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The odd Frèchet inverse exponential distribution with application

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Abstract

We introduce a new distribution with two parameters called the odd Frèchet inverse exponential (OFIE) distribution. The OFIE model can be more flexible. The cumulative density function (cdf) and the probability density function (pdf) are investigated. Some of its statistical properties are studied. The maximum likelihood (ML) estimation is employed for OFIE parameters. The importance of the OFIE model is assessed using one real data set.

Keywords: Odd Frèchet family, inverse exponential distribution, moments, maximum likelihood.

2010 MSC: 60E05, 62E10, 62N05.

1. Introduction

The one parameter exponential (E) distribution describes the time between events in a Poisson process. Its discrete analogue is the geometric distribution. Apart from its usage in Poisson processes, it has been used extensively in the literature for life testing. The E distribution is memoryless and has a constant failure rate; this latter property makes the distribution unsuitable for real life problems with bathtub failure rates (see Singh et al. [16] for details) and inverted bathtub failure rates, hence the need to generalize the E distribution in order to increase its flexibility and capability to model some other real life problems.

Keller and Kamath [6] studied the inverse exponential (IE) distribution. It has an inverted bathtub failure rate and it is an important competitive model for the E distribution. It has been identified and discussed by Lin et al. [10] as a lifetime model. If X is a non-negative E random variable, then the distribution of a random variable $Y = 1/X$ follows an IE distribution. Hence, if X denotes a random variable, the cdf and pdf of the IE distribution with a scale parameter α are respectively given by

$$g\left(x{:}\alpha\right)=\frac{\alpha}{x^2}e^{-\frac{\alpha}{x}},\quad x,\alpha>0, \tag{1.1}$$

and

$$G\left(x{:}\alpha\right)=e^{-\frac{\alpha}{x}},\quad x,\alpha>0. \tag{1.2}$$

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In the last years, in the literature some statisticians attempts to increase the modeling capacity of the IE distribution. Beta IE distribution is studied by Singh and Goel [15], Kumaraswamy IE is studied by Oguntunde et al. [11], Oguntunde et al. [12] proposed transmuted IE distribution, Oguntunde et al. [13] introduced exponentiated generalized IE distribution, and Weibull-inverted exponential distribution is studied by Oguntunde et al. [14].

Recently, *odd Frèchet generated family of distributions* (OF-G) has been proposed by Haq and Elgarhy [7]. The cdf and pdf of OF-G are

$$F(x\colon \theta,\ \xi) = \int_0^{\left[\frac{G(x;\xi)}{1-G(x;\xi)}\right]} \frac{\theta}{x^{\theta+1}} e^{-x^{-\theta}} dx = e^{-\left[\frac{1-G(x;\xi)}{G(x;\xi)}\right]^{\theta}}, \quad x \in R,\ \theta > 0 \tag{1.3}$$

and

$$f(x:\ \theta,\ \xi) = \frac{\theta g(x;\xi)\left[1-G(x;\xi)\right]^{\theta-1}}{G(x;\xi)^{\theta+1}} e^{-\left[\frac{1-G(x;\xi)}{G(x;\xi)}\right]^{\theta}}, \tag{1.4}$$

where $g(x:\ \xi)$ considers a pdf of baseline distribution. Hereafter, a random variable X with density function (1.4) is denoted by $X \sim OF-G(\theta,\ \xi)$.

In this article, we introduce a new lifetime model called the OFIE distribution. This paper is arranged as follows. In Section 2, we study the OFIE distribution. Statistical properties are calculated in Section 3. In Section 4, The ML method is applied to calculate the estimates of the model parameters. Simulation results are carried out to estimate the model parameters of OFIE distribution in Section 5. The analyses of one real data set is employed in Section 6. Finally, conclusions are appeared in Section 7.

