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Some properties of graded 2-absorbing and graded weakly 2-absorbing submodules

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Abstract

Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. In this paper we will obtain some results concerning the graded 2-absorbing and graded weakly 2-absorbing submodules of a graded modules over a commutative graded ring.

Keywords: Graded 2-absorbing submodule, graded weakly 2-absorbing submodule, graded submodules.

2010 MSC: 13A02, 16W50.

1. Introduction and preliminaries

Throughout this paper all rings are commutative with identity and all modules are unitary.

The concept of weakly prime ideals was initiated by Anderson and Smith in [8]. The concept of weakly 2-absorbing ideals was introduced in [14] as a generalization of the notion of weakly prime ideals. Badawi in [13] introduced the concept of 2-absorbing ideals of commutative rings that is a generalization of the concept of prime ideals. Later on, Anderson and Badawi in [7] generalized the concept of 2-absorbing ideals of commutative rings to the concept of n-absorbing ideals of commutative rings for every positive integer $n \geqslant 2$. In light of [7, 13] many authors studied the concept of 2-absorbing submodules and n-absorbing submodules, (see for example, [15, 18, 24, 26, 27]).

The scope of this paper is devoted to the theory of graded modules over graded commutative rings. One use of rings and modules with gradings is in describing certain topics in algebraic geometry. Graded prime ideals, and graded weakly prime ideals have been studied by various authors, (see for example [5, 10, 25]). The concept of graded 2-absorbing ideals and graded weakly 2-absorbing ideals, generalizations of graded prime ideals, and graded weakly prime ideals, respectively, were studied by Al-Zoubi and Abu-Dawwas, and other authors, (see [2, 19]). Graded prime submodules, and graded weakly prime submodules have been studied by various authors, (see for example [4, 6, 9, 11, 23]). The concept of

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graded 2-absorbing submodules and graded weakly 2-absorbing submodules, generalizations of graded prime submodules, and graded weakly prime submodules, respectively, were studied by Al-Zoubi and Abu-Dawwas in [1]. Later on, Hamoda and Ashour in [17] introduced the concept of graded n-absorbing submodules that is a generalization of the concept of graded prime ideals.

Here, we study several results concerning of graded 2-absorbing and graded weakly 2-absorbing submodules of graded modules over graded commutative rings.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [16, 20–22] for these basic properties and more information on graded rings and modules.

Let G be a group with identity e and R be a commutative ring with identity 1_R. Then R is a G-*graded ring* if there exist additive subgroups R_g of R such that $R = \bigoplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called to be *homogeneous* of degree g where the R_g's are additive subgroups of R indexed by the elements $g \in G$. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Moreover, $h(R) = \bigcup_{g\in G} R_g$. Let I be an ideal of R. Then I is called a *graded ideal* of (R, G) if $I = \bigoplus_{g\in G}(I \bigcap R_g)$. Thus, if $x \in I$, then $x = \sum_{g\in G} x_g$ with $x_g \in I$.

Let R be a G-graded ring and M an R-module. We say that M is a G-*graded* R-*module* (or *graded* R-*module*) if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = \bigoplus_{g\in G} M_g$ (as Abelian groups) and $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$. Here, R_gM_h denotes the additive subgroup of M consisting of all finite sums of elements r_gs_h with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \bigcup_{g\in G} M_g$ and the elements of $h(M)$ are called to be *homogeneous*. Let $M = \bigoplus_{g\in G} M_g$ be a graded R-module and N a submodule of M. Then N is called a *graded submodule* of M if $N = \bigoplus_{g\in G} N_g$, where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the g-*component* of N. Let R be a G-graded ring, M a graded R-module, and N a graded submodule of M. Then $(N :_R M)$ is defined as $(N :_R M) = \{r \in R | rM \subseteq N\}$. It is shown in [9, Lemma 2.1] that if N is a graded submodule of M, then $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of R. A proper graded submodule P of M is said to be *a graded prime submodule* (Resp. *graded weakly prime submodule*) of M if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in P$ (Resp. $0 \neq rm \in P$), then either $r \in (P :_R M)$ or $m \in P$ (see [9, 11]). A proper graded ideal I of R is said to be *a graded 2-absorbing ideal* (Resp. *a graded weakly 2-absorbing ideal*) of R if whenever $r, s, t \in h(R)$ with $rst \in I$ (Resp. $0 \neq rst \in I$), then $rs \in I$ or $rt \in I$ or $st \in I$ (see [2]). A proper graded submodule N of a graded R-module M is said to be *a graded 2-absorbing submodule* (Resp. *graded weakly 2-absorbing submodule*) of M if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$ (Resp. $0 \neq rsm \in N$), then $rs \in (N :_R M)$ or $rm \in N$ or $sm \in N$ (see [1]).

