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Hybrid iterative methods for two asymptotically nonexpansive semigroups in Hilbert spaces

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Abstract

The main objective of this work is to modify two hybrid projection algorithm. First, we prove the strongly convergence to common fixed points of a sequence $\{x_n\}$ generated by the hybrid projection algorithm of two asymptotically nonexpansive mappings, second, we prove the strongly convergence of a sequence $\{x_n\}$ generated by the hybrid projection algorithm of two asymptotically nonexpansive semigroups. Our main results extend and improve the results of Dong et al. [Q.-L. Dong, S. N. He, Y. J. Cho, Fixed Point Theory Appl., **2015** (2015), 12 pages].

Keywords: Asymptotically nonexpansive mappings, asymptotically nonexpansive semigroup, fixed point.

2010 MSC: 46C05, 47D03, 47H09, 47H10, 47H20.

1. Introduction

Let H be a real Hilbert space, C a nonempty closed convex subset of H and $T : C \to C$ a mapping. Recall that a self-mapping f of C is a contraction if $\|f(x) - f(y)\| \leqslant \alpha\|x - y\|$ for some $\alpha \in (0, 1)$ and T is a nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$, and T is asymptotically nonexpansive [2] if there exists a sequence $\{k_n\}$ with $k_n \geqslant 1$ for all n and $\lim_{n\to\infty} k_n = 1$ and such that $\|T^n x - T^n y\| \leqslant k_n\|x - y\|$ for all $n \geqslant 1$ and $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by Fix(T) the set of fixed points of T, that is, $\text{Fix}(T) = \{x \in C : Tx = x\}$.

Recall also that a one–parameter family $\mathcal{T} = \{T(t)|0 \leqslant t < \infty\}$ of self–mappings of a nonempty closed convex subset C of a Hilbert space H is said to be a (continuous) Lipschitzian semigroup on C (see, e.g., [12]) if the following conditions are satisfied:

(i) $T(0)x = x, \quad x \in C$;

(ii) $T(s + t)(x) = T(s)T(t), \quad s, t \geqslant 0, \ x \in C$;

(iii) for each $x \in C$, the map $t \mapsto T(t)x$ is continuous on $[0, \infty)$;

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(iv) there exists a bounded measurable function $L : [0,\infty) \to [0,\infty)$ such that, for each $t > 0$

$$\|T(t)x - T(t)y\| \leqslant L_t\|x - y\|, \quad x, y \in C.$$

A Lipschitzian semigroup $\mathcal{T}$ is called nonexpansive (or a contraction semigroup) if $L_t = 1$ for all $t > 0$, and asymptotically nonexpansive semigroup if $\limsup_{t\to\infty} L_t \leqslant 1$, respectively. We use $\mathrm{Fix}(\mathcal{T})$ to denote the common fixed point set of the semigroup, that is $\mathrm{Fix}(\mathcal{T}) = \{x \in C : T(t)x = x, t > 0\}$.

Fixed point iteration processes for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann and Ishikawa iteration processes have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities, see [4, 7, 9–11]. However, Mann and Ishikawa iterations processes have only weak convergence even in Hilbert space, see [5, 11].

Very recently, Takahashi et al. [11] proved the following strong convergence theorems by the hybrid method for nonexpansive mappings and nonexpansive semigroup in Hilbert space.

Theorem 1.1 ([11]). *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leqslant \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \leqslant \alpha_n \leqslant a < 1$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

In 2008, Inchan and Plubtieng [3] modified Ishikawa iteration process for two asymptotically nonexpansive mappings, for C is a nonempty closed convex subset of a Hilbert space H, let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}$, define $\{x_n\}$ as follows:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n z_n, \\ z_n = \beta_n x_n + (1-\beta_n)S^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leqslant \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases}$$

where $\theta_n = (1-\alpha_n)[(t_n^2 - 1) + (1-\beta_n)t_n^2(s_n^2 - 1)](\mathrm{diam}C)^2 \to 0$ as $n \to \infty$ and $0 \leqslant \alpha_n \leqslant a < 1$ and $0 < b \leqslant \beta_n \leqslant c < 1$ for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to common fixed points of two asymptotically nonexpansive mappings.

In 2015, Dong et al. [1], introduced a hybrid algorithm. Let T and S be two nonexpansive mappings into itself such that $F(T) \cap F(S) \neq \emptyset$, the sequence generated as follows:

$$\begin{cases} x_0 \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ z_n = \beta_n[\gamma_n y_n + (1-\gamma_n)x_n] + (1-\beta_n)Sy_n, \\ C_n = \{z \in C : \sigma\|z_n - z\|^2 + (1-\sigma)\|y_n - z\|^2 \leqslant \|x_n - z\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0\rangle \leqslant 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}x_0, \ n \geqslant 0, \end{cases}$$

for each $n \geqslant 0$, where $\alpha_n, \beta_n \in [0,1]$, $\delta \in [0,1), \gamma_n \in [0,1], \sigma \in (0,1)$. Then proved that $\{x_n\}$ converges in norm to $P_{F(T)\cap F(S)}x_0$.

Inspired and motivated by above, the purpose of this paper is to extend the results of Dong et al. [1] for S and T are two asymptotically nonexpansive mappings then we consider

$$\begin{cases} x_0 \in C = C_1, \quad x_1 = P_{C_1}x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ z_n = \beta_n[\gamma_n y_n + (1-\gamma_n)x_n] + (1-\beta_n)S^n y_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\|^2 + \|y_n - z\|^2 \leqslant 2\|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \geqslant 0, \end{cases} \tag{1.1}$$

where $\alpha_n, \beta_n, \gamma_n \in [0,1]$ and

$$\theta_n = [(1-\beta_n)(s_n^2 - 1) + (1-\alpha_n)(t_n^2 - 1) + (1-\beta_n)s_n^2(1-\aleph_n)(t_n^2-1)](\text{diam}C)^2 \to 0,$$

as $n \to \infty$ (here $t_n \to 1$ and $s_n \to 1$ as $n \to \infty$). Then, under some contral conditions we show the strongly convergence of $\{x_n\}$.