2. The OFIE model

The cdf of OFIE distribution with set of parameters $\varphi = (\alpha, \theta)$ is obtained by substituting (1.2) in (1.3) as follows

$$F(x;\ \theta,\ \alpha) = e^{-\left[e^{\frac{\alpha}{x}}-1\right]^{\theta}}, \quad x,\ \alpha, \theta > 0. \tag{2.1}$$

By inserting (1.1) and (1.2) in (1.4) we get the corresponding pdf to (2.1) which is given by

$$f(x;\ \theta,\ \alpha) = \frac{\theta\alpha}{x^2} e^{\frac{\alpha}{x}} \left[e^{\frac{\alpha}{x}}-1\right]^{\theta-1} e^{-\left[e^{\frac{\alpha}{x}}-1\right]^{\theta}}, \quad x, \alpha, \theta > 0. \tag{2.2}$$

Also, the survival function (sf), hrf, reversed hrf, and cumulative hrf of X are given, respectively, as follows:

$$R(x;\theta,\alpha) = 1 - e^{-\left[e^{\frac{\alpha}{x}}-1\right]^{\theta}}, \qquad h(x;\theta,\alpha) = \frac{\frac{\theta\alpha}{x^2} e^{\frac{\alpha}{x}} \left[e^{\frac{\alpha}{x}}-1\right]^{\theta-1} e^{-\left[e^{\frac{\alpha}{x}}-1\right]^{\theta}}}{1-e^{-\left[e^{\frac{\alpha}{x}}-1\right]^{\theta}}},$$

$$\tau(x;\theta,\alpha) = \frac{\theta\alpha}{x^2} e^{\frac{\alpha}{x}} \left[e^{\frac{\alpha}{x}}-1\right]^{\theta-1}, \qquad H(x;\theta,\alpha) = -\ln\left(1-e^{-\left[e^{\frac{\alpha}{x}}-1\right]^{\theta}}\right).$$

Figure 1 shows some descriptive pdf and hrf plots of $X \sim$ OFIE (φ) are illustrated below for specific parameter choices of φ.

From Figure 1, we conclude that pdf of OFIE distribution can be uni-model, decreasing and right skewed. Also, the hrf of OFIE distribution can be J-shaped, decreasing and increasing as seen from Figure 1.

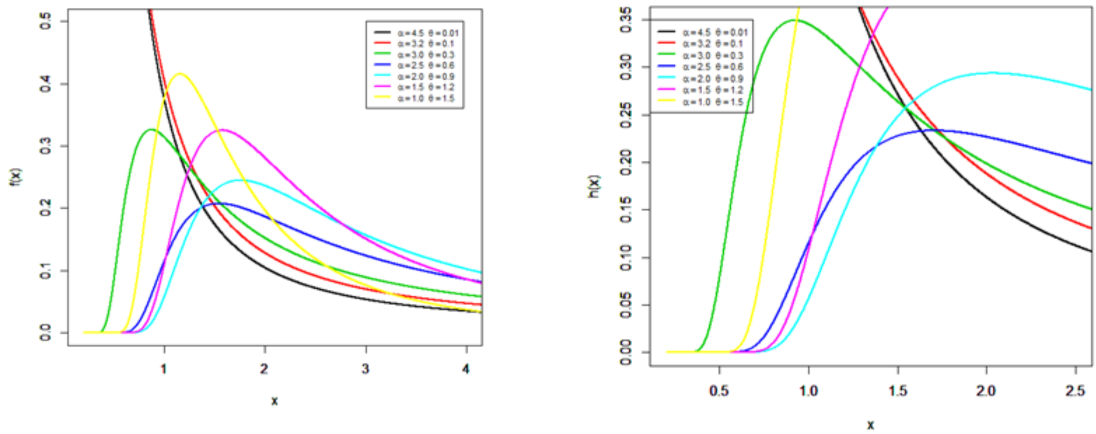

Figure 1: Plots of the pdf and hrf of the (OFIE) distribution for different values of parameters.

3. Some fundamental properties

Some fundamental properties of the OFIE distribution are obtained in this section.

