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O_1-convergence in partially ordered sets

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Abstract

Based on the introduction of notions of S^*-doubly continuous posets and B-topology in [T. Sun, Q. G. Li, L. K. Guo, Topology Appl., **207** (2016), 156–166], in this paper, we further propose the concept of B-consistent S^*-doubly continuous posets and prove that the O_1-convergence in a poset is topological if and only if the poset is a B-consistent S^*-doubly continuous poset. This is the main result which can be seen as a sufficient and necessary condition for the O_1-convergence in a poset being topological. Additionally, in order to present natural examples of posets which satisfy such condition, several special sub-classes of B-consistent S^*-doubly continuous posets are investigated.

Keywords: O_1-convergence, B-topology, S^*-doubly continuous poset, B-consistent S^*-doubly continuous poset.

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1. Introduction and preliminaries

The concept of O-convergence in partially ordered sets (posets, for short) was introduced by Birkhoff [1], Frink [5], and Mcshane [9]. It is defined as follows: a net $(x_i)_{i \in I}$ in a poset P is said to O-*converges to* $x \in P$ (we write $(x_i)_{i\in I} \xrightarrow{O} x$ in this paper) if there exist subsets D and F of P such that

(1) D is directed and F is filtered;
(2) $\sup D = x = \inf F$;
(3) for every $d \in D$ and $e \in F$, $d \leqslant x_i \leqslant e$ holds eventually, i.e., there exists $i_0 \in I$ such that $d \leqslant x_i \leqslant e$ for all $i \geqslant i_0$.

As what has been showed in [18], the O-convergence (Note: in [18], the O-convergence is called order-convergence) in a general poset P may not be topological, i.e., it is possible that P can not be endowed with a topology such that the O-convergence and the associated topological convergence are consistent. Hence, much works has been done to characterize those special posets in which the O-convergence is topological. The most recent result in [13] shows that the O-convergence in a poset which satisfies condition ($\triangle$) is

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topological if and only if the poset is $\mathcal{O}$-doubly continuous. This means that for a special class of posets, a sufficient and necessary condition for O-convergence being topological is obtained. For more results on O-convergence, the reader can refer to [8, 10, 14, 17].

The O_1-convergence is a special type of O-convergence in posets and was also introduced by Birkhoff [1]. In fact, the O_1-convergence in a general poset may also not be topological. To search those special posets in which the O_1-convergence is topological, Riecanová [11] proved that the O_1-convergence in any separable strongly compactly atomistic orthomodular lattice is topological. This result clarified a special condition of posets under which the O_1-convergence is topological. However, to the best of our knowledge, the equivalent characterization to the O_1-convergence in general posets being topological is still unknown.

We continue to consider the O_1-convergence in poset with the aim of establishing the equivalent characterization to the O_1-convergence in general posets being topological. More specifically, given a general poset P, we hope to clarify the order-theoretical condition of P which is sufficient and necessary for the associated O_1-convergence being topological. To this end, in Section 2, we propose the notion of B-consistent S^*-doubly continuous posets by introducing condition ($\star$), and then obtain the main result of this paper, that is: given a poset P, the O_1-convergence in P is topological if and only if P is a B-consistent S^*-doubly continuous poset if and only if the O_1-convergence and the topological convergence with respect to the B-topology on P are consistent. In Section 3, we study several special sub-classes of B-consistent S^*-doubly continuous posets.

Some conventional notions will be used in the sequel. Throughout this paper, given a set X, $F \sqsubseteq X$ means that F is a finite subset of X. Let P be a poset and $x \in P$, $\uparrow x$ and $\downarrow x$ are always used to denote the *principal filter* $\{y \in P : y \geqslant x\}$ and the *principal ideal* $\{z \in P : z \leqslant x\}$ of P, respectively. P is said to be *bounded* if it has the least element $\perp$ and the largest element $\top$. Given a poset P and $A \subseteq P$, by writing $\sup A$ we mean that the least upper bound of A in P exists and equals to $\sup A \in P$; dually, by writing $\inf A$ we mean that the greatest lower bound of A in P exists and equals to $\inf A \in P$. And the set A is called *an upper set* if $A = \uparrow A = \{b \in P : (\exists a \in A)\ a \leqslant b\}$, *the lower set* is defined dually.