2. Graded 2-absorbing submodules

Lemma 2.1. *Let R be a G-graded ring, M a graded R-module, and N a graded 2-absorbing submodule of M. Let $I = \bigoplus_{g\in G} I_g$ be a graded ideal of R. Then for every $r \in h(R)$, $m \in h(M)$ and $g \in G$ with $rI_gm \subseteq N$, either $rm \in N$ or $I_gm \subseteq N$ or $rI_g \subseteq (N :_R M)$.*

Proof. Let $r \in h(R)$, $m \in h(M)$ and $g \in G$ such that $rI_gm \subseteq N$, $rm \notin N$ and $rI_g \nsubseteq (N :_R M)$. We have to show that $I_gm \subseteq N$. As $rI_g \nsubseteq (N :_R M)$, there exists $i_g \in I_g$ such that $ri_g \notin (N :_R M)$. Since N is a graded 2-absorbing submodule, $ri_gm \in N$, $rm \notin N$ and $ri_g \notin (N :_R M)$, we have $i_gm \in N$. Now, let $i'_g \in I_g$. By $i_g + i'_g \in I_g$ it follows that $r(i_g + i'_g)m \in N$. Then either $(i_g + i'_g)m \in N$ or $r(i_g + i'_g) \in (N :_R M)$ as N is a graded 2-absorbing submodule. If $(i_g + i'_g)m \in N$, then we get $i'_gm \in N$ since $i_gm \in N$. If $r(i_g + i'_g) \in (N :_R M)$, then we get $ri'_g \notin (N :_R M)$ since $ri_g \notin (N :_R M)$, but $ri'_gm \in N$, so $i'_gm \in N$ since N is a graded 2-absorbing submodule, $rm \notin N$ and $ri'_g \notin (N :_R M)$. Hence $I_gm \subseteq N$. □

Theorem 2.2. *Let R be a G-graded ring, M a graded R-module, and N a graded 2-absorbing submodule of M. Let $I = \bigoplus_{g\in G} I_g$ and $J = \bigoplus_{g\in G} J_g$ be a graded ideals of R. Then for every $m \in h(M)$ and $g, h \in G$ with $I_gJ_hm \subseteq N$, either $I_gm \subseteq N$ or $J_hm \subseteq N$ or $I_gJ_h \subseteq (N :_R M)$.*

Proof. Let $m \in h(M)$ and $g, h \in G$ such that $I_g J_h m \subseteq N$, $I_g m \nsubseteq N$ and $J_h m \nsubseteq N$. We have to show that $I_g J_h \subseteq (N :_R M)$. Let $i_g \in I_g$ and $j_h \in J_h$. As $I_g m \nsubseteq N$ and $J_h m \nsubseteq N$, there exist $i'_g \in I_g$ and $j'_h \in J_h$ such that $i'_g m \notin N$ and $j'_h m \notin N$. Since $i'_g J_h m \subseteq N$, $i'_g m \notin N$ and $J_h m \nsubseteq N$, by Lemma 2.1, we get $i'_g J_h \subseteq (N :_R M)$. Also since $j'_h I_g m \subseteq N$, $j'_h m \notin N$ and $I_g m \nsubseteq N$, we get $j'_h I_g \subseteq (N :_R M)$, which implies $(I_g \backslash (N :_R m))J_h \subseteq (N :_R M)$ and $(J_h \backslash (N :_R m))I_g \subseteq (N :_R M)$. Hence we have $i'_g j'_h \in (N :_R M)$, $i'_g j_h \in (N :_R M)$ and $i_g j'_h \in (N :_R M)$. By $(i_g + i'_g) \in I_g$ and $(j_h + j'_h) \in J_h$ it follows that $(i_g + i'_g)(j_h + j'_h)m \in N$. Since N is a graded 2-absorbing submodule, we get either $(i_g + i'_g)m \in N$ or $(j_h + j'_h)m \in N$ or $(i_g + i'_g)(j_h + j'_h) \in (N :_R M)$. If $(i_g + i'_g)m \in N$, then $i_g m \notin N$ since $i'_g m \notin N$, so $i_g \in I_g \backslash (N :_R m)$. Hence $i_g j_h \in (N :_R M)$. Similarly, if $(j_h + j'_h)m \in N$, then $j_h m \notin N$ since $j'_h m \notin N$, therefore $j_h \in J_h \backslash (N :_R m)$. Hence $i_g j_h \in (N :_R M)$. If $(i_g + i'_g)(j_h + j'_h) \in (N :_R M)$, i.e, $i_g j_h + i_g j'_h + i'_g j_h + i'_g j'_h \in (N :_R M)$, then $i_g j_h \in (N :_R M)$ since $i_g j'_h$, $i'_g j_h$, $i'_g j'_h \in (N :_R M)$. Thus $I_g J_h \subseteq (N :_R M)$. □