3.1. *Important representation*

In this subsection important representations of the pdf and cdf for OFIE distribution are studied. Haq and Elgarhy [7] expressed the equation (1.4) as

$$f(x) = \sum_{k=0}^{\infty} \eta_k g(x, \xi) G(x, \xi)^k, \tag{3.1}$$

where

$$\eta_k = \sum_{i,j=0}^{\infty} \frac{\theta(-1)^{i+k}}{i!} \binom{\theta(i+1)+j}{j} \binom{\theta(i+1)+j-1}{k}.$$

By inserting equations (2.1) and (2.2) in equation (3.1) we can rewrite the OFIE as a linear combination of IE distribution as

$$f(x) = \sum_{k=0}^{\infty} \frac{w_k}{x^2} e^{-\frac{\alpha(k+1)}{x}}, \tag{3.2}$$

where $w_k = \alpha\eta_k$.

3.2. *Moments*

If X has the pdf (3.2), then its r^{th} moment can be calculated from the following equation

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x; \varphi) dx. \tag{3.3}$$

Substituting (3.2) into (3.3) yields

$$\mu_r' = E(X^r) = \sum_{k=0}^{\infty} w_k \int_0^{\infty} x^{r-2} e^{-\alpha(k+1)x^{-1}} dx.$$

Let $y = x^{-1}$, then

$$\mu_r' = \sum_{k=0}^{\infty} w_k \int_0^{\infty} y^{-r} e^{-\alpha(k+1)y} dx.$$

Then, $\mu_r^{'}$ becomes

$$\mu_r^{'} = \sum_{k=0}^{\infty} \frac{w_k \Gamma(1-r)}{[\alpha(k+1)]^{1-r}}, \quad r < 1.$$

The moment generating function of OFIE distribution can be calculated from the following equation

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) = \sum_{r,k=0}^{\infty} \frac{t^r}{r!} \frac{w_k \Gamma(1-r)}{[\alpha(k+1)]^{1-r}}, \quad r < 1.$$

The incomplete moments, say $\varphi_s(t)$, is

$$\varphi_s(t) = \int_0^t x^s f(x; \varphi) dx.$$

Using (3.2), then $\varphi_s(t)$ can be written as follows

$$\varphi_s(t) = \sum_{k=0}^{\infty} w_k \int_0^t x^{s-2} e^{-\alpha(k+1)x^{-1}} dx,$$

then,

$$\varphi_s(t) = \sum_{k=0}^{\infty} w_k \frac{\nu\left(1-s, \alpha(k+1)t^{-1}\right)}{(\alpha(k+1))^{1-s}}, \quad s < 1,$$

where $\nu(s,t) = \int_0^t x^{s-1} e^{-x} dx$ is the lower incomplete gamma function.

Further, the conditional moments, say $\tau_s(t)$, is given by

$$\tau_s(t) = \int_t^{\infty} x^s f(x; \varphi) dx.$$

Hence, by using pdf (3.2), we can write

$$\tau_s(t) = \sum_{k=0}^{\infty} w_k \int_t^{\infty} x^{s-2} e^{-\alpha(k+1)x^{-1}} dx,$$

then,

$$\tau_s(t) = \sum_{k=0}^{\infty} w_k \frac{\Gamma\left(1-s, \alpha(k+1)t^{-1}\right)}{(\alpha(k+1))^{1-s}}, \quad s < 1,$$

where $\Gamma(s,t) = \int_t^{\infty} x^{s-1} e^{-x} dx$ is the upper incomplete gamma function.

3.3. *Quantile function*

The quantile function, say $Q(u) = F^{-1}(u)$ of X is given by

$$Q(u) = \frac{\alpha}{\ln\left(1 + \left[\ln\left(\frac{1}{u}\right)\right]^{\frac{1}{\theta}}\right)}, \tag{3.4}$$

where, u is considered as a uniform random variable on the unit interval $(0, 1)$.