Given a topological space $(X, \mathcal{T})$ [4, 7] and a net $(x_i)_{i \in I}$ in X, we take $(x_i)_{i\in I} \xrightarrow{\mathcal{T}} x \in X$ to mean that the net $(x_i)_{i\in I}$ converges to x with respect to the topology $\mathcal{T}$.

To make this paper self-contained, we briefly review the following notions and propositions.

Definition 1.1 ([6])**.** Let P be a poset and $x, y, z \in P$. We say $y \ll x$ if for every directed subset D of P with $x \leqslant \sup D$, there exists $d \in D$ such that $y \leqslant d$; dually, we say $z \rhd x$ if for every filtered subset F of P with $\inf F \leqslant x$, there exists $e \in F$ such that $e \leqslant z$.

Remark 1.2 ([6]). Let P be a poset and $x, y, z, a, b, c, d \in P$. Then

(1) $x \ll y \Rightarrow x \leqslant y$ and $z \rhd x \Rightarrow z \geqslant x$;
(2) $a \leqslant x \ll y \leqslant b \Rightarrow a \ll b$ and $c \geqslant z \rhd x \geqslant d \Rightarrow c \rhd d$.

Definition 1.3 ([16])**.** A poset P is called a *doubly continuous poset* if for any $x \in P$, the set $\{y \in P : y \ll x\}$ is directed, the set $\{z \in P : z \rhd x\}$ is filtered and $\sup\{y \in P : y \ll x\} = x = \inf\{z \in P : z \rhd x\}$.

Remark 1.4 ([15]). Let P be a doubly continuous poset and $x, y, z \in P$. If $x \ll y$, then there exists $a \in P$ such that $x \ll a \ll y$; dually, if $z \rhd x$, then there exists $b \in P$ such that $z \rhd b \rhd x$.

Example 1.5 ([3, 19])**.**

(1) Chains, antichains, and finite posets are all doubly continuous posets.
(2) Every completely distributive lattice is a doubly continuous lattice. But the converse may not be true. For example, every non-distributive finite lattice is a doubly continuous lattice, but not a completely distributive lattice. In fact, it has been shown that L is a completely distributive lattice if and only if it is a distributive doubly continuous lattice.

Definition 1.6 ([12]). Let P be a poset and $x, y, z \in P$. We define $y \ll_S x$ if for every directed subset D of P with $\sup D = x$, there exists $d \in D$ such that $y \leqslant d$; dually, we define $z \rhd_S x$ if for every filtered subset F of P with $\inf F = x$, there exists $e \in F$ such that $z \geqslant e$.

In what follows, for a poset P and $x \in P$, we denote

(1) $\Downarrow_S x = \{a \in P : a \ll_S x\}$, $\Uparrow_S x = \{b \in P : x \ll_S b\}$;
(2) $\downdownarrows_S x = \{c \in P : x \rhd_S c\}$, $\upuparrows_S x = \{d \in P : d \rhd_S x\}$.

Definition 1.7 ([12]). A poset P is said to be S-*doubly continuous* if for every $x \in P$, the sets $\Downarrow_S x$ and $\upuparrows_S x$ are directed and filtered, respectively, and $\sup \Downarrow_S x = x = \inf \upuparrows_S x$.

Definition 1.8 ([12]). An S-doubly continuous poset P is said to be S^*-*doubly continuous* if for every $x \in P$, $y \in \Downarrow_S x$ and $z \in \upuparrows_S x$, there exist $y_0 \in \Downarrow_S x$ and $z_0 \in \upuparrows_S x$ such that $\uparrow y_0 \cap \downarrow z_0 \subseteq \Uparrow_S y \cap \downdownarrows_S z$.

Proposition 1.9 ([12]). *If* P *is a doubly continuous poset, then* P *is* S^**-doubly continuous.*

Definition 1.10 ([6]). Let P be a poset and $U \subseteq P$. U is said to be *Scott open* if and only if the the following two conditions are satisfied:

(1) $U = \uparrow U$, that is to say, U is an upper set;
(2) $\sup D \in U$ implies $D \cap U \neq \emptyset$ for every directed subset D of P.

It can be formally verified that the collection of all Scott open subsets of P forms a topology on P, which is called the *Scott topology* and denoted by σ_P.

