta} \qquad \text{and} \qquad m_2 = -\frac{\alpha + \sqrt{\Delta}}{2\beta}. \tag{2.2}$$

We note in particular that in the case $\alpha = 0$, we have that

$$m_1 = \sqrt{\frac{c}{\beta}} = -m_2. \tag{2.3}$$

We observe that

$$0 < m_1 \leqslant -m_2 \qquad \text{and} \qquad m_1 - m_2 = \frac{\sqrt{\Delta}}{\beta}. \tag{2.4}$$

From (2.2), we see that the Green function associated with the traveling wave equation (2.1) is given by

$$G(t,s) = \begin{cases} e^{m_1(t-s)}, & -\infty \leqslant t \leqslant s \leqslant \infty, \\ e^{m_2(t-s)}, & -\infty \leqslant s \leqslant t \leqslant \infty. \end{cases} \tag{2.5}$$

Using (2.4), we see directly that for any $s \in \mathbb{R}$,

$$\lim_{|t|\to\infty} G(t,s) = 0.$$

On the other hand, we have that a travelling wave solution $u \in C_b(\mathbb{R},\mathbb{R})$ must satisfy the fixed point equation

$$u(t) = \int G(t,s) f(s,u(s))\, ds := Au(t). \tag{2.6}$$

2.2. Banach spaces with Bielecki's type norm

Let ν be a continuous function defined in $\mathbb{R}$ with positive real values and denote with E the set of continuous functions u defined on $\mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}} |u(x)|\, \nu(x) < \infty.$$

It is straightforward to see that E is a normed linear space with norm

$$\|u\|_\nu = \sup_{x\in\mathbb{R}} |u(x)|\, \nu(x).$$

It is not difficult to show that the space $\langle E, \|\cdot\|_\nu\rangle$ is in fact a Banach space since for any Cauchy sequence $\{u_n\} \subset \langle E, \|\cdot\|_\nu\rangle$, then the sequence $\{\tilde{u}_n\} \subset \langle C_b(\mathbb{R}), |\cdot|_\infty\rangle$, where $\tilde{u}_n = u_n \nu$ is also a Cauchy sequence. Then, there is $\tilde{u}_0 \in C_b(\mathbb{R})$ such that $\tilde{u}_n \to \tilde{u}$ in $C_b(\mathbb{R})$, so if we set $u_0 = \tilde{u}_0/\nu$, we see that $u_n \to \tilde{u}$ in $\langle E, \|\cdot\|_\nu\rangle$.

It is worth to mention that the space E was introduced by Zima in [15] and corresponds to a generalization of a space introduced by Bielecki in [3]. It is important to point out that even though the Arzelà-Ascoli theorem fails to work in the space E, there are sufficient conditions of compactness (see [1]).

Before we go further, we need to recall that the family $\Omega \subset E$ is almost equicontinuous on $\mathbb{R}$, if the family $\Omega \subset E$ is equicontinuous in each interval $[a,b]$ with $-\infty < a < b < \infty$. So, we state a compactness criterion which is a modification of the analogous result obtained by Zima in [15].

Proposition 2.4 ([15])**.** *Let q be a continuous positive function on $\mathbb{R}$ such that*

$$\lim_{|x|\to\infty} \frac{\nu(x)}{q(x)} = 0.$$

If the family $\Omega \subset E$ is almost equicontinuous on $\mathbb{R}$ and uniformly bounded in the sense of the norm

$$\|u\|_q = \sup_{x\in\mathbb{R}} |u(x)| q(x),$$

then Ω is relatively compact in $\langle E, \|\cdot\|_\nu\rangle$.

3. Existence of positive traveling waves

In this section, we establish the existence of positive traveling wave solutions for the model (2.1), by imposing some hypotheses on the nonlinear term f. The result will be a consequence of the Krasnosel'skii fixed point theorem and a compactness in a Banach space $\langle E, \|\cdot\|_\nu\rangle$, where the function ν is given by

$\nu(x) = e^{-\eta|x|}$, and $\eta > -m_2 \geqslant m_1$ equipped with the norm

$$\|u\|_\nu = \sup_{x\in\mathbb{R}} e^{-\eta|x|}|u(x)|. \tag{3.1}$$

For this space $\langle E, \|\cdot\|_\nu\rangle$ we have the following compactness criteria.

Proposition 3.1. *Let* q *be the function defined by*

$$q(x) = \begin{cases} e^{-m_1 x}, & x \geqslant 0, \\ e^{-m_2 x}, & x < 0. \end{cases}$$

If the family $\Omega \subset E$ *is almost equicontinuous on* $\mathbb{R}$ *and uniformly bounded in the sense of the norm*

$$\|u\|_q = \sup_{x\in\mathbb{R}} q(x)|u(x)|, \tag{3.2}$$

then Ω *is relatively compact in* $\langle E, \|\cdot\|_\nu\rangle$.

Proof. The proof is a direct consequence of Proposition 2.4, since the function q is positive and continuous on $\mathbb{R}$. From the fact $\eta > -m_2 \geqslant m_1$, we conclude that

$$\lim_{x\to-\infty} \frac{e^{\eta x}}{e^{-m_2 x}} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{e^{-\eta x}}{e^{-m_1 x}} = 0,$$

meaning that

$$\lim_{|x|\to\infty} \frac{\nu(x)}{q(x)} = 0. \qquad \square$$

The existence of positive traveling solutions of (2.1) will be a consequence of the compactness result contained in Proposition 3.1 in the Banach space $\langle E, \|\cdot\|_\nu\rangle$ where $\nu(x) = e^{-\eta|x|}$, and $\eta > -m_2 \geqslant m_1$. We impose some restrictions on the nonlinear function f in order to verify the hypotheses of the Krasnosel'skii fixed point theorem (see Theorem 2.3),

(H1) $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous non-negative function such that

$$f(t, u) \leqslant b(t) + a(t)|u|^\sigma, \tag{3.3}$$

where $\sigma \geqslant 1$ and a, b are continuous non-negative functions;

(H2) the integrals

$$M^- = \int_{-\infty}^0 e^{-m_2 s} b(s)\, ds \quad \text{and} \quad M^+ = \int_0^\infty e^{-m_1 s} b(s)\, ds,$$

are convergent;

(H3) the integrals

$$M^-_{\sigma,i} = \int_{-\infty}^0 e^{-(m_i+\eta\sigma)s} a(s)\, ds \quad \text{and} \quad M^+_{\sigma,i} = \int_0^\infty e^{-(m_i-\eta\sigma)s} a(s)\, ds,$$

are convergent for $i = 1, 2$, and $\sigma \geqslant 1$.