The median (M) can be calculated by setting $u = 0.5$ in (3.4). The M is given by

$$M = \frac{\alpha}{\ln\left(1 + [\ln(2)]^{\frac{1}{\theta}}\right)}.$$

4. ML estimation

Let $X_1, \dots, X_n$ be observed values from the OFIE distribution with set of parameters $\varphi = (\alpha,\ \theta)^T$. The total log-likelihood function for the vector of parameters φ can be expressed as

$$\ln L(\varphi) = n \ln \theta + n \ln \alpha - 2\sum_{i=1}^{n} \ln x_i + \alpha \sum_{i=1}^{n} \frac{1}{x_i} + (\theta - 1) \sum_{i=1}^{n} \ln\left(e^{\frac{\alpha}{x_i}} - 1\right) - \sum_{i=1}^{n} \left(e^{\frac{\alpha}{x_i}} - 1\right)^{\theta}.$$

The elements of the score function $U(\varphi) = (\,U_\alpha,\ U_\theta)$ are given by

$$U_\alpha = \frac{n}{\alpha} + \sum_{i=1}^{n} \frac{1}{x_i} + (\theta - 1) \sum_{i=1}^{n} \frac{\frac{1}{x_i} e^{\frac{\alpha}{x_i}}}{e^{\frac{\alpha}{x_i}} - 1} - \theta \sum_{i=1}^{n} \frac{1}{x_i} e^{\frac{\alpha}{x_i}} \left(e^{\frac{\alpha}{x_i}} - 1\right)^{\theta - 1},$$

and

$$U_\theta = \frac{n}{\theta} + \sum_{i=1}^{n} \ln\left(e^{\frac{\alpha}{x_i}} - 1\right) - \sum_{i=1}^{n} \left(e^{\frac{\alpha}{x_i}} - 1\right)^{\theta} \ln\left(e^{\frac{\alpha}{x_i}} - 1\right).$$

Then the ML estimators of the parameters α and θ are obtained by setting U_α and U_θ to be zero and solving them. Clearly, it is difficult to solve them, therefore we apply the Newton-Raphson's iteration method and use the computer packages such as Maple or R or other software.

5. Simulations

A simulation result is assessed to evaluate and compare the behavior of the estimators with respect to their *mean square errors* (MSEs). We generate 10000 random samples $X_1, \dots, X_n$ of sizes $n = (30, 50, 100)$ from OFIE distribution. Six selected sets of parameters are considered as: set 1:(0.5, 0.5), set 2:(0.5, 0.75), set 3:(0.5, 1.25), set 4:(0.75, 0.5), set 5:(0.75, 0.75), and set 6:(0.75, 1.25) .

The ML estimates of α and θ are calculated. Then, the MSEs of the estimates of the unknown parameters are calculated. Simulated outcomes are listed in Table 1 and the following observations are detected. The MSEs decrease as sample sizes increase for all estimates.

Table 1: The parameter estimation from OFIE model using MLE.

N	Par	set 1:(0.5, 0.5)		set 2:(0.5, 0.75)		set 3:(0.5, 1.25)	
		MLE	MSE	MLE	MSE	MLE	MSE
30	A	0.5172	0.0093	0.5266	0.0169	0.5898	0.0905
	Θ	0.5292	0.0197	0.8003	0.0482	1.3444	0.1244
50	A	0.5118	0.0059	0.5202	0.0096	0.5496	0.0338
	Θ	0.5173	0.0102	0.7812	0.0249	1.3026	0.0691
100	A	0.5068	0.0030	0.5081	0.0047	0.5186	0.0113
	Θ	0.5105	0.0050	0.7635	0.0108	1.2742	0.0306

n	Par	set 4:(0.75, 0.5)		set 5:(0.75, 0.75)		set 6:(0.75, 1.25)	
		MLE	MSE	MLE	MSE	MLE	MSE
30	A	0.7812	0.0201	0.7947	0.0394	0.8675	0.3122
	Θ	0.5313	0.0195	0.7986	0.0447	1.3251	0.1215
50	A	0.7646	0.0135	0.7751	0.0218	0.8214	0.0758
	Θ	0.5181	0.0106	0.7803	0.0243	1.3027	0.0663
100	A	0.7580	0.0064	0.7669	0.0111	0.7804	0.0276
	Θ	0.5103	0.0052	0.7658	0.0109	1.2715	0.0294