2. Preliminaries

In this section, we collect and give some useful lemmas that will be used for our main result in the next section.

Lemma 2.1. *Let* H *be a real Hilbert space, then the following hold:*

(i) $\|x+y\|^2 \leqslant \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \ \forall x, y \in H$;

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \ t \in [0,1], \ \forall x, y \in H.$

Lemma 2.2 ([5])**.** *Let* C *be a nonempty bounded closed convex subset of real Hilbert space* H *and let*

$$\mathcal{T} := \{T(s) : 0 \leqslant s < \infty\},$$

an asymptotically nonexpansive semigroup on C*. If* $\{x_n\}$ *is a sequence in* C *satisfying the properties*

(i) $x_n \rightharpoonup z$;

(ii) $\limsup_{t\to\infty} \limsup_{n\to\infty} \|T(t)x_n - x_n\| = 0$;

then $z \in Fix(\mathcal{T})$.

Lemma 2.3 ([5])**.** *Let* C *be a nonempty bounded closed convex subset of real Hilbert space* H *and let*

$$\mathcal{T} := \{T(s) : 0 \leqslant s < \infty\},$$

an asymptotically nonexpansive semigroup on C*. Then for any* $u \geqslant 0$,

$$\limsup_{u\to\infty} \limsup_{t\to\infty} \sup_{x\in C} \|\frac{1}{t}\int_0^t T(s)x ds - T(u)(\frac{1}{t}\int_0^t T(s)x ds)\| = 0.$$

Lemma 2.4 ([6])**.** *Let* T *be an asymptotically nonexpansive mapping defined on a bounded convex subset* C *of a Hilbert space* H*. If* $\{x_n\}$ *is a sequence in* C *such that* $x_n \rightharpoonup x$ *and* $Tx_n - x_n \to 0$*, then* $x \in F(T)$.

Lemma 2.5 ([8])**.** *Let* C *be a nonempty closed convex subset of* H*. Let* $\{x_n\}$ *be a sequence in* H *and* $u \in H$*. Let* $q = P_C u$*. If* $\{x_n\}$ *is such that* $\omega_w(x_n) \subset C$ *and satisfies the condition*

$$\|x_n - u\| \leqslant \|u - q\|,$$

for all $n \geqslant 1$*, then* $x_n \to q$.

3. Main result

In this section we introduce two theorems. We first prove the strong convergence theorem of modified the hybrid method of of asymptotically nonexpansive mappings into $P_{F(T)\cap F(S)}x_0$. As the second part, we prove the strong convergence of modified the hybrid method of asymptotically nonexpansive semigroups into $P_{\mathfrak{I}}x_0$.

3.1. Strong convergence theorem of asymptotically nonexpansive mappings

In this section, we prove the strong convergence theorem of the algorithm (1.1) into $P_{F(T)\cap F(S)}x_0$.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and $T, S : C \to C$ be two asymptotically nonexpansive mappings with the sequences $\{t_n\}$ and $\{s_n\}$, respectively, such that $F(T)\cap F(S) \neq \emptyset$. Assume that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $[0,1]$ such that $\alpha_n, \beta_n \leqslant 1-\delta$ for some $\delta \in (0,1]$. Then the sequence $\{x_n\}$ generated by (1.1) converges in norm to $P_{F(T)\cap F(S)}x_0$.*

Proof. Putting $t_\infty \sup\{t_n : n \geqslant 1\} < \infty$ and $s_\infty \sup\{s_n : n \geqslant 1\} < \infty$. We first show by induction that $F(T)\cap F(S) \subseteq C_n$ for all $n \in \mathbb{N}$. It is obvious that $F(T)\cap F(S) \subseteq C_1$. Suppose that $F(T)\cap F(S) \subseteq C_k$ for each $k \in \mathbb{N}$. Let $u \in F(T)\cap F(S) \subseteq C_k$, then from Lemma 2.1, we have

$$\begin{aligned}
\|y_k - u\|^2 &= \|\alpha_k x_k + (1-\alpha_k)T^k x_k - u\|^2 \\
&= \|\alpha_k(x_k - u) + (1-\alpha_k)(T^k x_k - u)\|^2 \\
&= \alpha_k\|x_k - u\|^2 + (1-\alpha_k)\|T^k x_k - u\|^2 - \alpha_k(1-\alpha_k)\|x_k - T^k x_k\|^2 \\
&\leqslant \alpha_k\|x_k - u\|^2 + (1-\alpha_k)\|T^k x_k - u\|^2 \\
&\leqslant \alpha_k\|x_k - u\|^2 + (1-\alpha_k)t_k^2\|x_k - u\|^2 \\
&= \|x_k - u\|^2 - \|x_k - u\|^2 + \alpha_k\|x_k - u\|^2 + (1-\alpha_k)t_k^2\|x_k - u\|^2 \\
&= \|x_k - u\|^2 - (1-\alpha_k)\|x_k - u\|^2 + (1-\alpha_k)t_k^2\|x_k - u\|^2 \\
&= \|x_k - u\|^2 + (1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2.
\end{aligned} \tag{3.1}$$