3.1. *Boundedness of the operator* A

We first consider the boundedness of the operator A in the case:

$$b(t) = 0, \quad \sigma > 1.$$

Lemma 3.2. *Let* G *be the Green's function given by* (2.5) *and* $\sigma > 1$. *If* f *satisfies the hypothesis* (**H1**) *with* $b(t) = 0$ *and* $M^\pm_{\sigma,i}$ *for* $i = 1, 2$, *satisfies the hypothesis* (**H3**), *then the integral operator* A *satisfies*

$$\sup_{t\in\mathbb{R}} |(Au)(t)|\, e^{-\eta|t|} < \infty.$$

Moreover, we also have the estimate

$$|Au(t)|\, q(t) \leqslant \|u\|_q^\sigma \cdot \max\left\{M^-_{\sigma,1} + M^+_{\sigma,1}, M^-_{\sigma,2} + M^+_{\sigma,2}\right\}.$$

Proof. First, we consider $t \geqslant 0$, then

$$\begin{aligned}
|(Au)(t)|\, e^{-\eta t} &\leqslant e^{-\eta t}\left[\int_{-\infty}^{t} G(t,s)f(s,u(s))\, ds + \int_{t}^{\infty} G(t,s)f(s,u(s))\, ds\right]\\
&\leqslant e^{-\eta t}\left[\int_{-\infty}^{t} e^{m_2(t-s)}a(s)(u(s))^{\sigma}\, ds + \int_{t}^{\infty} e^{m_1(t-s)}a(s)(u(s))^{\sigma}\, ds\right] = e^{-\eta t}\left[I_1 + I_2\right].
\end{aligned}$$

So, by hypothesis (**H1**), (3.3), when $b(t) = 0$ and $\sigma > 1$, we have

$$\begin{aligned}
I_1 &= e^{m_2 t}\left[\int_{-\infty}^{0} e^{-m_2 s}e^{-\eta\sigma s}a(s)(e^{\eta s}u(s))^{\sigma}\, ds + \int_{0}^{t} e^{-m_2 s}e^{\eta\sigma s}a(s)(e^{-\eta s}u(s))^{\sigma}\, ds\right]\\
&\leqslant e^{m_2 t}\left[\left(\sup_{r\leqslant 0} e^{\eta r}u(r)\right)^{\sigma}\int_{-\infty}^{0} e^{-(m_2+\eta\sigma)s}a(s)\, ds + \left(\sup_{r\geqslant 0} e^{-\eta r}u(r)\right)^{\sigma}\int_{0}^{t} e^{-(m_2-\eta\sigma)s}a(s)\, ds\right]\\
&\leqslant \|u\|_{\nu}^{\sigma}\, e^{m_2 t}\left[\int_{-\infty}^{0} e^{-(m_2+\eta\sigma)s}a(s)\, ds + \int_{0}^{t} e^{-(m_2-\eta\sigma)s}a(s)\, ds\right]
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= e^{m_1 t}\int_{t}^{\infty} e^{-m_1 s}e^{\eta\sigma s}a(s)(e^{-\eta s}u(s))^{\sigma}\, ds\\
&\leqslant e^{m_1 t}\left(\sup_{r\geqslant 0} e^{-\eta r}u(r)\right)^{\sigma}\int_{t}^{\infty} e^{-(m_1-\eta\sigma)s}a(s)\, ds \leqslant \|u\|_{\nu}^{\sigma}\, e^{m_1 t}\int_{t}^{\infty} e^{-(m_1-\eta\sigma)s}a(s)\, ds.
\end{aligned}$$

On the other hand, as $0 < m_1 \leqslant -m_2 < \eta$, then $-\eta + m_1 < 0$, $-\eta + m_2 < 0$, and $m_1 - m_2 > 0$. Thus, for $t \geqslant 0$ and $s \geqslant 0$ we have the estimates

$$e^{(-\eta+m_1)t} \leqslant 1, \qquad e^{(-\eta+m_2)t} \leqslant 1, \quad \text{and} \quad e^{-(m_1-\eta\sigma)s} \leqslant e^{-(m_2-\eta\sigma)s}.$$

From these estimates, we have that

$$\begin{aligned}
|(Au)(t)|\, e^{-\eta t} &\leqslant \|u\|_{\nu}^{\sigma}\, e^{-\eta t}\Big(e^{m_2 t}\left[\int_{-\infty}^{0} e^{-(m_2+\eta\sigma)s}a(s)\, ds + \int_{0}^{t} e^{-(m_2-\eta\sigma)s}a(s)\, ds\right]\\
&\qquad + e^{m_1 t}\int_{t}^{\infty} e^{-(m_1-\eta\sigma)s}a(s)\, ds\Big)\\
&\leqslant \|u\|_{\nu}^{\sigma}\left(\int_{-\infty}^{0} e^{-(m_2+\eta\sigma)s}a(s)\, ds + \int_{0}^{t} e^{-(m_2-\eta\sigma)s}a(s)\, ds + \int_{t}^{\infty} e^{-(m_1-\eta\sigma)s}a(s)\, ds\right)\\
&\leqslant \|u\|_{\nu}^{\sigma}\left(M_{\sigma,2}^{-} + M_{\sigma,2}^{+}\right).
\end{aligned} \tag{3.4}$$

We follow an analogous procedure in the case $t \leqslant 0$. In this case, we have that

$$\begin{aligned}
|(Au)(t)|\, e^{\eta t} &\leqslant e^{\eta t}\left[\int_{-\infty}^{t} G(t,s)f(s,u(s))\, ds + \int_{t}^{\infty} G(t,s)f(s,u(s))\, ds\right]\\
&\leqslant e^{\eta t}\left[\int_{-\infty}^{t} e^{m_2(t-s)}a(s)(u(s))^{\sigma}\, ds + \int_{t}^{\infty} e^{m_1(t-s)}a(s)(u(s))^{\sigma}\, ds\right]\\
&= e^{\eta t}\left[I_1 + I_2\right].
\end{aligned}$$

Again, by hypothesis (**H1**), (3.3), when $b(t) = 0$ and $\sigma > 1$, we have

$$I_1 = e^{m_2 t}\left[\int_{-\infty}^{t} e^{-m_2 s}e^{-\eta\sigma s}a(s)(e^{\eta s}u(s))^{\sigma}\, ds\right]$$

$$\leqslant e^{m_2 t}\left(\sup_{r\leqslant 0} e^{\eta r}u(r)\right)^{\sigma}\int_{-\infty}^{t} e^{-(m_2+\eta\sigma)s}a(s)\,ds \leqslant \|u\|_{\nu}^{\sigma}\, e^{m_2 t}\int_{-\infty}^{t} e^{-(m_2+\eta\sigma)s}a(s)\,ds$$

and

$$\begin{aligned}
I_2 &= e^{m_1 t}\left[\int_t^0 e^{-m_1 s}e^{-\eta\sigma s}a(s)(e^{\eta s}u(s))^{\sigma}\,ds + \int_0^{\infty} e^{-m_1 s}e^{\eta\sigma s}a(s)(e^{-\eta s}u(s))^{\sigma}\,ds\right]\\
&\leqslant e^{m_1 t}\left[\left(\sup_{r\leqslant 0} e^{\eta r}u(r)\right)^{\sigma}\int_t^0 e^{-(m_1+\eta\sigma)s}a(s)\,ds + \left(\sup_{r\geqslant 0} e^{-\eta r}u(r)\right)^{\sigma}\int_0^{\infty} e^{-(m_1-\eta\sigma)s}a(s)\,ds\right]\\
&\leqslant \|u\|_{\nu}^{\sigma}\, e^{m_1 t}\left[\int_t^0 e^{-(m_1+\eta\sigma)s}a(s)\,ds + \int_0^{\infty} e^{-(m_1-\eta\sigma)s}a(s)\,ds\right].
\end{aligned}$$

On the other hand, as $0 < m_1 \leqslant -m_2 < \eta$, then $\eta + m_1 > 0$, $\eta + m_2 > 0$, and $m_1 - m_2 > 0$. Thus, for $t \leqslant 0$ and $s \leqslant 0$ we have that

$$e^{(\eta+m_1)t} \leqslant 1, \qquad e^{(\eta+m_2)t} \leqslant 1, \quad \text{and} \quad e^{-(m_2+\eta\sigma)s} \leqslant e^{-(m_1+\eta\sigma)s}.$$