Theorem 2.3. *Let R be a G-graded ring, M a graded R-module, and N be a proper graded submodule of M. Let* $I = \bigoplus_{g \in G} I_g$ *and* $J = \bigoplus_{g \in G} J_g$ *be graded ideals of R and* $U = \bigoplus_{g \in G} U_g$ *be a graded submodule of M. Then the following statements are equivalent:*

(i) *N is a graded 2-absorbing submodule of M;*
(ii) *for every* $g, h, \lambda \in G$ *with* $I_g J_h U_\lambda \subseteq N$, *either* $I_g U_\lambda \subseteq N$ *or* $J_h U_\lambda \subseteq N$ *or* $I_g J_h \subseteq (N :_R M)$.

Proof.

(i)⇒(ii) Assume that N is a graded 2-absorbing submodule of M. Let $g, h, \lambda \in G$ such that $I_g J_h U_\lambda \subseteq N$ and $I_g J_h \nsubseteq (N :_R M)$. By Theorem 2.2 for all $u_\lambda \in U_\lambda$ we have either $I_g u_\lambda \subseteq N$ or $J_h u_\lambda \subseteq N$. If $I_g u_\lambda \subseteq N$ for all $u_\lambda \in U_\lambda$, then $I_g U_\lambda \subseteq N$. Similarly, if $J_h u_\lambda \subseteq N$ for all $u_\lambda \in U_\lambda$, then $J_h U_\lambda \subseteq N$. Assume that there exist u_λ, $u'_\lambda \in U_\lambda$ such that $I_g u_\lambda \nsubseteq N$ and $J_h u'_\lambda \nsubseteq N$. Since $I_g J_h u_\lambda \subseteq N$, $I_g u_\lambda \nsubseteq N$ and $I_g J_h \nsubseteq (N :_R M)$, by Theorem 2.2, we get $J_h u_\lambda \subseteq N$. Also since $I_g J_h u'_\lambda \subseteq N$, $J_h u'_\lambda \nsubseteq N$ and $I_g J_h \nsubseteq (N :_R M)$, we get $I_g u'_\lambda \subseteq N$. By $u_\lambda + u'_\lambda \in U_\lambda$ it follows that $I_g J_h (u_\lambda + u'_\lambda) \subseteq N$. By Theorem 2.2, we get either $I_g(u_\lambda + u'_\lambda) \subseteq N$ or $J_h(u_\lambda + u'_\lambda) \subseteq N$. If $I_g(u_\lambda + u'_\lambda) \subseteq N$, then $I_g u_\lambda \subseteq N$ since $I_g u'_\lambda \subseteq N$ which is a contradiction. Similarly if $J_h(u_\lambda + u'_\lambda) \subseteq N$, then $J_h u'_\lambda \subseteq N$ since $J_h u_\lambda \subseteq N$, a contradiction. Therefore either $I_g U_\lambda \subseteq N$ or $J_h U_\lambda \subseteq N$.