Similarly, we note that from Lemma 2.1 and (3.1), we have

$$\begin{aligned}
\|z_k - u\|^2 &= \|\beta_k[\gamma_k y_k + (1-\gamma_k)x_k] + (1-\beta_k)S^k y_k - u\|^2 \\
&= \|\beta_k([\gamma_k y_k + (1-\gamma_k)x_k] - u) + (1-\beta_k)(S^k y_k - u)\|^2 \\
&= \beta_k\|[\gamma_k y_k + (1-\gamma_k)x_k] - u\|^2 + (1-\beta_k)\|S^k y_k - u\|^2 \\
&\leqslant \beta_k[\gamma_k\|y_k - u\|^2 + (1-\gamma_k)\|x_k - u\|^2] + (1-\beta_k)\|S^k y_k - u\|^2 \\
&\leqslant \beta_k\gamma_k\|y_k - u\|^2 + \beta_k(1-\gamma_k)\|x_k - u\|^2 + (1-\beta_k)s_k^2\|y_k - u\|^2 \\
&\leqslant \beta_k\gamma_k[\|x_k - u\|^2 + (1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2] + \beta_k(1-\gamma_k)\|x_k - u\|^2 \\
&\quad + (1-\beta_k)s_k^2[\|x_k - u\|^2 + (1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2] \\
&\leqslant \beta_k\|x_k - u\|^2 + (1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2 + (1-\beta_k)s_k^2\|x_k - u\|^2 \\
&= \|x_k - u\|^2 - (1-\beta_k)\|x_k - u\|^2 + (1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2 + (1-\beta_k)s_k^2\|x_k - u\|^2 \\
&\quad + (1-\beta_k)s_k^2(1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2 \\
&= \|x_k - u\|^2 + (1-\beta_k)(s_k^2 - 1)\|x_k - u\|^2 + (1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2 \\
&\quad + (1-\beta_k)s_k^2(1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2.
\end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we obtain that

$$\|z_k - u\|^2 + \|y_k - u\|^2 \leqslant \|x_k - u\|^2 + (1-\beta_k)(s_k^2 - 1)\|x_k - u\|^2 + (1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2$$

$$
\begin{aligned}
&+(1-\beta_k)s_k^2(1-\alpha_k)(t_k^2-1)\|x_k-u\|^2+\|x_k-u\|^2\\
&+(1-\alpha_k)(t_k^2-1)\|x_k-u\|^2\\
\leqslant\ & 2\|x_k-u\|^2+[(1-\beta_k)(s_k^2-1)+2(1-\alpha_k)(t_k^2-1)\\
&+(1-\beta_k)s_k^2(1-\alpha_k)](\mathrm{diam}C)^2\\
=\ & 2\|x_k-u\|^2+\theta_k,
\end{aligned}
$$

where $\theta_k=[(1-\beta_k)(s_k^2-1)+2(1-\alpha_k)(t_k^2-1)+(1-\beta_k)s_k^2(1-\alpha_k)](\mathrm{diam}C)^2\to 0$ as $n\to\infty$. It follows that $u\in C_{k+1}$ and then $F(T)\cap F(S)\subseteq C_n$ for all $n\in\mathbb{N}$. Next, we show that C_n is closed and convex for all $n\in\mathbb{N}$. It is obvious that $C_1=C$ is closed and convex. Suppose that C_k is closed and convex for each $k\in\mathbb{N}$. Let $\{z_m\}_{m=1}^{\infty}\subseteq C_{k+1}\subseteq C_k$ with $z_m\to z$ as $m\to\infty$. Since C_k is closed and $z_m\in C_{k+1}$, we have $z_m\in C_k$ and $\|z_k-z_m\|^2+\|y_k-z_m\|^2\leqslant 2\|z_m-x_k\|^2+\theta_k$. From Lemma 2.1, we have

$$
\begin{aligned}
\|z_k-z\|^2+\|y_k-z\|^2 &= \|z_k-z_m+z_m-z\|^2+\|y_k-z_m+z_m-z\|^2\\
&\leqslant [\|z_k-z_m\|^2+\|z_m-z\|^2+2\langle z_k-z_m,z_m-z\rangle]\\
&\quad+[\|y_k-z_m\|^2+\|z_m-z\|^2+2\langle y_k-z_m,z_m-z\rangle]\\
&= \|z_k-z_m\|^2+\|y_k-z_m\|^2+2\|z_m-z\|^2\\
&\quad+2(\|z_k-z_m\|\|z_m-z\|+\|y_k-z_m\|\|z_m-z\|)\\
&\leqslant 2\|x_k-z_m\|^2+\theta_k\\
&\quad+2(\|z_m-z\|^2+\|z_k-z_m\|\|z_m-z\|+\|y_k-z_m\|\|z_m-z\|).
\end{aligned}
$$

Taking $m\to\infty$, it follows that

$$\|z_k-z\|^2+\|y_k-z\|^2\leqslant 2\|x_k-z\|^2+\theta_k.$$

Then $z\in C_{k+1}$ and hence C_{k+1} is closed. Let $x,y\in C_{k+1}\subseteq C_k$ with $z=\alpha x+(1-\alpha)y$ where $\alpha\in[0,1]$. Since C_k is convex, $z\in C_k$. Thus, we have

$$\|z_k-x\|^2+\|y_k-x\|^2\leqslant 2\|x_k-x\|^2+\theta_k,$$

and

$$\|z_k-y\|^2+\|y_k-y\|^2\leqslant 2\|x_k-y\|^2+\theta_k.$$

Hence

$$
\begin{aligned}
\|z_k-z\|^2+\|y_k-z\|^2 &= \|z_k-(\alpha x+(1-\alpha)y)\|^2+\|y_k-(\alpha x+(1-\alpha)y)\|^2\\
&= \|\alpha(z_k-x)+(1-\alpha)(z_k-y)\|^2+\|\alpha(y_k-x)+(1-\alpha)(y_k-y)\|^2\\
&= \alpha\|z_k-x\|^2+(1-\alpha)\|z_k-y\|^2-\alpha(1-\alpha)\|x-y\|^2\\
&\quad+\alpha\|y_k-x\|^2+(1-\alpha)\|y_k-y\|^2-\alpha(1-\alpha)\|x-y\|^2\\
&= \alpha(\|z_k-x\|^2+\|y_k-x\|^2)+(1-\alpha)(\|z_k-y\|^2+\|y_k-y\|^2)\\
&\quad-2\alpha(1-\alpha)\|x-y\|^2\\
&\leqslant \alpha(2\|x_k-x\|^2+\theta_k)+(1-\alpha)(2\|x_k-y\|^2+\theta_k)-2\alpha(1-\alpha)\|x-y\|^2\\
&= 2[\alpha\|x_k-x\|^2+(1-\alpha)\|x_k-y\|^2-\alpha(1-\alpha)\|x-y\|^2]\\
&\quad+\alpha\theta_k+(1-\alpha)\theta_k\\
&= 2\|\alpha(x_k-x)+(1-\alpha)(x_k-y)\|^2+\theta_k\\
&= 2\|x_k-(\alpha x+(1-\alpha)y)\|^2\\
&= 2\|x_k-z\|^2+\theta_k.
\end{aligned}
$$