Therefore

$$\begin{aligned}
|(Au)(t)|\, e^{\eta t} &\leqslant \|u\|_{\nu}^{\sigma}\, e^{\eta t}\Big(e^{m_2 t}\int_{-\infty}^{t} e^{-(m_2+\eta\sigma)s}a(s)\,ds\\
&\qquad + e^{m_1 t}\left[\int_t^0 e^{-(m_1+\eta\sigma)s}a(s)\,ds + \int_0^{\infty} e^{-(m_1-\eta\sigma)s}a(s)\,ds\right]\Big)\\
&\leqslant \|u\|_{\nu}^{\sigma}\left(\int_{-\infty}^{t} e^{-(m_2+\eta\sigma)s}a(s)\,ds + \int_t^0 e^{-(m_1+\eta\sigma)s}a(s)\,ds + \int_0^{\infty} e^{-(m_1-\eta\sigma)s}a(s)\,ds\right)\\
&\leqslant \|u\|_{\nu}^{\sigma}\left(M_{\sigma,1}^{-} + M_{\sigma,1}^{+}\right).
\end{aligned} \tag{3.5}$$

So, from the previous estimates (see (3.4) and (3.5)) we have that for $t \in \mathbb{R}$, $b(t) = 0$, and $\sigma > 1$,

$$|(Au)(t)|\, e^{-\eta|t|} \leqslant \|u\|_{\nu}^{\sigma}\cdot \max\left\{M_{\sigma,1}^{-} + M_{\sigma,1}^{+}, M_{\sigma,2}^{-} + M_{\sigma,2}^{+}\right\},$$

which implies that

$$\sup_{t\in\mathbb{R}} |(Au)(t)|\, e^{-\eta|t|} < \infty.$$

Proceeding analogously, we see that the integral operator A also satisfies

$$|(Au)(t)|\, q(t) \leqslant \|u\|_{q}^{\sigma}\cdot \max\left\{M_{\sigma,1}^{-} + M_{\sigma,1}^{+}, M_{\sigma,2}^{-} + M_{\sigma,2}^{+}\right\}. \qquad \square$$

From previous Lemma 3.2, we have that the family $(Au)_{u\in E}$ is uniformly bounded with respect to the norms $\|\cdot\|_{\nu}$ and $\|\cdot\|_{q}$, respectively (see equations (3.1) and (3.2)).

Now, we consider boundedness of the operator A in the case:

$$\sigma = 1.$$

Lemma 3.3. *Let G be the Green's function given by (2.5) and $\sigma = 1$. If f satisfies the hypothesis* (**H1**)*, $M^{\pm}$ satisfies the hypothesis* (**H2**)*, and $M_{1,i}^{\pm}$ for $i = 1, 2$, satisfies the hypothesis* (**H3**)*, then the integral operator A satisfies*

$$\sup_{t\in\mathbb{R}} |(Au)(t)|\, e^{-\eta|t|} < \infty.$$

Moreover, we also have the estimate

$$|(Au)(t)|\, q(t) \leqslant M^{-} + M^{+} + \|u\|_{q}\cdot \max\left\{M_{1,1}^{-} + M_{1,1}^{+},\ M_{1,2}^{-} + M_{1,2}^{+}\right\}.$$

Proof. First, we have that

$$|(Au)(t)|\,e^{-\eta|t|} \leqslant e^{-\eta|t|}\int_{-\infty}^{\infty} G(t,s)f(s,u(s))\,ds$$
$$\leqslant e^{-\eta|t|}\left[\int_{-\infty}^{\infty} G(t,s)b(s)\,ds + \int_{-\infty}^{\infty} G(t,s)a(s)u(s)\,ds\right] = e^{-\eta|t|}\left[I_0 + I\right].$$

We consider $t \geqslant 0$, then, by (2.5) we have

$$\begin{aligned} I_0 &= \int_{-\infty}^{t} e^{m_2(t-s)}b(s)\,ds + \int_{t}^{\infty} e^{m_1(t-s)}b(s)\,ds \\ &\leqslant \int_{-\infty}^{0} e^{m_2(t-s)}b(s)\,ds + \int_{0}^{t} e^{m_2(t-s)}b(s)\,ds + \int_{t}^{\infty} e^{m_1(t-s)}b(s)\,ds \\ &\leqslant \int_{-\infty}^{0} e^{m_2(t-s)}b(s)\,ds + \int_{0}^{t} e^{m_1(t-s)}b(s)\,ds + \int_{t}^{\infty} e^{m_1(t-s)}b(s)\,ds \\ &\leqslant e^{m_2 t}\int_{-\infty}^{0} e^{-m_2 s}b(s)\,ds + e^{m_1 t}\int_{0}^{\infty} e^{-m_1 s}b(s)\,ds. \end{aligned}$$

Then,

$$e^{-\eta t}I_0 \leqslant e^{(m_2-\eta)t}\int_{-\infty}^{0} e^{-m_2 s}b(s)\,ds + e^{(m_1-\eta)t}\int_{0}^{\infty} e^{-m_1 s}b(s)\,ds \leqslant M^- + M^+. \tag{3.6}$$

Now, in the case $t \leqslant 0$, we have that

$$\begin{aligned} I_0 &= \int_{-\infty}^{t} e^{m_2(t-s)}b(s)\,ds + \int_{t}^{\infty} e^{m_1(t-s)}b(s)\,ds \\ &\leqslant \int_{-\infty}^{t} e^{m_2(t-s)}b(s)\,ds + \int_{t}^{0} e^{m_1(t-s)}b(s)\,ds + \int_{0}^{\infty} e^{m_1(t-s)}b(s)\,ds \\ &\leqslant \int_{-\infty}^{t} e^{m_2(t-s)}b(s)\,ds + \int_{t}^{0} e^{m_2(t-s)}b(s)\,ds + \int_{0}^{\infty} e^{m_1(t-s)}b(s)\,ds \\ &\leqslant e^{m_2 t}\int_{-\infty}^{0} e^{-m_2 s}b(s)\,ds + e^{m_1 t}\int_{0}^{\infty} e^{-m_1 s}b(s)\,ds. \end{aligned}$$

Then,

$$e^{\eta t}I_0 \leqslant e^{(m_2+\eta)t}\int_{-\infty}^{0} e^{-m_2 s}b(s)\,ds + e^{(m_1+\eta)t}\int_{0}^{\infty} e^{-m_1 s}b(s)\,ds \leqslant M^- + M^+. \tag{3.7}$$

Therefore, by (3.6) and (3.7), we have

$$e^{-\eta|t|}I_0 \leqslant M^- + M^+. \tag{3.8}$$

Now, following a similar procedure as in the proof of Lemma 3.2 with $\sigma = 1$, we obtain that

$$e^{-\eta|t|}I = \|u\|_v \cdot \max\left\{M_{1,1}^- + M_{1,1}^+,\ M_{1,2}^- + M_{1,2}^+\right\}. \tag{3.9}$$

So, from the previous estimates (see (3.8) and (3.9)) we have that for $t \in \mathbb{R}$, and $\sigma = 1$,

$$|(Au)(t)|\,e^{-\eta|t|} \leqslant M^- + M^+ + \|u\|_v \cdot \max\left\{M_{1,1}^- + M_{1,1}^+,\ M_{1,2}^- + M_{1,2}^+\right\},$$

which implies that

$$\sup_{t\in\mathbb{R}} |(Au)(t)|\,e^{-\eta|t|} < \infty.$$

Proceeding analogously, we see that the integral operator A also satisfies, when $\sigma = 1$, that

$$|(Au)(t)|\,q(t) \leqslant M^- + M^+ + \|u\|_q \cdot \max\left\{M_{1,1}^- + M_{1,1}^+,\ M_{1,2}^- + M_{1,2}^+\right\}.$$

□

From previous Lemma 3.3, we have that when $\sigma = 1$, the family $(Au)_{u\in E}$ is uniformly bounded with respect to the norms $\|\cdot\|_\nu$ and $\|\cdot\|_q$, respectively (see equations (3.1) and (3.2)).