(ii)⇒(i) Assume that (ii) holds. Let r_g, $s_h \in h(R)$ and $m_\lambda \in h(M)$ such that $r_g s_h m_\lambda \in N$. Let $I = r_g R$ and $J = s_h R$ be graded ideals of R generated by r_g, s_h, respectively and $U_\lambda = m_\lambda R$ be a graded submodule of M generated by m_λ. Then $I_g J_h U_\lambda \subseteq N$. By our assumption we obtain $I_g U_\lambda \subseteq N$ or $J_h U_\lambda \subseteq N$ or $I_g J_h \subseteq (N :_R M)$. Hence $r_g m_\lambda \in N$ or $s_h m_\lambda \in N$ or $r_g s_h \in (N :_R M)$. Therefore N is a graded 2-absorbing submodule of M. □

3. Graded weakly 2-absorbing submodules

Let N be a graded submodule of M and let $g \in G$. We say that N_g is a weakly g-2-absorbing submodule of R_e-module M_g, if $N_g \neq M_g$; and whenever $r, s \in R_e$ and $m \in M_g$ with $0 \neq rsm \in N_g$, then either $rs \in (N_g :_{R_e} M_g)$ or $rm \in N_g$ or $sm \in N_g$ (see [1]).

Lemma 3.1. *Let R be a G-graded ring, M a graded R-module and N a graded weakly 2-absorbing submodule of M, and* $g \in G$. *If* $r_e s_e U \subseteq N_g$ *and* $0 \neq 2r_e s_e U$ *for some* $r_e, s_e \in R_e$ *and some submodule U of* M_g, *then either* $r_e s_e \in (N_g :_{R_e} M_g)$ *or* $r_e U \subseteq N_g$ *or* $s_e U \subseteq N_g$.

Proof. By [1, Lemma 3.2], N_g is a weakly g-2-absorbing R_e-submodule of M_g for every $g \in G$. Assume that $r_e s_e U \subseteq N_g$, $0 \neq 2r_e s_e U$ and $r_e s_e \notin (N_g :_{R_e} M_g)$ for some $r_e, s_e \in R_e$ and some submodule U of M_g. We have to show that $U \subseteq (N_g :_{M_g} r_e) \cup (N_g :_{M_g} s_e)$. Let $u_g \in U \subseteq M_g$. If $0 \neq r_e s_e u_g$, then either $r_e u_g \in N_g$ or $s_e u_g \in N_g$ since N_g is a weakly g-2-absorbing R_e-submodule of M_g and $r_e s_e \notin (N_g :_{R_e} M_g)$. So $u_g \in (N_g :_{M_g} r_e) \cup (N_g :_{M_g} s_e)$. Suppose that $r_e s_e u_g = 0$. Since $0 \neq 2r_e s_e U$, there exists $u'_g \in U \subseteq M_g$ such that $0 \neq 2r_e s_e u'_g$, hence $0 \neq r_e s_e u'_g \in N_g$. Since N_g is a weakly g-2-absorbing R_e-submodule of M_g, we have either $r_e u'_g \in N_g$ or $s_e u'_g \in N_g$. Let $v_g = u_g + u'_g$. Hence $0 \neq r_e s_e v_g \in N_g$. Then $r_e v_g \in N_g$ or $s_e v_g \in N_g$ as N_g is a weakly g-2-absorbing R_e-submodule of M_g. Now, we consider three cases.

Case 1: $r_e u'_g \in N_g$ and $s_e u'_g \notin N_g$. On the contrary let $r_e u_g \notin N_g$. Then $r_e v_g \notin N_g$ and hence $s_e v_g \in N_g$. This yields that $r_e(v_g + u'_g) \notin N_g$ and $s_e(v_g + u'_g) \notin N_g$. So $0 = r_e s_e(v_g + u'_g) = 2r_e s_e u'_g$ since N_g is a weakly g-2-absorbing submodule and $r_e s_e \notin (N_g :_{R_e} M_g)$, which is a contradiction. Thus $r_e u_g \in N_g$.

Case 2: $r_e u'_g \notin N_g$ and $s_e u'_g \in N_g$. The proof is similar to that of Case 1.

Case 3: $r_e u'_g \in N_g$ and $s_e u'_g \in N_g$. Since $r_e v_g \in N_g$ or $s_e v_g \in N_g$, we get $r_e u_g \in N_g$ or $s_e u_g \in N_g$. Thus $u_g \in (N_g :_{M_g} r_e) \cup (N_g :_{M_g} s_e)$. □

Theorem 3.2. *Let* R *be a* G*-graded ring,* M *a graded* R*-module, and* N *a graded weakly 2-absorbing submodule of* M *and* $g \in G$. *If* $r_e IU \subseteq N_g$ *and* $0 \neq 4r_e IU$ *for some* $r_e \in R_e$, I *ideal of* R_e *and some submodule* U *of* M_g, *then either* $r_e I \subseteq (N_g :_{R_e} M_g)$ *or* $r_e U \subseteq N_g$ *or* $IU \subseteq N_g$.