It follows that $z \in C_{k+1}$ and hence C_{k+1} is convex. Therefore, C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined. Since $x_n = P_{C_n}x_0$, it follows that

$$\langle x_0 - x_n, x_n - y\rangle \geqslant 0,$$

for all $y \in F(T) \cap F(S)$ and $n \in \mathbb{N}$. So $u \in F(T) \cap F(S)$, we have

$$\begin{aligned}0 &\leqslant \langle x_0 - x_n, x_n - u\rangle = -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - u\rangle \\ &\leqslant -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - u\|.\end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \leqslant \|x_0 - x_n\|\|x_0 - u\|,$$

and hence

$$\|x_0 - x_n\| \leqslant \|x_0 - u\|,$$

for all $u \in F(T) \cap F(S)$ and $n \in \mathbb{N}$. From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subseteq C_n$, we obtain that

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geqslant 0, \tag{3.3}$$

for all $n \in \mathbb{N}$. So, for all $x_{n+1} \in C_{n+1}$, for $n \in \mathbb{N}$, we have

$$\begin{aligned}0 &\leqslant \langle x_0 - x_n, x_n - x_{n+1}\rangle = -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\leqslant -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|.\end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \leqslant \|x_0 - x_n\|\|x_0 - x_{n+1}\|,$$

and hence

$$\|x_0 - x_n\| \leqslant \|x_0 - x_{n+1}\|,$$

for all $n \in \mathbb{N}$. Since $\{\|x_0 - x_n\|\}$ is bounded, $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Next, we claim that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0.$$

From (3.3), we have

$$\begin{aligned}\|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &\leqslant \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2.\end{aligned}$$

Since, $\lim_{n\to\infty} \|x_n - x_0\|$ exists, we have $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. Next, we now claim that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0 = \lim_{n\to\infty} \|Sx_n - x_n\|.$$

Since $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\|^2 + \|y_n - x_{n+1}\|^2 \leqslant 2\|x_n - x_{n+1}\|^2 + \theta_n.$$

From $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ and $\theta_n \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \|z_n - x_{n+1}\| = 0 = \lim_{n\to\infty} \|y_n - x_{n+1}\|,$$

which yields

$$\|z_n - x_n\| \leqslant \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0, \quad \text{as } n \to \infty,$$

and then we have

$$\|y_n - x_n\| \leqslant \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0, \quad \text{as } n \to \infty.$$

By definition of y_n, we have $y_n - x_n = (1-\alpha_n)(T^n x_n - x_n)$, we obtain

$$\|T^n x_n - x_n\| = \frac{1}{1-\alpha_n}\|y_n - x_n\|.$$

Since $\alpha_n \leqslant 1-\delta$, then we have

$$\|T^n x_n - x_n\| \to 0,$$

as $n \to \infty$. From $z_n = \beta_n[\gamma_n y_n + (1-\gamma_n)x_n] + (1-\beta_n)S^n y_n$, we have

$$\begin{aligned}\|S^n y_n - z_n\| &= \frac{\beta_n}{1-\beta_n}\|\gamma_n(y_n - z_n) + (1-\gamma_n)(x_n - z_n)\| \\ &\leqslant \frac{1}{1-\beta_n}(\gamma_n\|y_n - z_n\| + (1-\gamma_n)\|x_n - z_n\|),\end{aligned}$$

which yields

$$\|S^n y_n - z_n\| \to 0, \quad \text{as } n \to \infty,$$

and so

$$\begin{aligned}\|Tx_n - x_n\| &\leqslant \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leqslant t_\infty\|x_n - T^n x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + (1+t_\infty)\|x_n - x_{n+1}\| \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Similarly, we have

$$\|Sx_n - x_n\| \to 0, \quad \text{as } n \to \infty.$$

By Lemma 2.4, and boundedness of $\{x_n\}$, we have $\emptyset \neq \omega_w(x_n) \subset F(T) \cap F(S)$. Since

$$z_0 = P_{F(T)\cap F(S)}x_0, z_0 \in F(T) \cap F(S) \subset C,$$

and Lemma 2.5 guarantees the strong convergence of $\{x_n\}$ to $P_{F(T)\cap F(S)}x_0$. This completes the proof. □

3.2. *Strong convergence theorem of asymptotically nonexpansive semigroups*

First, we study some examples for relationship between a nonexpansive semigroup and an asymptotically nonexpansive semigroup for motivation of this work.

Example 3.2. Let $H_1 = H_2 = \mathbb{R}$ and let $\mathcal{T} := \{T(s) : 0 \leqslant s < \infty\}$, where

$$T(s)x = \frac{1}{1+2s}x, \quad \forall x \in \mathbb{R}.$$

We see that for any $x, y \in \mathbb{R}$

$$\|T(s)x - T(s)y\| = \|(\frac{1}{1+2s})x - (\frac{1}{1+2s})y\| = (\frac{1}{1+2s})\|x-y\|,$$

then we have $\mathcal{T}$ is nonexpansive semigroup. If $L_s = 1$ we have $\limsup_{s\to\infty} L_s = 1$, then $\mathcal{T}$ is asymptotically nonexpansive semigroup.