Now, we consider $\delta > \gamma > 0$, $m = e^{m_2\delta}$ and the set

$$K = \left\{ u \in E : u(t) \geqslant 0 \text{ on } \mathbb{R} \text{ and } \min_{t\in[\gamma,\delta]} u(t) \geqslant m \|u\|_\nu \right\} \subset E. \tag{3.10}$$

The first remark is that K is a cone in E. In fact, if $k > 0$ and $u \in K$, then

$$ku(t) \geqslant 0 \qquad \text{and} \qquad \min_{t\in[\gamma,\delta]} ku(t) \geqslant m \|ku\|_\nu ,$$

therefore $ku \in K$.

Lemma 3.4. *Let $\delta > \gamma > 0$ and $m = e^{m_2\delta}$. Then*

$$\min_{t\in[\gamma,\delta]} (Au)(t) \geqslant m \|Au\|_\nu .$$

Proof. Let $\tau \geqslant 0$. First we consider the case $t \leqslant \tau$. So, we have that

$$\begin{aligned}(Au)(t) &= \int_{-\infty}^{t} G(t,s)f(s,u(s))\,ds + \int_{t}^{\infty} G(t,s)f(s,u(s))\,ds \\ &= e^{m_2 t}\int_{-\infty}^{t} e^{-m_2 s}f(s,u(s))\,ds + e^{m_1 t}\int_{t}^{\infty} e^{-m_1 s}f(s,u(s))\,ds = I_1 + I_2.\end{aligned}$$

Using that $-\infty \leqslant s \leqslant t \leqslant \tau$, we conclude that

$$I_1 = e^{m_2 t}\int_{-\infty}^{t} e^{-m_2 s}e^{-m_2(\tau-s)}G(\tau,s)f(s,u(s))\,ds = e^{-m_2(\tau-t)}\int_{-\infty}^{t} G(\tau,s)f(s,u(s))\,ds.$$

Now, we split the integral I_2 from t to ∞ into the integrals t to τ and τ to ∞:

$$\begin{aligned}I_2 &= e^{m_1 t}\left[e^{-m_2\tau}\int_{t}^{\tau} e^{-(m_1-m_2)s}G(\tau,s)f(s,u(s))\,ds + e^{-m_1\tau}\int_{\tau}^{\infty} G(\tau,s)f(s,u(s))\,ds\right] \\ &\geqslant e^{m_1 t}\left[e^{-m_2\tau}e^{-(m_1-m_2)\tau}\int_{t}^{\tau} G(\tau,s)f(s,u(s))\,ds + e^{-m_1\tau}\int_{\tau}^{\infty} G(\tau,s)f(s,u(s))\,ds\right] \\ &\geqslant e^{m_1 t}\left[e^{-m_1\tau}\int_{t}^{\infty} G(\tau,s)f(s,u(s))\,ds\right] = e^{-m_1(\tau-t)}\int_{t}^{\infty} G(\tau,s)f(s,u(s))\,ds.\end{aligned}$$

We conclude for $t \in [\gamma,\delta]$, $t \leqslant \tau$ that

$$(Au)(t) \geqslant e^{-m_2(\tau-t)}\int_{-\infty}^{t} G(\tau,s)f(s,u(s))\,ds + e^{-m_1(\tau-t)}\int_{t}^{\infty} G(\tau,s)f(s,u(s))\,ds.$$

We assume now that $\tau \leqslant t$. We split the integral I_1 from $-\infty$ to t into the integrals $-\infty$ to τ and τ to t:

$$\begin{aligned}I_1 &= e^{m_2 t}\left[e^{-m_2\tau}\int_{-\infty}^{\tau} G(\tau,s)f(s,u(s))\,ds + e^{-m_1\tau}\int_{\tau}^{t} e^{(m_1-m_2)s}G(\tau,s)f(s,u(s))\,ds\right] \\ &\geqslant e^{m_2 t}\left[e^{-m_2\tau}\int_{-\infty}^{\tau} G(\tau,s)f(s,u(s))\,ds + e^{-m_1\tau}e^{(m_1-m_2)\tau}\int_{\tau}^{t} G(\tau,s)f(s,u(s))\,ds\right] \\ &\geqslant e^{m_2 t}e^{-m_2\tau}\int_{-\infty}^{t} G(\tau,s)f(s,u(s))\,ds = e^{-m_2(\tau-t)}\int_{-\infty}^{t} G(\tau,s)f(s,u(s))\,ds\end{aligned}$$

and

$$I_2 = e^{m_1 t}\int_t^{\infty} e^{-m_1 s}e^{-m_1(\tau-s)}G(\tau,s)f(s,u(s))\,ds = e^{-m_1(\tau-t)}\int_t^{\infty} G(\tau,s)f(s,u(s))\,ds.$$

We conclude for $t \in [\gamma,\delta]$, $\tau \leqslant t$ that

$$(Au)(t) \geqslant e^{-m_2(\tau-t)}\int_{-\infty}^{t} G(\tau,s)f(s,u(s))\,ds + e^{-m_1(\tau-t)}\int_t^{\infty} G(\tau,s)f(s,u(s))\,ds.$$

So, we have shown for any t and τ that

$$(Au)(t) \geqslant e^{-m_2(\tau-t)}\int_{-\infty}^{t} G(\tau,s)f(s,u(s))\,ds + e^{-m_1(\tau-t)}\int_t^{\infty} G(\tau,s)f(s,u(s))\,ds.$$

Now, for $t \in [\gamma,\delta]$, we have that

$$e^{m_2 t} \geqslant e^{m_2\delta} \qquad \text{and} \qquad e^{m_1 t} \geqslant e^{m_1\gamma}.$$

So, we conclude that

$$(Au)(t) \geqslant m\left(e^{-m_2\tau}\int_{-\infty}^{t} G(\tau,s)f(s,u(s))\,ds + e^{-m_1\tau}\int_t^{\infty} G(\tau,s)f(s,u(s))\,ds\right),$$

where $m = \min\{e^{m_2\delta}, e^{m_1\gamma}\} = e^{m_2\delta}$. Now, we remember that $0 < m_1 \leqslant -m_2 < \eta$, then for $\tau \geqslant 0$ we have that

$$e^{(\eta-m_2)\tau} \geqslant 1 \qquad \text{and} \qquad e^{(\eta-m_1)\tau} \geqslant 1,$$

meaning that

$$\begin{aligned}(Au)(t) &\geqslant m\left(e^{(\eta-m_2)\tau}e^{-\eta\tau}\int_{-\infty}^{t} G(\tau,s)f(s,u(s))\,ds + e^{(\eta-m_1)\tau}e^{-\eta\tau}\int_t^{\infty} G(\tau,s)f(s,u(s))\,ds\right)\\ &\geqslant me^{-\eta|\tau|}\int_{-\infty}^{\infty} G(\tau,s)f(s,u(s))\,ds.\end{aligned}$$

Now, for $\tau \leqslant 0$ we have that

$$\begin{aligned}(Au)(t) &\geqslant m\left(e^{-(\eta+m_2)\tau}e^{\eta\tau}\int_{-\infty}^{t} G(\tau,s)f(s,u(s))\,ds + e^{-(\eta+m_1)\tau}e^{\eta\tau}\int_t^{\infty} G(\tau,s)f(s,u(s))\,ds\right)\\ &\geqslant me^{-\eta|\tau|}\int_{-\infty}^{\infty} G(\tau,s)f(s,u(s))\,ds.\end{aligned}$$

Note that as $0 < m_1 \leqslant -m_2 < \eta$, then $m_1 + \eta > 0$ and $m_2 + \eta > 0$, then for $\tau \leqslant 0$ we have that

$$e^{-(\eta+m_2)\tau} \geqslant 1 \qquad \text{and} \qquad e^{-(\eta+m_1)\tau} \geqslant 1.$$

Therefore, we have that

$$\min_{t\in[\gamma,\delta]} (Au)(t) \geqslant e^{m_2\delta}\,\|Au\|_{\nu}\,. \qquad \square$$

In the same fashion, we are able to establish an analogous result to Lemma 3.4.