Proof. By [1, Lemma 3.2], N_g is a weakly g-2-absorbing R_e-submodule of M_g for every $g \in G$. Assume that $r_e IU \subseteq N_g$, $0 \neq 4r_e IU$, $r_e I \nsubseteq (N_g :_{R_e} M_g)$ and $r_e U \nsubseteq N_g$ for some $r_e \in R_e$, I ideal of R_e and some submodule U of M_g. We have to show that $IU \subseteq N_g$. By [3, Lemma 2.15], there exists $s_e \in I$ such that $0 \neq 2r_e s_e U + 2r_e s_e U$ and $r_e s_e \notin (N_g :_{R_e} M_g)$. Hence $0 \neq 2r_e s_e U$ and $r_e s_e U \subseteq N_g$. Thus $s_e U \subseteq N_g$ by Lemma 3.1. Let $i_e \in I$. Assume that $0 \neq 2r_e i_e U$. Since $r_e i_e U \subseteq N_g$ and $r_e U \nsubseteq N_g$, by Lemma 3.1, we have either $r_e i_e \in (N_g :_{R_e} M_g)$ or $i_e U \subseteq N_g$. Thus $i_e \in ((N_g :_{R_e} M_g) :_{R_e} r_e) \cup (N_g :_{R_e} U)$. Now, let $2r_e i_e U = 0$. This yields that $0 \neq 2r_e s_e U = 2r_e(s_e + i_e)U$ and $r_e(s_e + i_e)U \subseteq N_g$. It follows that either $(s_e + i_e)U \subseteq N_g$ or $r_e(s_e + i_e) \in (N_g :_{R_e} M_g)$ by Lemma 3.1. If $(s_e + i_e)U \subseteq N_g$, then since $s_e U \subseteq N_g$, we get $i_e U \subseteq N_g$. Let $r_e(s_e + i_e) \in (N_g :_{R_e} M_g)$ and $(s_e + i_e)U \nsubseteq N_g$. Then $(s_e + i_e + s_e)U \nsubseteq N_g$ since $s_e U \subseteq N_g$. As $r_e s_e \notin (N_g :_{R_e} M_g)$ and $r_e(s_e + i_e) \in (N_g :_{R_e} M_g)$, then $r_e(s_e + i_e + s_e) \notin (N_g :_{R_e} M_g)$. Since $2r_e(s_e + i_e + s_e)U = 4r_e s_e U \neq 0$ and $r_e(s_e + i_e + s_e)U \subseteq N_g$, by Lemma 3.1, we get $r_e U \subseteq N_g$, which is a contradiction. Hence $(s_e + i_e)U \subseteq N_g$ and so $i_e U \subseteq N_g$. Thus $I \subseteq ((N_g :_{R_e} M_g) :_{R_e} r_e) \cup (N_g :_{R_e} U)$. This yields that $IU \subseteq N_g$ since $r_e I \nsubseteq (N_g :_{R_e} M_g)$. □

Let R_i be a graded commutative ring with identity and M_i be a graded R_i-module for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is a graded R-module and each graded submodule of M is of the form $N = N_1 \times N_2$ for some graded submodules N_1 of M_1 and N_2 of M_2.

Theorem 3.3. *Let* $R = R_1 \times R_2$ *be a* G*-graded ring and* $M = M_1 \times M_2$ *be a graded* R*-module where* M_1 *is a graded* R_1*-module and* M_2 *is a graded* R_2*-module. Let* N_1 *be a proper graded submodule of* M_1. *Then the following statements are equivalent:*

(i) N_1 *is a graded 2-absorbing of* M_1;
(ii) $N_1 \times M_2$ *is a graded 2-absorbing submodule of* M;
(iii) $N_1 \times M_2$ *is a graded weakly 2-absorbing submodule of* M.

Proof.