Example 3.3. Let $H_1 = H_2 = \mathbb{R}$ and let $\mathcal{T} := \{T(s) : 0 \leqslant s < \infty\}$, where

$$T(s)x = \frac{2+2s}{1+2s}x, \quad \forall x \in \mathbb{R}.$$

We see that for any $x, y \in \mathbb{R}$

$$\|T(s)x - T(s)y\| = \|(\frac{2+2s}{1+2s})x - (\frac{2+2s}{1+2s})y\| = (\frac{2+2s}{1+2s})\|x-y\|,$$

put $L_s = (\frac{2+2s}{1+2s})$ we have $\limsup_{s\to\infty} L_s = \limsup_{s\to\infty}(\frac{2+2s}{1+2s}) = 1$, then $\mathcal{T}$ is asymptotically nonexpansive semigroup. If we let $s = 1$ we have $\frac{2+2s}{1+2s} = \frac{4}{3} \not< 1$, then $\mathcal{T}$ is not necessarily nonexpansive semigroup.

From above example we see that a mapping $\mathcal{T}$ is a nonexpansive semigroup then $\mathcal{T}$ is asymptotically nonexpansive semigroup. But $\mathcal{T}$ is an asymptotically nonexpansive semigroup is not nonexpansive semigroup.

In 2008, Takahashi et al. [11] proved the strong convergence theorems by the hybrid method for nonexpansive semigroup in Hilbert space.

Theorem 3.4 ([11])**.** *Let* H *be a Hilbert space and* C *be a nonempty closed convex subset of* H*. Let*

$$\mathcal{T} = \{T(s) : 0 \leqslant s < \infty\},$$

be a one-parameter nonexpansive mapping semigroup on C *such that* $F(\mathcal{T}) \neq \emptyset$ *and let* $x_0 \in H$*. For* $C_1 = C$ *and* $u_1 = P_{C_1}x_0$*, define a sequence* $\{u_n\}$ *of* C *as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)u_n ds, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leqslant \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \leqslant \alpha_n \leqslant a < 1$, $0 < \lambda_n < \infty$ *for all* $n \in \mathbb{N}$ *and* $\lambda_n \to \infty$*. Then* $\{u_n\}$ *converges strongly to* $z_0 = P_{F(\mathcal{T})}x_0$*.*

In the same year, Inchan and Plubtieng [3] modified Ishikawa iteration process for two asymptotically nonexpansive semigroups for C is a nonempty closed convex subset of a Hilbert space H,

$$\mathcal{T} = \{T(t) : 0 \leqslant t < \infty\},$$

and

$$\mathcal{S} = \{S(t) : 0 \leqslant t < \infty\},$$

be two asymptotically nonexpansive semigroups on C such that $\mathcal{F} = F(\mathcal{T}) \cap F(\mathcal{S}) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}$, define $\{x_n\}$ as follows:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(t)z_n dt, \\ z_n = \beta_n x_n + (1-\beta_n)\frac{1}{t_n}\int_0^{t_n} S(t)x_n dt, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leqslant \|x_n - z\|^2 + \widetilde{\theta_n}\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases}$$

where

$$\widetilde{\theta_n} = (1-\alpha_n)[(\widetilde{t}_n^2 - 1) + (1-\beta_n)\widetilde{t}_n^2(\widetilde{s}_n^2 - 1)](\text{diam}C)^2 \to 0,$$

(here $\widetilde{t}_n = \frac{1}{t_n}\int_0^{t_n} L_t^T dt$ and $\widetilde{s}_n \frac{1}{s_n}\int_0^{s_n} L_t^S dt$ and $0 \leqslant \alpha_n \leqslant a < 1$ and $0 < b \leqslant \beta_n \leqslant c < 1$ for all $n \in \mathbb{N}$ and $\widetilde{t}_n \to \infty, \widetilde{s}_n \to \infty$).

As the second part of this work, we extend the results of Dang, et al. [1] for $\mathcal{T}$ is an asymptotically nonexpansive semigroup, then we consider

$$\begin{cases} x_0 \in C = C_1, x_1 = P_{C_1}x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(t)z_n dt, \\ z_n = \beta_n[\gamma_n y_n + (1-\gamma_n)x_n] + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt, \\ C_{n+1} = \{z \in C_n : \|z_n - z\|^2 + \|y_n - z\|^2 \leqslant 2\|x_n - z\|^2 + \widetilde{\theta_n}\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \geqslant 0, \end{cases} \tag{3.4}$$

where $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ and

$$\widetilde{\theta_n} = [(1-\beta_n)(\widetilde{s}_n^2 - 1) + (1-\alpha_n)(\widetilde{t}_n^2 - 1) + (1-\beta_n)\widetilde{s}_n^2(1-\alpha_n)(\widetilde{t}_n^2 - 1)](\text{diam}C)^2 \to 0,$$

as $n \to \infty$ (here $\widetilde{t}_n = \frac{1}{t_n}\int_0^{t_n} L_t^T dt \to 1$ and $\widetilde{s}_n \frac{1}{s_n}\int_0^{s_n} L_t^S dt \to 1$ as $n \to \infty$).

Theorem 3.5. *Let* H *be a Hilbert space and let* C *be a nonempty closed bounded subset of* H. *Let*

$$\mathfrak{T} = \{T(t) : 0 \leqslant t < \infty\},$$

and

$$\mathfrak{S} = \{S(t) : 0 \leqslant t < \infty\},$$

be two asymptotically nonexpansive semigroups on C *such that* $\mathfrak{F} = F(\mathfrak{T}) \cap F(\mathfrak{S}) \neq \emptyset$ *and let* $x_0 \in C$. *Let* $C_1 = C, x_1 = P_{C_1}x_0$ *and* $\{x_n\}$ *be a sequence generated by* (3.4) *with satisfies* $\alpha_n, \beta_n, \gamma_n \in [0, 1]$, $0 \leqslant \alpha_n \leqslant a < 1$ *and* $0 < b \leqslant \beta_n \leqslant c < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $t_n \to \infty, s_n \to \infty$. *Then* $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathfrak{F}}x_0$.