2. B-consistent S^*-doubly continuous posets

In this section, the O_1-convergence in posets is reviewed. Then, a special class of S^*-doubly continuous posets, named B-consistent S^*-doubly continuous posets, is introduced. Finally, we present a sufficient and necessary condition of a general poset which can precisely serve as an order-theoretical condition for the associated O_1-convergence being topological.

Definition 2.1 ([1]). Let P be a poset. A net $(x_i)_{i\in I}$ in P is said to O_1-*converges* to $x \in P$ if there exist nets $(u_i)_{i\in I}$ and $(v_i)_{i\in I}$ in P such that

(O1) $(u_i)_{i\in I}$ is an increasing net, i.e., $u_{i_1} \leqslant u_{i_2}$ for any $i_1, i_2 \in I$ with $i_1 \leqslant i_2$, and $(v_i)_{i\in I}$ is a decreasing net, i.e., $v_{i_1} \geqslant v_{i_2}$ for any $i_1, i_2 \in I$ with $i_1 \leqslant i_2$;
(O2) $u_i \leqslant x_i \leqslant v_i$ for all $i \in I$;
(O3) $\sup\{u_i : i \in I\} = x = \inf\{v_i : i \in I\}$.

In this case, we write $(x_i)_{i\in I} \xrightarrow{O_1} x$.

It is worth noting that if $(x_i)_{i\in I}$ is a net in a poset P, then $(x_i)_{i\in I} \xrightarrow{O_1} x \in P$ implies $(x_i)_{i\in I} \xrightarrow{O} x$. But the converse implication may not be true. This fact can be illustrated by the following Example 2.2.

Example 2.2. Let $P = \{a, b, x\}$ with $a \leqslant x$ and $b \leqslant x$, and $I = \{1, 2, 3\}$ with $1 \leqslant 2 \leqslant 3$. And let $(x_i)_{i\in I}$ be the net defined by $x_1 = a, x_2 = b$ and $x_3 = x$. By the definition of O-convergence, it is easy to verify that $(x_i)_{i\in I} \xrightarrow{O} x$. Suppose $(x_i)_{i\in I} \xrightarrow{O_1} x$. Then there exist an increasing net $(u_i)_{i\in I}$ and a decreasing net $(v_i)_{i\in I}$ which satisfy the conditions (O2) and (O3) in Definition 2.1. By (O2), we have $u_1 \leqslant x_1 = a \leqslant v_1$ and $u_2 \leqslant x_2 = b \leqslant v_2$, which implies that $u_1 = a, u_2 = b$ and $u_1 \nleqslant u_2$. This contradicts the fact that $(u_i)_{i\in I}$ is an increasing net. Thus, the net $(x_i)_{i\in I}$ does not O_1-converge to x.

Remark 2.3. Let P be a poset and $(x_i)_{i\in I}$ a net in P.

(1) If $(x_i)_{i\in I}$ is a constant net in P with value x, then $(x_i)_{i\in I} \xrightarrow{O_1} x$ since the increasing net $(u_i)_{i\in I}$ defined by $u_i = x$ for all $i \in I$ and the decreasing net $(v_i)_{i\in I}$ defined by $v_i = x$ for all $i \in I$ satisfy the conditions (O2) and (O3) in Definition 2.1.

(2) Suppose $(x_i)_{i\in I} \xrightarrow{O_1} x \in P$. Then there exist an increasing net $(u_i)_{i\in I}$ and a decreasing net $(v_i)_{i\in I}$ in P that satisfy the conditions (O2) and (O3) in Definition 2.1. This can imply that the subsets $\{u_i : i \in I\}$ and $\{v_i : i \in I\}$ of P are directed and filtered, respectively.

(3) The O_1-convergent point of a net $(x_i)_{i\in I}$ in P, if exists, is unique. Indeed, suppose $(x_i)_{i\in I} \xrightarrow{O_1} x_1 \in P$ and $(x_i)_{i\in I} \xrightarrow{O_1} x_2 \in P$. Then we have $(x_i)_{i\in I} \xrightarrow{O} x_1$ and $(x_i)_{i\in I} \xrightarrow{O} x_2$. By Remark 2.3 (2) (the O-convergent point of a net $(x_i)_{i\in I}$ in P, if exists, is unique.) in [19], $x_1 = x_2$.