(i)⇒(ii) Assume that $(r_1, r_2)(s_1, s_2)(m_1, m_2) = (r_1 s_1 m_1, r_2 s_2 m_2) \in N_1 \times M_2$, where $r_1, s_1 \in h(R_1), r_2, s_2 \in h(R_2)$, $m_1 \in h(M_1)$, $m_2 \in h(M_2)$. Then $r_1 s_1 m_1 \in N_1$. Since N_1 is a graded 2-absorbing of M_1, we get either $r_1 m_1 \in N_1$ or $s_1 m_1 \in N_1$ or $r_1 s_1 \in (N_1 :_{R_1} M_1)$. If $r_1 m_1 \in N_1$, then $(r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2) \in N_1 \times M_2$. Similarly, if $s_1 m_1 \in N_1$, then $(s_1, s_2)(m_1, m_2) = (s_1 m_1, s_2 m_2) \in N_1 \times M_2$. Again, if $r_1 s_1 \in (N_1 :_{R_1} M_1)$, then $(r_1, r_2)(s_1, s_2) = (r_1 s_1, r_2 s_2) \in (N_1 \times M_2 :_R M)$. Thus $N_1 \times M_2$ is a graded 2-absorbing submodule of M.

(ii)⇒(iii) It is obvious.

(iii)⇒(i) Let $r_1 s_1 m_1 \in N_1$ for $r_1, s_1 \in h(R_1)$ and $m_1 \in h(M_1)$. Then for each $0 \neq m_2 \in h(M_2)$, we have $(0, 0) \neq (r_1, 1)(s_1, 1)(m_1, m_2) = (r_1 s_1 m_1, m_2) \in N_1 \times M_2$. Since $N_1 \times M_2$ is a graded weakly 2-absorbing submodule of M, we get either $(r_1, 1)(m_1, m_2) = (r_1 m_1, m_2) \in N_1 \times M_2$ or $(s_1, 1)(m_1, m_2) = (s_1 m_1, m_2) \in N_1 \times M_2$ or $(r_1, 1)(s_1, 1) = (r_1 s_1, 1) \in (N_1 \times M_2 :_R M)$. It follows that either $r_1 m_1 \in N_1$ or $s_1 m_1 \in N_1$ or $r_1 s_1 \in (N_1 :_{R_1} M_1)$. □

Theorem 3.4. *Let $R = R_1 \times R_2$ be a G-graded ring and $M = M_1 \times M_2$ be a graded R-module where M_1 is a nonzero graded R_1-module and M_2 is a nonzero graded R_2-module. Let N_1 and N_2 be proper graded submodules of M_1 and M_2, respectively.*

(i) *If $N = N_1 \times N_2$ is a graded weakly 2-absorbing submodule of M, then N_1 is a graded weakly prime submodule of M_1; moreover, if $0 \neq N_2$, then N_1 is a graded classical prime submodule of M_1.*

(ii) *If $N = N_1 \times N_2$ is a graded weakly 2-absorbing submodule of M and $(N_1 :_{R_1} M_1)M_1 \neq 0$, then N_2 is a graded prime submodule of M_2.*

Proof.

(i) Assume that $N = N_1 \times N_2$ is a graded weakly 2-absorbing submodule of M. We show that N_1 is a graded weakly prime submodule of M_1. Since $N_2 \neq M_2$, there exists $m_2 \in h(M_2)\backslash N_2$. Let $0 \neq rm_1 \in N_1$ for $r \in h(R_1)$ and $m_1 \in h(M_1)$. Then $(0,0) \neq (r,1)(1,0)(m_1,m_2) = (rm_1,0) \in N = N_1 \times N_2$. Since $N = N_1 \times N_2$ is a graded weakly 2-absorbing submodule of M and $m_2 \notin N_2$, either $(r,1)(1,0) = (r,0) \in (N :_R M) = (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)$ or $(1,0)(m_1,m_2) = (m_1,0) \in N = N_1 \times N_2$. Hence either $m_1 \in N_1$ or $r \in (N_1 :_R M_1)$ which shows that N_1 is a graded weakly prime submodule of M_1. Now assume that $0 \neq N_2$ and let $rsm_1 \in N_1$ for r, $s \in h(R_1)$ and $m_1 \in h(M_1)$. Let $0 \neq n_2 \in N_2 \cap h(M_2)$. Then $(0,0) \neq (r,1)(s,1)(m_1,n_2) = (rsm_1,n_2) \in N = N_1 \times N_2$. Since $N = N_1 \times N_2$ is a graded weakly 2-absorbing submodule of M and $1 \notin (N_2 :_R M_2)$, we get either $(r,1)(m_1,n_2) = (rm_1,n_2) \in N = N_1 \times N_2$ or $(s,1)(m_1,n_2) = (sm_1,n_2) \in N = N_1 \times N_2$. Hence, either $rm_1 \in N_1$ or $sm_1 \in N_1$. Therefore N_1 is a graded classical prime submodule of M_1.