Proof. First observe that $\mathfrak{F} \subseteq C_n$ for all $n \in \mathbb{N}$. For $\mathfrak{F} \subset C = C_1$ suppose that $\mathfrak{F} \subset C_k$ for each $k \in \mathbb{N}$. Let $u \in \mathfrak{F} \subset C_k$. Then we have

$$\begin{aligned}
\|y_k - u\|^2 &= \|\alpha_k x_k + (1-\alpha_k)\frac{1}{t_k}\int_0^{t_k} T(t)x_k dt - u\|^2 \\
&= \|\alpha_k(x_k - u) + (1-\alpha_k)(\frac{1}{t_k}\int_0^{t_k} T(t)x_k dt - u)\|^2 \\
&\leqslant \alpha_k\|x_k - u\|^2 + (1-\alpha_k)\|\frac{1}{t_k}\int_0^{t_k} T(t)x_k dt - u\|^2 \\
&\leqslant \alpha_k\|x_k - u\|^2 + (1-\alpha_k)(\frac{1}{t_k}\int_0^{t_k} \|T(t)x_k - u\| dt)^2 \\
&\leqslant \alpha_k\|x_k - u\|^2 + (1-\alpha_k)(\frac{1}{t_k}\int_0^{t_k} L_t^T\|x_k - u\| dt)^2 \\
&\leqslant \alpha_k\|x_k - u\|^2 + (1-\alpha_k)(\frac{1}{t_k}\int_0^{t_k} L_t^T dt)^2\|x_k - u\|^2 \\
&= \|x_k - u\|^2 + (1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2.
\end{aligned} \tag{3.5}$$

By Lemma 2.1 again, we have

$$\begin{aligned}
\|z_k - u\|^2 &= \|\beta_k[\gamma_k y_k + (1-\gamma_k)x_k] + (1-\beta_k)\frac{1}{s_k}\int_0^{s_k} S(t)y_k dt - u\|^2 \\
&= \|\beta_k([\gamma_k y_k + (1-\gamma_k)x_k] - u) + (1-\beta_k)(\frac{1}{s_k}\int_0^{s_k} S(t)y_k dt - u)\|^2 \\
&\leqslant \beta_k\|[\gamma_k y_k + (1-\gamma_k)x_k] - u\|^2 + (1-\beta_k)\|\frac{1}{s_k}\int_0^{s_k} S(t)y_k dt - u\|^2 \\
&\leqslant \beta_k\|\gamma_k(y_k - u) + (1-\gamma_k)(x_k - u)\|^2 + (1-\beta_k)(\frac{1}{s_k}\int_0^{s_k} \|S(t)y_k - u\| dt)^2 \\
&\leqslant \beta_k\gamma_k\|y_k - u\|^2 + \beta_k(1-\gamma_k)\|x_k - u\|^2 + (1-\beta_k)(\frac{1}{s_k}\int_0^{s_k} L_t^S\|y_k - u\| dt)^2 \\
&\leqslant \beta_k\gamma_k\|y_k - u\|^2 + \beta_k(1-\gamma_k)\|x_k - u\|^2 + (1-\beta_k)(\frac{1}{s_k}\int_0^{s_k} L_t^S dt)^2\|y_k - u\|^2 \\
&\leqslant \beta_k\gamma_k[\|x_k - u\|^2 + (1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2] + \beta_k(1-\gamma_k)\|x_k - u\|^2 \\
&\quad + (1-\beta_k)\widetilde{s_k}^2[\|x_k - u\|^2 + (1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2] \\
&\leqslant \beta_k\|x_k - u\|^2 + (1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2 + (1-\beta_k)\widetilde{s_k}^2\|x_k - u\|^2 \\
&\quad + (1-\beta_k)\widetilde{s_k}^2(1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2 \\
&= \|x_k - u\|^2 + (1-\beta_k)(\widetilde{s_k}^2 - 1)\|x_k - u\|^2 + (1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2 \\
&\quad + (1-\beta_k)\widetilde{s_k}^2(1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2.
\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we obtain that

$$\begin{aligned}
\|z_k-u\|^2+\|y_k-u\|^2 &\leqslant \|x_k-u\|^2+(1-\beta_k)(\widetilde{s_k}^2-1)\|x_k-u\|^2+(1-\alpha_k)(\widetilde{t_k}^2-1)\|x_k-u\|^2 \\
&\quad+(1-\beta_k)\widetilde{s_k}^2(1-\alpha_k)(\widetilde{t_k}^2-1)\|x_k-u\|^2+\|x_k-u\|^2 \\
&\quad+(1-\alpha_k)(\widetilde{t_k}^2-1)\|x_k-u\|^2+(1-\beta_k)s_k^2(1-\alpha_k)(t_k^2-1)\|x_k-u\|^2 \\
&\quad+\|x_k-u\|^2+(1-\alpha_k)(t_k^2-1)\|x_k-u\|^2 \\
&\leqslant 2\|x_k-u\|^2+[(1-\beta_k)(\widetilde{s_k}^2-1)+2(1-\alpha_k)(\widetilde{t_k}^2-1) \\
&\quad+(1-\beta_k)\widetilde{s_k}^2(1-\alpha_k)(\widetilde{t_k}^2-1)](\mathrm{diam}C)^2 \\
&= 2\|x_k-u\|^2+\widetilde{\theta_k},
\end{aligned}$$

where

$$\widetilde{\theta_k}=[(1-\beta_k)(\tilde{s}_k^2-1)+2(1-\alpha_k)(\tilde{t}_k^2-1)+(1-\beta_k)\tilde{s}_k^2(1-\alpha_k)(\tilde{t}_k^2-1)](\mathrm{diam}C)^2\to 0,$$