Lemma 3.5. *Let* $\delta > \gamma > 0$. *Then*

$$\min_{t\in[-\delta,-\gamma]} (Au)(t) \geqslant e^{-m_1\delta}\,\|Au\|_{\nu}\,.$$

The following result presents an important property that satisfies the operator A defined by (2.6).

Theorem 3.6. *Let* G *be the Green's function given by* (2.5), f *satisfies the hypothesis* (**H1**), $M^{\pm}$ *satisfies the hypothesis* (**H2**), $M^{\pm}_{\sigma,i}$ *satisfies the hypothesis* (**H3**) *for* $i = 1, 2$ *and* $\sigma \geqslant 1$, *and* K *be the set defined by* (3.10). *Then, given a bounded set* $\Omega \subset E$, *we have that* $A : \overline{\Omega} \cap K \to K$ *is completely continuous.*

Proof. Let $\Omega \subset E$ be a bounded set. First, we prove that A maps $\overline{\Omega} \cap K$ into K. From the hypothesis (**H1**) and since $G(s,t) \geqslant 0$ for $s, t \in \mathbb{R}$, we see that

$$(Au)(t) \geqslant 0, \qquad u \in \overline{\Omega} \cap K. \tag{3.11}$$

Now, from Lemma 3.2 in the case $\sigma > 1$, and Lemma 3.3 in the case $\sigma = 1$, we know that for any $u \in \overline{\Omega} \cap K$, $\sup_{t \in \mathbb{R}} |(Au)(t)| e^{-\eta |t|} < \infty$. Thus

$$Au \in E, \qquad \forall u \in \overline{\Omega} \cap K. \tag{3.12}$$

Moreover, Lemma 3.4 implies that for $t \in [\gamma, \delta]$, $m = e^{m_2 \delta}$, and $u \in \overline{\Omega} \cap K$

$$\min_{t \in [\gamma,\delta]} (Au)(t) \geqslant m \|Au\|_{\nu}, \qquad \forall u \in \overline{\Omega} \cap K. \tag{3.13}$$

The equations (3.11), (3.12), and (3.13) imply that $A(\overline{\Omega} \cap K) \subset K$ as desired.

Now let us prove that $A : \overline{\Omega} \cap K \to K$ is completely continuous. For Lemma 3.2 in the case $\sigma > 1$, and Lemma 3.3 in the case $\sigma = 1$, we have that $\|Au\|_{\nu} \leqslant k_1 \|u\|_{\nu}$ and $\|Au\|_q \leqslant k_2 \|u\|_q$, i.e., the operator A is continuous with respect to the norms $\|\cdot\|_{\nu}$ and $\|\cdot\|_q$, respectively (see equations (3.1) and (3.2)). Let $u \in \overline{\Omega} \cap K$. We also have that the functions Au are uniformly bounded with respect to the norm $\|\cdot\|_q$. Moreover, for any $T > 0$, the fact that G and f are continuous, implies that $\{Au : u \in \overline{\Omega} \cap K\}$ are equicontinuous in each compact interval $[-T, T]$ of $\mathbb{R}$. Thus, by Proposition 3.1, $A(\overline{\Omega} \cap K)$ is a precompact set in E (relative compact), i.e., operator A maps bounded sets into precompact sets. Therefore, A is completely continuous. □

We set M_{σ} as

$$M_{\sigma} = \max\left\{ M^{-}_{\sigma,1} + M^{+}_{\sigma,1},\ M^{-}_{\sigma,2} + M^{+}_{\sigma,2} \right\}.$$

From the assumptions **(H1)-(H3)**, we have the following existence result for positive traveling solution.

Theorem 3.7. *Let* G, f, $M^{\pm}_{\sigma,i}$ *for* $i = 1, 2$ *and* $\sigma \geqslant 1$, *and* $M^{\pm}$ *be as in Theorem* 3.6. *If there exist* $\lambda > 0$, $\delta > \gamma > 0$, *and* $t_0 > 0$ *such that for* $t \in [\gamma, \delta]$ *and* $u \in \left[m\lambda, \lambda e^{\eta\delta}\right]$, *we have that*

$$f(t,u) \geqslant \lambda e^{\eta t_0} \left(\int_{\gamma}^{\delta} G(t_0, s)\, ds \right)^{-1}, \tag{3.14}$$

then there is a positive traveling solution for (2.1), *provided that* $\lambda^{1-\sigma} \neq M_{\sigma}$ *in the case* $b(t) = 0, \sigma > 1$, *and* $\lambda \neq \dfrac{M^{-} + M^{+}}{1 - M_1}$, *in the case* $\sigma = 1$.

Proof. Note that every fixed point of the integral operator A given by (2.6), is a solution of (2.1). We will show that A satisfies the assumptions of Theorem 2.3. From Theorem 3.6 we know that $A : \overline{\Omega} \cap K \to K$ is completely continuous for any bounded set $\Omega \subset E$. Now, we consider the case $b(t) = 0$ and $\sigma > 1$. Fix

$$r^{1-\sigma} = M_{\sigma} \qquad \text{and} \qquad R = \lambda.$$

Assume without loss of generality that $r < R$, and define

$$\Omega_1 = \{u \in E : \|u\|_{\nu} < r\} \qquad \text{and} \qquad \Omega_2 = \{u \in E : \|u\|_{\nu} < R\}.$$

If $u \in K \cap \partial\Omega_1$ then from (3.4) and (3.5) we have that

$$|(Au)(t)|\, e^{-\eta|t|} \leqslant r^{\sigma} M_{\sigma} = r,$$

hence $\|Au\|_{\nu} \leqslant \|u\|_{\nu}$. Now, if $u \in K \cap \partial\Omega_2$ then

$$\min_{t\in[\gamma,\delta]} u(t) \geqslant m\lambda \qquad \text{and} \qquad \sup_{t\in\mathbb{R}} u(t)e^{-\eta|t|} = \lambda.$$

Thus, for $t \in [\gamma, \delta]$, we have $m\lambda \leqslant u(t) \leqslant \lambda\, e^{\eta\delta}$. By (3.14),

$$\begin{aligned}(Au)(t_0) = \int_{-\infty}^{\infty} G(t_0,s)f(s,u(s))\,ds &\geqslant \int_{\gamma}^{\delta} G(t_0,s)f(s,u(s))\,ds \\ &\geqslant \int_{\gamma}^{\delta} G(t_0,s)\left(\int_{\gamma}^{\delta} G(t_0,y)\,dy\right)^{-1} \lambda\, e^{\eta t_0}\,ds = \lambda\, e^{\eta t_0},\end{aligned}$$

which implies $(Au)(t_0)\, e^{-\eta t_0} \geqslant \lambda$. This gives $\|Au\|_{\nu} \geqslant \|u\|_{\nu}$ for $u \in K \cap \partial\Omega_2$. By Theorem 2.3, the operator A has at least one fixed point in the set $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which means that the problem (2.1) has a positive solution u such that $r \leqslant \|u\|_{\nu} \leqslant R$.