(ii) Assume that $N = N_1 \times N_2$ is a graded weakly 2-absorbing submodule of M and $(N_1 :_{R_1} M_1)M_1 \neq 0$. Let $rm_2 \in N_2$ for $r \in h(R_2)$ and $m_2 \in h(M_2)$. If $(N_1 :_{R_1} M_1)m_1 = 0$ for each $m_1 \in M_1\backslash N_1$, then $(M_1\backslash N_1) \subseteq (0 :_{M_1} (N_1 :_{R_1} M_1))$. Thus $M_1 = N_1 \cup (M_1\backslash N_1) \subseteq N_1 \cup (0 :_{M_1} (N_1 :_{R_1} M_1))$ and since $M_1 \nsubseteq N_1$, we get $M_1 \subseteq (0 :_M (N_1 :_{R_1} M_1))$ by [12, Lemma 2.2]. Hence $(N_1 :_{R_1} M_1)M_1 = 0$, which is a contradiction. Thus there exist $t \in (N_1 :_{R_1} M_1) \cap h(R_1)$ and $m_1 \in h(M_1)\backslash N_1$ with $tm_1 \neq 0$. Then $(0,0) \neq (t,1)(1,r)\ (m_1,m_2) = (tm_1,rm_2) \in N = N_1 \times N_2$. Since $N = N_1 \times N_2$ is a graded weakly 2-absorbing submodule of M and $m_1 \notin N_1$, we get $(t,1)(1,r) = (t,r) \in (N :_R M) = (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)$ or $(t,1)(m_1,m_2) = (tm_1,m_2) \in N$. It follows that either $r \in (N_2 :_{R_2} M_2)$ or $m_2 \in N_2$. Therefore N_2 is a graded prime submodule of M_2. □

Theorem 3.5. *Let $R = R_1 \times R_2$ be a G-graded ring and $M = M_1 \times M_2$ be a graded R-module where M_1 is a nonzero graded R_1-module and M_2 is a nonzero graded R_2-module. Let $0 \neq N_1$ be a proper graded submodule of M_1 and $(N_1 :_{R_1} M_1)M_1 \neq 0$. Then the graded submodule $N_1 \times 0$ is a graded weakly 2-absorbing submodule of M_1 if and only if N_1 is a graded weakly prime submodule of M_1 and 0 is a graded prime submodule of M_2.*

Proof.

($\Rightarrow$) By Theorem 3.4.

($\Leftarrow$) Assume that $(0,0) \neq (r_1,r_2)(s_1,s_2)(m_1,m_2) = (r_1s_1m_1,r_2s_2m_2) \in N_1 \times 0$, where $r_1,s_1 \in h(R_1), r_2,s_2 \in h(R_2)$, $m_1 \in h(M_1)$, $m_2 \in h(M_2)$. Then $0 \neq r_1s_1m_1 \in N_1$ and $r_2s_2m_2 = 0$. Since N_1 is a graded weakly prime submodule of M_1, we get either $r_1 \in (N_1 :_{R_1} M_1)$ or $s_1 \in (N_1 :_{R_1} M_1)$ or $m_1 \in N_1$. Since 0 is a graded prime submodule of M_2 and $r_2s_2m_2 = 0$, we get either $r_2 \in (0 :_{R_2} M_2)$ or $s_2 \in (0 :_{R_2} M_2)$ or $m_2 = 0$. It is easy to see that in any of the above cases $(r_1,r_2)(s_1,s_2) \in (N_1 \times 0 :_R M)$ or $(r_1,r_2)(m_1,m_2) \in N_1 \times 0$ or $(s_1,s_2)\ (m_1,m_2) \in N_1 \times 0$. Thus $N_1 \times 0$ is a graded weakly 2-absorbing submodule of M_1. □

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