6. Application

In this section, we give an application to a real data set to evaluate the flexibility of the OFIE model. In order to compare the OFIE model with other fitted distributions, we compare the fits of the OFIE distribution with the beta generalized inverse Weibull geometric distribution (BGIWGc) (Elbatal et al., [5]), McDonald log-logistic (McLL) (Tahir et al., [17]), McDonald Weibull (McW) (Cordeiro et al., [4]), new modified Weibull (NMW) (Almalki and Yuan, [3]), transmuted complementary Weibull-geometric (TCWG) (Afify et al., [1]), beta Weibull (BW) (Lee et al., [9]), and exponentiated transmuted generalized Rayleigh (ETGR) (Afify et al., [2]) distributions.

The data set is taken from (Gross and Clark, [6]) on the relief times of twenty patients receiving an analgesic.

The ML estimates along with their standard error (SE) of the model parameters are provided in Tables 2 and 3. In the same tables, the analytical measures including Anderson Darling statistic (A^*), Cramér-von Mises statistic (W^*), Akaike Information Criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC) are presented.

Table 2 lists the MLEs of the model parameters and their corresponding standard whereas errors the values of $-2\log L$, AIC, CAIC, HQIC, A^*, and W^* are given in Table 3.

Table 2: MLEs and their standard errors (in parentheses) for the relief times data.

Model	MLE and SE					
OFIE (α, θ)	1.073 (0.0618)	2.929 (0.518)	-	-	-	-
BGIWGc (α, γ, θ, p, a, b)	19.1874 (33.3)	20.5968 (43.241)	1.4346 (0.837)	9.8485 (2.001)	39.2308×10^{-5} (63.252)	5.8015 (4.346)
McLL (α, β, a, b, c)	0.8811 (0.109)	2.0703 (3.693)	19.2254 (22.341)	32.0332 (43.077)	1.9263 (5.165)	-
McW (α, β, a, b, c)	2.7738 (6.38)	0.3802 (0.188)	79.108 (119.131)	17.8976 (39.511)	3.0063 (13.968)	-
NMW (α, β, γ, δ, θ)	0.1215 (0.056)	2.7837 (20.37)	8.227×10^{-5} (1.512×10^{-5})	0.0003 (0.025)	2.7871 (0.428)	-
TCWG (α, β, γ, λ)	43.6627 (45.459)	5.1271 (0.814)	0.2823 (0.042)	-0.2713 (0.656)	-	-
BW (α, β, a, b)	0.8314 (0.954)	0.6126 (0.34)	29.9468 (40.413)	11.6319 (21.9)	-	-
ETGR (α, β, λ, δ)	0.1033 (0.436)	0.6917 (0.086)	-0.342 (1.971)	23.5392 (105.371)	-	-

Table 3: Measures of goodness-of-fit statistics for the relief times data.

Model	**AIC**	**CAIC**	**BIC**	**HQIC**	A^*	W^*
OFIE	35.078	35.784	33.68	35.467	0.19209	0.0334
BGIWGc	43.662	50.124	39.468	44.828	0.24665	0.0434
McLL	43.854	48.14	40.359	44.826	0.46199	0.07904
McW	43.907	48.193	40.412	44.879	0.46927	0.08021
NMW	51.173	55.459	47.678	52.145	1.0678	0.17585
TCWG	41.607	44.274	38.811	42.385	0.43603	0.07252
BW	42.396	45.063	39.6	43.174	0.51316	0.0873
ETGR	44.856	47.523	42.06	45.634	0.79291	0.13629

Table 3 compares the fits of the OFIE distribution with the other known distributions. Table 3 shows that the OFIE model has the lowest values for AIC, CAIC, HQIC, A^*, and W^* among all fitted distributions. So, the OFIE is the best model. The fitted pdf and pp plot for the OFIE model are displayed in Figure 2. Figure 3 shows the estimated cdf and sf for the OFIE model. From these plots it is evident that the OFIE model provides close fit to the data.