as $n\to\infty$ (here $\tilde{t}_k=\frac{1}{t_k}\int_0^{t_k}L_t^T dt\to 1$ and $\tilde{s}_k\frac{1}{s_k}\int_0^{s_k}L_t^S dt\to 1$ as $k\to\infty$). It follows that $u\in C_{k+1}$ and then $F(T)\cap F(S)\subseteq C_n$ for all $n\in\mathbb{N}$. Again, by using the same argument in the proof in Theorem 3.1, we can show that C_n is closed and convex for all $n\in\mathbb{N}$. It is obvious that $C_1=C$ is closed and convex. Suppose that C_k is closed and convex for each $k\in\mathbb{N}$. Let $\{z_m\}_{m=1}^{\infty}\subseteq C_{k+1}\subseteq C_k$ with $z_m\to z$ as $m\to\infty$. Since C_k is closed and $z_m\in C_{k+1}$, we have $z\in C_k$ and $\|z_k-z_m\|^2\leqslant\|z_m-x_k\|^2+\tilde{\theta}_k$. From Lemma 2.1, we have

$$\begin{aligned}
\|z_k-z\|^2+\|y_k-z\|^2 &= \|z_k-z_m+z_m-z\|^2+\|y_k-z_m+z_m-z\|^2 \\
&\leqslant [\|z_k-z_m\|^2+\|z_m-z\|^2+2\langle z_k-z_m,z_m-z\rangle] \\
&\quad+[\|y_k-z_m\|^2+\|z_m-z\|^2+2\langle y_k-z_m,z_m-z\rangle] \\
&= \|z_k-z_m\|^2+\|y_k-z_m\|^2+2\|z_m-z\|^2 \\
&\quad+2(\|z_k-z_m\|\|z_m-z\|+\|y_k-z_m\|\|z_m-z\|) \\
&\leqslant 2\|x_k-z_m\|^2+\tilde{\theta}_k+2(\|z_m-z\|^2 \\
&\quad+\|z_k-z_m\|\|z_m-z\|+\|y_k-z_m\|\|z_m-z\|).
\end{aligned}$$

Taking $m\to\infty$, it follows that

$$\|z_k-z\|^2+\|y_k-z\|^2\leqslant 2\|x_k-z\|^2+\tilde{\theta}_k.$$

Then $z\in C_{k+1}$ and hence C_{k+1} is closed. Let $x,y\in C_{k+1}\subseteq C_k$ with $z=\alpha x+(1-\alpha)y$ where $\alpha\in[0,1]$. Since C_k is convex, $z\in C_k$. Thus, we have

$$\|z_k-x\|^2+\|y_k-x\|^2\leqslant 2\|x_k-x\|^2+\tilde{\theta}_k,$$

and

$$\|z_k-y\|^2+\|y_k-y\|^2\leqslant 2\|x_k-y\|^2+\tilde{\theta}_k.$$

Hence

$$\begin{aligned}
\|z_k-z\|^2+\|y_k-z\|^2 &= \|z_k-(\alpha x+(1-\alpha)y)\|^2+\|y_k-(\alpha x+(1-\alpha)y)\|^2 \\
&= \|\alpha(z_k-x)+(1-\alpha)(z_k-y)\|^2+\|\alpha(y_k-x)+(1-\alpha)(y_k-y)\|^2 \\
&= \alpha\|z_k-x\|^2+(1-\alpha)\|z_k-y\|^2-\alpha(1-\alpha)\|x-y\|^2 \\
&\quad+\alpha\|y_k-x\|^2+(1-\alpha)\|y_k-y\|^2-\alpha(1-\alpha)\|x-y\|^2
\end{aligned}$$

$$\begin{aligned}
&= \alpha(\|z_k - x\|^2 + \|y_k - x\|^2) + (1-\alpha)(\|z_k - y\|^2 + \|y_k - y\|^2) \\
&\quad - 2\alpha(1-\alpha)\|x - y\|^2 \\
&\leqslant \alpha(2\|x_k - x\|^2 + \widetilde{\theta}_k) + (1-\alpha)(2\|x_k - y\|^2 + \widetilde{\theta}_k) - 2\alpha(1-\alpha)\|x - y\|^2 \\
&= 2[\alpha\|x_k - x\|^2 + (1-\alpha)\|x_k - y\|^2 - \alpha(1-\alpha)\|x - y\|^2] + \alpha\widetilde{\theta}_k + (1-\alpha)\widetilde{\theta}_k \\
&= 2\|\alpha(x_k - x) + (1-\alpha)(x_k - y)\|^2 + \widetilde{\theta}_k \\
&= 2\|x_k - (\alpha x + (1-\alpha)y)\|^2 \\
&= 2\|x_k - z\|^2 + \widetilde{\theta}_k.
\end{aligned}$$

It follows that $z \in C_{k+1}$ and hence C_{k+1} is convex. Therefore, C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined. Since $x_n = P_{C_n}x_0$, it follows that

$$\langle x_0 - x_n, x_n - y\rangle \geqslant 0,$$

for all $y \in F(T) \cap F(S)$ and $n \in \mathbb{N}$. So $u \in F(T) \cap F(S)$, we have

$$\begin{aligned}
0 \leqslant \langle x_0 - x_n, x_n - u\rangle &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - u\rangle \\
&\leqslant -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - u\|.
\end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \leqslant \|x_0 - x_n\|\|x_0 - u\|,$$

and hence

$$\|x_0 - x_n\| \leqslant \|x_0 - u\|,$$

for all $u \in F(T) \cap F(S)$ and $n \in \mathbb{N}$. From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subseteq C_n$, we obtain that

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geqslant 0, \tag{3.7}$$

for all $n \in \mathbb{N}$. So, for all $x_{n+1} \in C_{n+1}$, for $n \in \mathbb{N}$, we have

$$\begin{aligned}
0 \leqslant \langle x_0 - x_n, x_n - x_{n+1}\rangle &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\
&\leqslant -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|.
\end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \leqslant \|x_0 - x_n\|\|x_0 - x_{n+1}\|,$$

and hence

$$\|x_0 - x_n\| \leqslant \|x_0 - x_{n+1}\|,$$

for all $n \in \mathbb{N}$. Since $\{\|x_0 - x_n\|\}$ is bounded, $\lim_{n\to\infty}\|x_n - x_0\|$ exists. Next, we claim that