Finally, in the case $\sigma = 1$ we consider

$$r = \frac{M^- + M^+}{1 - M_1}, \quad R = \lambda,$$

which completes the proof of theorem. Note that under the conditions of Theorem 3.7 there are no constant traveling waves. In fact, suppose that $u(t) = k$ for all $t \in \mathbb{R}$, with $0 < k < \infty$, then replacing in the equation (2.1) we have

$$-ck + f(t,k) = 0.$$

Now, for $\sigma > 1$ and $b(t) = 0$, $f(t,k)$ satisfies (3.3), then for any $t \in \mathbb{R}$,

$$ck \leqslant a(t)k^{\sigma} \quad \Leftrightarrow \quad c \leqslant a(t)k^{\sigma-1},$$

but we also have that $a(t) \to 0$ as $t \to \infty$. Then we reach a contradiction. Now, for $\sigma = 1$, $f(t,k)$ satisfies (3.3), then for any $t \in \mathbb{R}$,

$$ck \leqslant b(t) + a(t)k \quad \Leftrightarrow \quad 0 \leqslant b(t) + (a(t) - c)k,$$

but we also have that $a(t) \to 0$ and $b(t) \to 0$ as $t \to \infty$. Then we reach a contradiction. □

We note that Theorem 3.7 also holds in the case that there is $t_0 < 0$, in this case $t \in [-\delta, -\gamma]$ and $u \in \left[e^{-m_1\delta}\lambda, e^{-\eta\gamma}\right]$, and we have that

$$f(t,u) \geqslant \lambda e^{-\eta t_0} \left(\int_{-\delta}^{-\gamma} G(t_0,s)\,ds\right)^{-1}.$$

4. Application to the existence result

In this section we present some basic examples to illustrate the applicability of our main result.

Example 4.1. Consider the following problem

$$-cu(t) + \alpha u'(t) + \beta u''(t) + e^{-\rho|t|}|u(t)|^{\sigma} = 0, \quad t \in \mathbb{R}, \tag{4.1}$$

with $\sigma > 1$. For the hypothesis **(H3)** to be fulfilled on $M^{\pm}_{\sigma,i}$, it is necessary that

$$\rho > \max\{m_1 + \eta\sigma, m_2 + \eta\sigma, \eta\sigma - m_1, \eta\sigma - m_2\}.$$

Under this condition, we have that

$$M^{-}_{\sigma,1} = \frac{1}{\rho-(m_1+\eta\sigma)}, \quad M^{-}_{\sigma,2} = \frac{1}{\rho-(m_2+\eta\sigma)}, \quad M^{+}_{\sigma,1} = \frac{1}{\rho+(m_1-\eta\sigma)}, \quad M^{+}_{\sigma,2} = \frac{1}{\rho+(m_2-\eta\sigma)}.$$

We consider then $M_\sigma = \max\left\{M^{-}_{\sigma,1}+M^{+}_{\sigma,1},\ M^{-}_{\sigma,2}+M^{+}_{\sigma,2}\right\}$. Now, let $\gamma = t_0 > 0$ and $\delta > \gamma$, then $m = \min\{e^{m_2\delta}, e^{m_1\gamma}\} = e^{m_2\delta}$ and

$$\int_\gamma^\delta G(t_0,s)ds = \int_\gamma^\delta e^{m_1(\gamma-s)}ds = \frac{1}{m_1}\left(1-e^{m_1(\gamma-\delta)}\right).$$

Now, for $t \in [\gamma,\delta]$ and $m\lambda \leqslant u \leqslant \lambda\, e^{\eta\delta}$, we have

$$\begin{aligned}\left(\int_\gamma^\delta G(t_0,s)\,ds\right) f(t,u) &\geqslant \frac{1}{m_1}\left(1-e^{m_1(\gamma-\delta)}\right) e^{-\rho\delta}\,(m\lambda)^\sigma \\ &\geqslant \frac{1}{m_1}\left(1-e^{m_1(\gamma-\delta)}\right) e^{-\rho\delta} e^{m_2\sigma\delta}\lambda^\sigma \geqslant \lambda\, e^{\eta t_0} \geqslant \lambda\, e^{\eta\gamma}.\end{aligned}$$

So, if we take $\sigma > 1$, then

$$\lambda^{\sigma-1} \geqslant \frac{m_1 e^{(m_1+\rho-m_2\sigma)\delta+\eta\gamma}}{e^{m_1\delta}-e^{m_1\gamma}}.$$

Hence, the function

$$f(t,u) = e^{-\rho t|}|u|^\sigma$$

satisfies the assumptions of Theorem 3.7 with $\lambda^{\sigma-1} = \dfrac{m_1 e^{(m_1+\rho-m_2\sigma)\delta+\eta\gamma}}{e^{m_1\delta}-e^{m_1\gamma}}$. Note $\lambda^{\sigma-1} \neq M_\sigma^{-1}$. By Theorem 3.7, the problem (4.1) has a positive solution u such that

$$M_\sigma^{\frac{1}{1-\sigma}} \leqslant \|u\|_\nu \leqslant \left(\frac{m_1 e^{(m_1+\rho-m_2\sigma)\delta+\eta\gamma}}{e^{m_1\delta}-e^{m_1\gamma}}\right)^{\frac{1}{\sigma-1}}.$$

Example 4.2. Consider the following problem

$$-cu(t)+\alpha u'(t)+\beta u''+b(t)+a(t)|u(t)| = 0, \quad t \in \mathbb{R}, \tag{4.2}$$

where $b(t) = e^{-\rho_1|t|}$ and $a(t) = e^{-\rho_2|t|}$ are continuous non-negative functions for $t \in \mathbb{R}$. For the hypotheses (**H2**) and (**H3**) to be fulfilled on $M^\pm$ and $M^\pm_{1,i}$, it is necessary that

$$-\rho_1 < \max\{m_1,-m_2\} = -m_2$$

and

$$\rho_2 > \max\{m_1+\eta, m_2+\eta, \eta-m_1, \eta-m_2\} = \eta-m_2.$$

Under this condition, we have that

$$\begin{aligned} M^- &= \frac{1}{\rho_1-m_2}, & M^-_{1,1} &= \frac{1}{\rho_2-(m_1+\eta)}, & M^-_{1,2} &= \frac{1}{\rho_2-(m_2+\eta)}, \\ M^+ &= \frac{1}{\rho_1+m_1}, & M^+_{1,1} &= \frac{1}{\rho_2+(m_1-\eta)}, & M^+_{1,2} &= \frac{1}{\rho_2+(m_2-\eta)}. \end{aligned}$$

We consider then $M_1 = \max\left\{M^+_{1,1}+M^-_{1,1}, M^+_{1,2}+M^-_{1,2}\right\}$. So that $M_1 < 1$ we need that

$$\frac{1}{\rho_2-(m_2+\eta)} + \frac{1}{\rho_2+(m_2-\eta)} = \frac{2(\rho_2-\eta)}{(\rho_2-\eta)^2-m_2^2} < 1.$$

This implies that

$$\rho_2 > 1+\eta+\sqrt{1+m_2^2}.$$

Now, let $\gamma = t_0 > 0$ and $\delta > \gamma$, then $m = \min\{e^{m_2\delta}, e^{m_1\gamma}\} = e^{m_2\delta}$ and

$$\int_\gamma^\delta G(t_0,s)ds = \int_\gamma^\delta e^{m_1(\gamma-s)}ds = \frac{1}{m_1}\left(1-e^{m_1(\gamma-\delta)}\right).$$

Therefore, for $t\in[\gamma,\delta]$ and $m\lambda \leqslant u \leqslant \lambda\, e^{\eta\delta}$ we have

$$\frac{\left(1-e^{m_1(\gamma-\delta)}\right)e^{-\rho_1\delta}}{m_1e^{\eta\gamma}-\left(1-e^{m_1(\gamma-\delta)}\right)e^{(m_2-\rho_2)\delta}} \geqslant \lambda.$$

Hence, the function

$$f(t,u) = e^{-\rho_1|t|}+e^{-\rho_2|t|}|u|$$

satisfies the assumptions of Theorem 3.7 with $\lambda = \frac{\left(1-e^{m_1(\gamma-\delta)}\right)e^{-\rho_1\delta}}{m_1e^{\eta\gamma}-\left(1-e^{m_1(\gamma-\delta)}\right)e^{(m_2-\rho_2)\delta}}$. Note $\lambda < \frac{M^-+M^+}{1-M_1}$. By Theorem 3.7, the problem (4.2) has a nontrivial solution u such that

$$\frac{\left(1-e^{m_1(\gamma-\delta)}\right)e^{-\rho_1\delta}}{m_1e^{\eta\gamma}-\left(1-e^{m_1(\gamma-\delta)}\right)e^{(m_2-\rho_2)\delta}} \leqslant \|u\|_\nu \leqslant \frac{M^-+M^+}{1-M_1}.$$

5. On the dispersive models

As we pointed out above, there are some dispersive models with explicit travelling wave solutions. We will see that in some sense those solutions can be captured in our model.