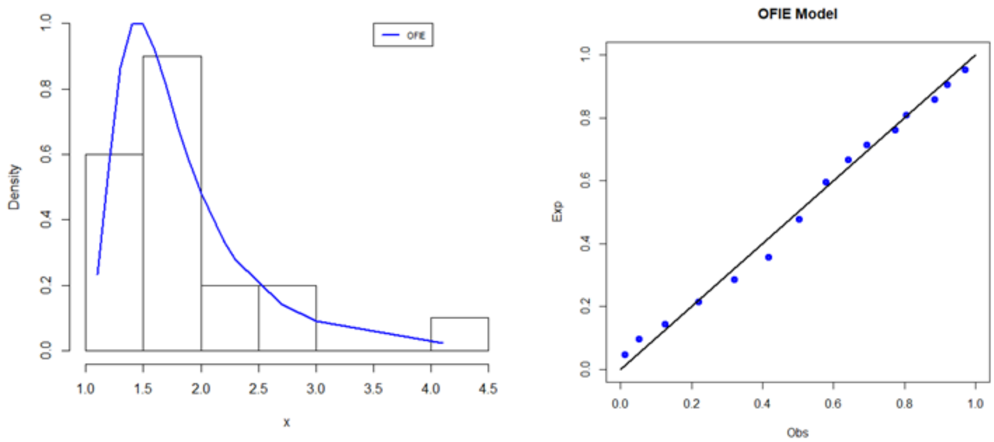

Figure 2: The empirical pdf and pp plot of the OFIE model.

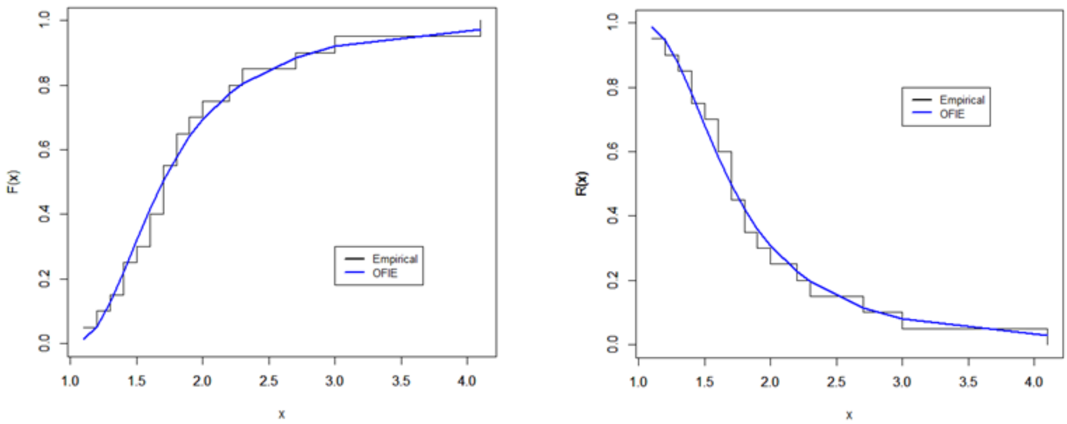

Figure 3: The empirical cdf and sf of the OFIE model.

7. Conclusions

In this article, we study a new two-parameter distribution named the odd Frèchet inverse exponential (OFIE) distribution. We derive explicit expressions for some of its statistical properties. We derive ML estimation. Simulation results are carried to evaluate the accuracy and performance of different estimators. The OFIE model provides better fits than some other competitive models using a real data set.

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