(4) Let D be a directed subset of P and F a filtered subset of P such that $\sup D$ and $\inf F$ exist. Define the net $(x_d)_{d\in D}$ by $x_d = d$ for all $d \in D$ and the net $(y_e)_{e\in F^{op}}$ by $y_e = e$ for all $e \in F$. Then $(x_d)_{d\in D} \xrightarrow{O_1} \sup D$ and $(y_e)_{e\in F^{op}} \xrightarrow{O_1} \inf F$.

Next we recall the definition and the fundamental properties of B-topology on posets, which has been introduced in [12].

Definition 2.4 ([12]). Given a poset P, a subset $U \subseteq P$ is called a *B-open set* if for any filter $\mathcal{F}$ in P that order-converges to $x \in U$, there exists $F \in \mathcal{F}$ such that $F \subseteq U$. (Note: for the definition of order-convergence in posets, one can refer to Definition 2.1 in [12].)

For a poset P, let $\mathcal{T}_P$ denote the set of all B-open subsets of P. It is routine to check that $\mathcal{T}_P$ forms a topology on P. And this topology is called the *B-Topology* on P.

Proposition 2.5 ([12]). *Let* P *be a poset and* $U \subseteq P$. *Then* $U \in \mathcal{T}_P$ *if and only if for any directed subset* D *of* P *and any filtered subset* F *of* P *with* $\sup D = \inf F = x \in U$, *there exist* $d_0 \in D$ *and* $e_0 \in F$ *such that* $\uparrow d_0 \cap \downarrow e_0 \subseteq U$.

Recall that given a topological space $(X, \mathcal{T})$, a subfamily $\mathcal{B}$ of $\mathcal{T}$ is called an open base for the topological space $(X, \mathcal{T})$ (sometimes, called an open base of $\mathcal{T}$) if $\mathcal{B} \subseteq \mathcal{T}$ and for every point $x \in X$ and every neighborhood V of x there exists $U \in \mathcal{B}$ such that $x \in U \subseteq V$.

Theorem 2.6 ([12]). *If* P *is* S^**-doubly continuous, then* $\mathcal{B}_P = \{\Uparrow_S y \cap \Downarrow_S z : y, z \in P\}$ *forms an open base of the B-topology* $\mathcal{T}_P$.

Definition 2.7 ([16]). Let P be a poset. A family $\overline{\sigma}_P$ of subsets of P is called the *Bi-Scott topology on* P if $\overline{\sigma}_P = \sigma_P \vee \sigma_{P^{op}}$, where P^{OP} is the dual poset of P. Obviously, $\overline{\sigma}_P$ has an open base $\mathcal{B} = \{U \cap V : U \in \sigma_P, V \in \sigma_{P^{op}}\}$.

Theorem 2.8 ([12]). *Let* P *be a doubly continuous poset. Then* $\overline{\sigma}_P = \mathcal{T}_P$.

Depending on the introduction and discussion of B-topology on posets [12], we are now in the position to clarify the order-theoretical condition for posets, under which the O_1-convergence is topological.

Definition 2.9. The O_1-convergence in a poset P is said to be *topological* if there exists a topology $\mathcal{T}$ on P such that

$$(x_i)_{i\in I} \xrightarrow{\mathcal{T}} x \in P \Longleftrightarrow (x_i)_{i\in I} \xrightarrow{O_1} x.$$

Definition 2.10. An S^*-doubly continuous poset P is said to be *consistent with respect to the* O_1*-convergence* (for short, B-consistent) if it satisfies the following condition for any net $(x_i)_{i\in I}$ in P and any $x \in P$:

$$x_i \in \uparrow y \cap \downarrow z \text{ eventually for any } y \in \Downarrow_S x \text{ and } z \in \Uparrow_S x \Rightarrow (x_i)_{i\in I} \xrightarrow{O_1} x. \qquad (\star)$$

Lemma 2.11. *If* P *is a B-consistent* S^**-doubly continuous poset, then the* O_1*-convergence in* P *is topological.*

Proof. By Remark 2.3 (2) and Proposition 2.5, it is not difficult to show a net

$$(x_i)_{i\in I} \xrightarrow{O_1} x \in P \Longrightarrow (x_i)_{i\in I} \xrightarrow{\mathcal{T}_P} x.$$