Example 5.1. Consider the IVP for the generalized KdV equation (gKdV)

$$\begin{cases} u_t+\beta u_{xxx}+u^p\,u_x = 0, & t,x\in\mathbb{R},\\ u(x,0) = u_0\,(x)\,. \end{cases}$$

In this case, assuming decay at infinity, the traveling wave solutions $u(x,t) = u(x-ct)$ satisfies the equation

$$-cu+\beta u''+\frac{1}{p+1}u^{p+1} = 0, \tag{5.1}$$

which has an explicit solution of the form

$$u_p\,(x) = \left(\frac{c(p+1)(p+2)}{2}\right)^{\frac{1}{p}}\operatorname{sech}^{\frac{2}{p}}\left(\frac{p}{2}\sqrt{\frac{c}{\beta}}x\right).$$

Moreover, solitary wave solutions take the form

$$u_p\,(x,t,c) = \left(\frac{c(p+1)(p+2)}{2}\right)^{\frac{1}{p}}\operatorname{sech}^{\frac{2}{p}}\left(\frac{p}{2}\sqrt{\frac{c}{\beta}}\,(x-ct)\right).$$

We observe that

$$u^{p+1} = u_p^\sigma u_p^{p+1-\sigma} = u_p^\sigma a_p, \quad \text{with} \quad a_p = \frac{1}{p+1}u_p^{p+1-\sigma},$$

meaning that u_p satisfies the equation (2.1) where $f(x,u) = u^\sigma a_p(x)$ and $\alpha = 0$. In this case, from the estimate (2.3), we have that $m_1 = \sqrt{\frac{c}{\beta}} = -m_2$.

We also have that

$$u_p(x) \sim e^{-m_1|x|}, \quad a_p(x) = \tfrac{1}{p+1} u_p^{p+1-\sigma}(x) \sim e^{-m_1(p+1-\sigma)|x|}.$$

Now, note that in order to fulfill the hypothesis (**H3**) on $M^{\pm}_{\sigma,i}$, it is necessary that

$$m_1(p+1-\sigma) = \rho > \max\{m_1+\eta\sigma, m_2+\eta\sigma\} = m_1+\eta\sigma > m_1+m_1\sigma = m_1(1+\sigma),$$

i.e., $p+1-\sigma > 1+\sigma > 2$, therefore $p > 2$. So, we choose $p = 3$ to exhibit an example. We take then $\sigma = \frac{5}{4}$ and $\eta = \frac{6}{5}m_1$, for example. From this, we have that $m_1+\eta\sigma = \frac{5}{2}m_1$ and $m_2+\eta\sigma = \frac{1}{2}m_1$.

To verify the hypotheses on $M^{\pm}_{\sigma,i}$, we note that

$$\int_0^\infty e^{\alpha s}\operatorname{sech}^{2q}(\beta x)\, ds = \left(\frac{2^{2q}}{2\beta q-\alpha}\right) {}_2F_1\left[2q, q-\tfrac{\alpha}{2\beta}, q-\tfrac{\alpha}{2\beta}+1, -1\right],$$

where the hypergeometric function ${}_2F_1[a,b,c,z]$ is defined by

$${}_2F_1[a,b,c,z] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt,$$

with $\Gamma(z)$ being the Gamma function (see [5]).

Now, we have that $a_3(x) = \frac{1}{4}u_3^{4-\sigma}(x) = \frac{1}{4}(10c)^{\frac{11}{12}}\operatorname{sech}^{\frac{11}{6}}\left(\frac{3}{2}m_1 x\right)$. Therefore, as $m_1 = -m_2$ and $a_3(x)$ is even, we have that

$$M^-_{\sigma,1} = M^+_{\sigma,2} = \int_0^\infty e^{(m_1+\eta\sigma)s}a_3(s)\, ds = \frac{(10c)^{\frac{11}{12}}}{4}\left(\frac{2^{\frac{23}{6}}}{m_1}\right) {}_2F_1\left[\tfrac{11}{6}, \tfrac{1}{12}, \tfrac{13}{12}, -1\right] \approx 26.9316\,\frac{c^{\frac{11}{12}}}{m_1},$$

$$M^-_{\sigma,2} = M^+_{\sigma,1} = \int_0^\infty e^{(-m_1+\eta\sigma)s}a_3(s)\, ds = \frac{(10c)^{\frac{11}{12}}}{4}\left(\frac{2^{\frac{23}{6}}}{9m_1}\right) {}_2F_1\left[\tfrac{11}{6}, \tfrac{3}{4}, \tfrac{7}{4}, -1\right] \approx 1.9018\,\frac{c^{\frac{11}{12}}}{m_1}.$$

We consider then

$$M_\sigma = \max\left\{M^+_{\sigma,1}+M^-_{\sigma,1}, M^+_{\sigma,2}+M^-_{\sigma,2}\right\} \approx 28.8334\,\frac{c^{\frac{11}{12}}}{m_1} = 28.8334\, c^{\frac{5}{12}}\sqrt{\beta}.$$

Now, let $\gamma = t_0 = 1$ and $\delta = 2$, then $m = \min\{e^{m_2\delta}, e^{m_1\gamma}\} = \min\{e^{2m_2}, e^{m_1}\} = e^{-2m_1}$ and

$$\int_1^2 G(1,s)ds = \int_1^2 e^{m_1(1-s)}ds = \frac{1}{m_1}\left(1-e^{-m_1}\right).$$

Therefore, for $t \in [\gamma, \delta]$ and $m\lambda \leqslant u \leqslant \lambda e^{\eta\delta} = \lambda e^{2.4m_1}$, we have

$$\lambda^3 \geqslant \frac{4m_1 e^{10.2m_1}}{e^{m_1}-1}.$$

Hence, the function

$$f(x,u) = a_3(x)|u|^\sigma,$$

with $a_3(x) = \frac{1}{4}(u_3(x))^{4-\sigma}$, $u_3(x) = (10c)^{\frac{1}{3}}\operatorname{sech}^{\frac{2}{3}}\left(\frac{3}{2}\sqrt{\frac{c}{\beta}}\,x\right)$, and $\sigma = \frac{5}{4}$, satisfies the assumptions of Theorem 3.7 with $\lambda = \left(\frac{4m_1 e^{10.2m_1}}{e^{m_1}-1}\right)^{1/3}$. Note that $\lambda^{1-\sigma} \neq M_\sigma$ for $\sigma = \frac{5}{4}$. In fact

$$\min_{c>0,\beta>0}\left(\frac{4\sqrt{\frac{c}{\beta}}\, e^{10.2\sqrt{\frac{c}{\beta}}}}{1-e^{-\sqrt{\frac{c}{\beta}}}}\right)^{1/3} - \left(28.8334\, c^{\frac{5}{12}}\sqrt{\beta}\right)^{-4} > 0,$$

then $\lambda \neq M_{\sigma}^{-4}$. By Theorem 3.7, the problem (5.1) has a nontrivial solution u_3 such that

$$\frac{c^{-\frac{5}{3}}\beta^{-2}}{(28.8333)^4} \leqslant \|u_3\|_{\nu} \leqslant \left(\frac{4m_1 e^{10.2m_1}}{1-e^{-m_1}}\right)^{1/3}, \qquad \text{with} \quad \nu(t) = e^{-\frac{6}{5}m_1|t|}.$$