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0.$$

From (3.7), we have

$$\begin{aligned}
\|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&\leqslant \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\
&= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2.
\end{aligned}$$

Since, $\lim_{n\to\infty}\|x_n - x_0\|$ exists, we have $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$. Since $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\|^2 + \|y_n - x_{n+1}\|^2 \leqslant 2\|x_n - x_{n+1}\|^2 + \widetilde{\theta}_n.$$

From $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$ and $\theta_n \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty}\|z_n - x_{n+1}\| = 0 = \lim_{n\to\infty}\|y_n - x_{n+1}\|,$$

which yields

$$\|z_n - x_n\| \leqslant \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0, \quad \text{as } n \to \infty,$$

and then we have

$$\|y_n - x_n\| \leqslant \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0, \quad \text{as } n \to \infty.$$

We now claim that

$$\limsup_{r\to\infty}\limsup_{n\to\infty}\|T(r)x_n - x_n\| = 0 = \limsup_{r\to\infty}\limsup_{n\to\infty}\|S(r)x_n - x_n\|.$$

Indeed, by definition of y_n and $x_{n+1} \subset C_n$ we have

$$\|\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt - x_n\| = \frac{1}{1-\alpha_n}\|y_n - x_n\| \to 0, \quad \text{as } n \to \infty.$$

From $z_n = \beta_n[\gamma_n y_n + (1-\gamma_n)x_n] + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt$, we have

$$\begin{aligned}\|\frac{1}{s_n}\int_0^{s_n} S(t)y_n dt - z_n\| &= \frac{\beta_n}{1-\beta_n}\|\gamma_n(y_n - z_n) + (1-\gamma_n)(x_n - z_n)\| \\ &\leqslant \frac{\beta_n}{1-\beta_n}(\gamma_n\|y_n - z_n\| + (1-\gamma_n)\|x_n - z_n\|) \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

It follows that

$$\begin{aligned}\|\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt - x_n\| &\leqslant \|\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt - \frac{1}{s_n}\int_0^{s_n} S(t)y_n dt\| + \|\frac{1}{s_n}\int_0^{s_n} S(t)y_n dt - z_n\| + \|z_n - x_n\| \\ &\leqslant \frac{1}{s_n}\int_0^{s_n}\|S(t)x_n - S(t)y_n\|dt + \|\frac{1}{s_n}\int_0^{s_n} S(t)y_n dt - z_n\| + \|z_n - x_n\| \\ &\leqslant \frac{1}{s_n}\int_0^{s_n} L_t^S dt\|x_n - y_n\| + \|\frac{1}{s_n}\int_0^{s_n} S(t)y_n dt - z_n\| + \|z_n - x_n\| \\ &\leqslant \widetilde{s}_n\|x_n - y_n\| + \|\frac{1}{s_n}\int_0^{s_n} S(t)y_n dt - z_n\| + \|z_n - x_n\| \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

For all $0 \leqslant r < \infty$, we note that

$$\begin{aligned}\|S(r)x_n - x_n\| &\leqslant \|S(r)x_n - S(r)(\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt)\| + \|S(r)(\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt) - \frac{1}{s_n}\int_0^{s_n} S(t)x_n dt\| \\ &\quad + \|\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt - x_n\| \\ &\leqslant (L_\infty + 1)\|\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt - x_n\| + \|S(r)(\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt) - \frac{1}{s_n}\int_0^{s_n} S(t)x_n dt\| \\ &:= (L_\infty + 1)A_n^S(r) + B_n^S(r),\end{aligned}$$

where $A_n^S := \|\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt - x_n\|$ and $B_n^S := \|S(r)(\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt) - \frac{1}{s_n}\int_0^{s_n} S(t)x_n dt\|$. By Lemma 2.3, we have $\limsup_{n\to\infty} A_n^S(r) = 0 = \limsup_{n\to\infty} B_n^S(r)$. We can deduce that for all $0 \leqslant r < \infty$,

$$\|T(r)x_n - x_n\| \leqslant \|T(r)x_n - T(r)(\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt)\| + \|T(r)(\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt) - \frac{1}{t_n}\int_0^{t_n} T(t)x_n dt\|$$

$$
\begin{aligned}
&+\|\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt - x_n\| \\
&\leqslant (L_\infty+1)\|\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt - x_n\| + \|T(r)(\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt) - \frac{1}{t_n}\int_0^{t_n} T(t)x_n dt\| \\
&:= (L_\infty+1)A_n^T(r) + B_n^T(r),
\end{aligned}
$$

where $A_n^T(r) := \|\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt - x_n\|$ and $B_n^T(r) := \|T(r)(\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt) - \frac{1}{t_n}\int_0^{t_n} T(t)x_n dt\|$. By Lemma 2.3, we have $\limsup_{n\to\infty} A_n^T(r) = 0 = \limsup_{n\to\infty} B_n^T(r)$. Then we obtain

$$
\limsup_{r\to\infty}\limsup_{n\to\infty}\|T(r)x_n - x_n\| = 0 = \limsup_{r\to\infty}\limsup_{n\to\infty}\|S(r)x_n - x_n\|.
$$

We note by Lemma 2.2, that every weak limit of $\{x_n\}$ is a member of $\mathfrak{F}$. From $x_n \rightharpoonup z \in P_{\mathfrak{F}}x_0$, we have $x_0 - x_n \rightharpoonup x_0 - z_0$, form H satisfies the Kadec-Klee property, it follows that

$$
x_0 - x_n \to x_0 - z_0.
$$

So, we have

$$
\|x_n - z_0\| = \|x_n - x_0 - (z_0 - z_0)\| \to 0, \quad \text{as } n \to \infty.
$$

Hence, $x_n \to z_0$. This completes the proof. □

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