To prove the theorem, it suffices to show a net

$$(x_i)_{i\in I} \xrightarrow{\mathcal{T}_P} x \in P \Longrightarrow (x_i)_{i\in I} \xrightarrow{O_1} x.$$

Now we suppose the net $(x_i)_{i\in I} \xrightarrow{\mathcal{T}_P} x$. For any $y \in \Downarrow_S x$ and $z \in \Uparrow_S x$, since P is an S^*-doubly continuous poset, there exist $y_0 \in \Downarrow_S x$ and $z_0 \in \Uparrow_S x$ such that $\uparrow y_0 \cap \downarrow z_0 \subseteq \Uparrow_S y \cap \Downarrow_S z$. According to Theorem 2.6, we have that $x_i \in \Uparrow_S y_0 \cap \Downarrow_S z_0$ eventually. Thus $x_i \in \Uparrow_S y_0 \cap \Downarrow_S z_0 \subseteq \uparrow y_0 \cap \downarrow z_0 \subseteq \Uparrow_S y \cap \Downarrow_S z \subseteq \uparrow y \cap \downarrow z$ holds eventually. It follows from the assumption of P satisfying condition ($\star$) that the net $(x_i)_{i\in I} \xrightarrow{O_1} x$. □

Conversely, we have the following Lemma.

Lemma 2.12. *If the O_1-convergence in a poset P is topological, then P is a B-consistent S^*-doubly continuous poset.*

The proof of Lemma 2.12 is divided into the following three steps (Facts 2.13, 2.14, and 2.15).

Fact 2.13. If the O_1-convergence in a poset P is topological, then P is an S-doubly continuous poset.

Proof. Suppose that the O_1-convergence in P is topological. Then there exists a topology $\mathcal{T}$ on P such that a net

$$(x_i)_{i\in I} \xrightarrow{O_1} x \Longleftrightarrow (x_i)_{i\in I} \xrightarrow{\mathcal{T}} x$$

for every $x \in P$. Let $\mathcal{D} = \{(w, W) \in (\cup \mathcal{N}_x) \times \mathcal{N}_x : w \in W\}$, where $\mathcal{N}_x = \{W \in \mathcal{T} : x \in W\}$. Define the order $\leqslant$ on $\mathcal{D}$ by

$$(\forall (w_1, W_1), (w_2, W_2) \in \mathcal{D})\ (w_1, W_1) \leqslant (w_2, W_2) \Longleftrightarrow W_2 \subseteq W_1.$$

It can be verified straightforwardly that $\leqslant$ is a preorder and $\mathcal{D}$ is directed. Let $x_{(w,W)} = w$ for every $(w, W) \in \mathcal{D}$. Then for any $V \in \mathcal{N}_x$, we have that $x_{(w,W)} = w \in W \subseteq V$ for every $(w, W) \geqslant (x, V)$. This means the net $(x_{(w,W)})_{(w,W)\in\mathcal{D}} \xrightarrow{\mathcal{T}} x$. Hence $(x_{(w,W)})_{(w,W)\in\mathcal{D}} \xrightarrow{O_1} x$. And thus, there exist an increasing net $(u_{(w,W)})_{(w,W)\in\mathcal{D}}$ and a decreasing net $(v_{(w,W)})_{(w,W)\in\mathcal{D}}$ satisfying the conditions (O2) and (O3) in Definition 2.1.