Example 5.2. Consider the IVP for the generalized p-Gardner equation

$$\begin{cases} u_t + \beta u_{xxx} + bu^p u_x + au^{2p}u_x = 0, & t, x \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases}$$

In this case, assuming decay at infinity, the traveling wave solution $u(x,t) = u(x-ct)$ satisfies the equation

$$-cu + \beta u'' + \frac{b}{p+1}u^{p+1} + \frac{a}{2p+1}u^{2p+1} = 0. \tag{5.2}$$

Hamdi et al. in [8], using symbolic computation for an elliptic Riccati equation, obtained an exact travelling wave solution $u(x,t) = u(x-ct)$ for the generalized p-Gardner equation (5.2) of the form

$$u_p(x) = \left(\frac{(p+1)(p+2)c}{b}\right)^{\frac{1}{p}} \left(1 + \sqrt{1 + \left(\frac{ac(p+1)(p+2)^2}{(2p+1)b^2}\right)} \cosh\left(p\sqrt{\frac{c}{\beta}}x\right)\right)^{-\frac{1}{p}}.$$

We observe that the equation (5.2) is a particular case of the equation (2.1) where $\alpha = 0$ and

$$f(x,u) = \frac{b}{p+1}u^{p+1} + \frac{a}{2p+1}u^{2p+1}.$$

In this case we have that $m_1 = \sqrt{\frac{c}{\beta}} = -m_2$. We first see that

$$\frac{b}{p+1}u_p^{p+1} + \frac{a}{2p+1}u_p^{2p+1} = \tilde{b}(x) + \tilde{a}(x)u, \qquad \tilde{b}(x) = \frac{b}{p+1}(u_p(x))^{p+1}, \qquad \tilde{a}(x) = \frac{a}{2p+1}(u_p(x))^{2p}.$$

Now, from the definition of u_p we get that

$$u_p(x) \sim e^{-m_1|x|}, \qquad \tilde{b}(x) \sim e^{-(p+1)m_1|x|}, \qquad \tilde{a}(x) \sim e^{-2pm_1|x|}.$$

Note that the hypothesis (**H2**) on $M^{\pm}$ is satisfied, since

$$e^{-m_2 s}\tilde{b}(s) \sim e^{(p+2)m_1 s} \to 0, \quad s \to -\infty, \qquad \text{and} \qquad e^{-m_1 s}\tilde{b}(s) \sim e^{-(p+2)m_1 s} \to 0, \quad s \to \infty.$$

In order to fulfill the hypothesis (**H3**) on $M_{1,i}^{\pm}$, we need that

$$2pm_1 > \max\{m_1 + \eta, m_2 + \eta, \eta - m_2, \eta - m_1\} = m_1 + \eta.$$

Therefore

$$(2p-1)m_1 > \eta \quad \wedge \quad \eta > m_1 \quad \Longrightarrow \quad p > 1.$$

So, to illustrate the analysis, we consider the case $p = 2$. To simplify, consider $a = b = c = 1$ and let us take $\eta = 2m_1$. Then

$$m_1 + \eta = 3\,m_1 \quad \text{and} \quad m_2 + \eta = m_1, \quad \text{where} \quad m_1 = \frac{1}{\sqrt{\beta}}.$$

Consequently, we have that

$$u_2(x) = (12)^{\frac{1}{2}} \left(1 + \sqrt{1 + \frac{48}{5}} \cosh(2m_1 x)\right)^{-\frac{1}{2}},$$

$$\tilde{b}(x)=\frac{1}{3}(u_2(x))^3=4\sqrt{12}\left(1+\sqrt{\frac{53}{5}}\cosh(2m_1x)\right)^{-\frac{3}{2}},$$

and

$$\tilde{a}(x)=\frac{1}{5}(u_2(x))^4=\frac{144}{5}\left(1+\sqrt{\frac{53}{5}}\cosh(2m_1x)\right)^{-2}.$$

Then, we have

$$M^-=\int_{-\infty}^{0}e^{-m_2s}\tilde{b}(s)\,ds=\frac{5}{\sqrt{12}m_1}\left(\sqrt{2}\sqrt[4]{\frac{53}{5}}-\sqrt{1+\sqrt{\frac{53}{5}}}\right),$$
$$M^+=\int_{0}^{\infty}e^{-m_1s}\tilde{b}(s)\,ds=\frac{5}{\sqrt{12}m_1}\left(\sqrt{2}\sqrt[4]{\frac{53}{5}}-\sqrt{1+\sqrt{\frac{53}{5}}}\right).$$

As $m_1=-m_2$ and $\tilde{a}(s)$ is even, we have that

$$M_{1,1}^-=M_{1,2}^+=\int_0^\infty e^{(m_1+\eta)s}\tilde{a}(s)\,ds\approx\frac{0.24}{m_1},\qquad M_{1,2}^-=M_{1,1}^+=\int_0^\infty e^{(-m_1+\eta)s}\tilde{a}(s)\,ds\approx\frac{0.048}{m_1}.$$

From previous estimates, we conclude that

$$M^-+M^+\approx\frac{1.4111}{m_1},\qquad M_1=\max\{M_{1,1}^++M_{1,1}^-,M_{1,2}^++M_{1,2}^-\}\approx\frac{0.288}{m_1}.$$

Now, let $\gamma=t_0=1$ and $\delta=2$, then $m=\min\{e^{m_2\delta},e^{m_1\gamma}\}=\min\{e^{2m_2},e^{m_1}\}=e^{-2m_1}$ and

$$\int_1^2 G(1,s)ds=\int_1^2 e^{m_1(1-s)}ds=\frac{1}{m_1}\left(1-e^{-m_1}\right).$$

Therefore, for $t\in[\gamma,\delta]$ and $m\lambda=e^{-2m_1}\lambda\leqslant u(t)\leqslant\lambda e^{\eta\delta}=\lambda e^{4m_1}$ we have that

$$\lambda\leqslant\frac{4\sqrt{12}\,(1-e^{-m_1})\left(1+\sqrt{\frac{53}{5}}\cosh(4m_1)\right)^{-\frac{3}{2}}}{m_1e^{2m_1}-\frac{144}{5}\,(1-e^{-m_1})\,e^{-2m_1}\left(1+\sqrt{\frac{53}{5}}\cosh(4m_1)\right)^{-2}}=\mathbf{r}.$$

Hence, we see that function f takes the form

$$f(x,u)=\tilde{b}+\tilde{a}u$$

and satisfies the assumptions of Theorem 3.7 with $\lambda=\mathbf{r}$. Note that $\mathbf{r}\neq\frac{M^-+M^+}{1-M_1}=\frac{1.4111}{m_1-0.288}$ (for $0<\beta\lesssim 12.056$). By Theorem 3.7, the problem (5.2) has a nontrivial solution u_2 such that

$$\mathbf{r}\leqslant\|u_2\|_\nu\leqslant\frac{1.4111}{m_1-0.288},\qquad\text{with}\quad\nu(t)=e^{-2m_1|t|}.$$

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