Let $D = \{u_{(w,W)} : (w, W) \in \mathcal{D}\}$ and $F = \{v_{(w,W)} : (w, W) \in \mathcal{D}\}$. Then by Definition 2.1 and Remark 2.3 (2), D is directed, F is filtered, and $\sup D = x = \inf F$. For every $d = u_{(w_1,W_1)} \in D$ and $e = v_{(w_2,W_2)} \in F$, as $(w, W_1 \cap W_2) \geqslant (w_1, W_1), (w_2, W_2)$ for every $w \in W_1 \cap W_2$, we can conclude that $d = u_{(w_1,W_1)} \leqslant u_{(w,W_1\cap W_2)} \leqslant x_{(w,W_1\cap W_2)} = w \leqslant v_{(w,W_1\cap W_2)} \leqslant v_{(w_2,W_2)} = e$ for each $w \in W_1 \cap W_2$, which implies $W_1 \cap W_2 \subseteq \uparrow d \cap \downarrow e$. Let D' be a directed subset of P with $\sup D' = x$. Consider the net $(x_{d'})_{d'\in D'}$ defined by $x_{d'} = d'$ for all $d' \in D'$. By Remark 2.3 (4), the net $(x_{d'})_{d'\in D'} \xrightarrow{O_1} x$. This implies the net $(x_{d'})_{d'\in D'} \xrightarrow{\mathcal{T}} x$. Thus, there exists $d'_0 \in D'$ such that $x_{d'} = d' \in W_1 \cap W_2 \subseteq \uparrow d \cap \downarrow e$ for every $d' \geqslant d'_0$. In particular, we have $x_{d'_0} = d'_0 \in \uparrow d \cap \downarrow e$, and thus $d \ll_S x$. Because d is arbitrarily taken from D, we have $D \subseteq \Downarrow_S x$ and $\sup D = \sup \Downarrow_S x = x$. For $y_1, y_2 \in \Downarrow_S x$, since D is directed and $\sup D = x$, by the definition of $\ll_S$, there exist $d_1, d_2 \in D$ such that $y_1 \leqslant d_1$ and $y_2 \leqslant d_2$. Hence there exists $d_0 \in D \subseteq \Downarrow_S x$ such that $y_1 \leqslant d_1 \leqslant d_0$ and $y_2 \leqslant d_2 \leqslant d_0$. Therefore, the set $\Downarrow_S x$ is directed. It can be similarly proved that the set $\Uparrow_S x$ is filtered and $\inf \Uparrow_S x = x$. Therefore, P is S-doubly continuous. □

Fact 2.14. If the O_1-convergence in a poset P is topological, then P is an S^*-doubly continuous poset.

Proof. For any $y \in \Downarrow_S x$ and $z \in \Uparrow_S x$, it follows from the definitions of $\ll_S$ and $\rhd_S$ that there exist $d_y = u_{(w_3,W_3)} \in D$ and $e_z = v_{(w_4,W_4)} \in F$ such that $y \leqslant d_y$ and $e_z \leqslant z$. By the discussion in Fact 2.13, we have $W_3 \cap W_4 \subseteq \uparrow d_y \cap \downarrow e_z \subseteq \uparrow y \cap \downarrow z$. $I_x = \{\uparrow a \cap \downarrow b : a \in \Downarrow_S x \ \&\ b \in \Uparrow_S x\}$ and $I_0 = \{(k, K) \in (\cup I_x) \times I_x : k \in K\}$. And define the order $\prec$ on I_0 by

$$(\forall (k_1, K_1), (k_2, K_2) \in I_0)\ (k_1, K_1) \prec (k_2, K_2) \Longleftrightarrow K_2 \subseteq K_1.$$

One can easily check that $\prec$ is a directed preoder. Let $x_{(k,K)} = k$, $u_{(k,K)} = a$ and $v_{(k,K)} = b$ for every $(k, K) = (k, \uparrow a \cap \downarrow b) \in I_0$. Then the nets $(x_{(k,K)})_{(k,K)\in I_0}$, $(u_{(k,K)})_{(k,K)\in I_0}$ (increasing) and$(v_{(k,K)})_{(k,K)\in I_0}$ (decreasing) satisfy the conditions (O2) and (O3) in Definition 2.1, and hence the net $(x_{(k,K)})_{(k,K)\in I_0} \xrightarrow{O_1} x$. Thus, the net $(x_{(k,K)})_{(k,K)\in I_0} \xrightarrow{\mathcal{T}} x$. This means that there exists $(k_0, K_0) = (k_0, \uparrow y_0 \cap \downarrow z_0) \in I_0$, where $y_0 \in \Downarrow_S x$ and $z_0 \in \Uparrow_S x$, such that $x_{(k,K)} = k \in W_3 \cap W_4 \subseteq \uparrow d_y \cap \downarrow e_z \subseteq \uparrow y \cap \downarrow z$ for all $(k, K) \succ (k_0, K_0)$. In particular, it holds that $(k, K_0) \succ (k_0, K_0)$ for all $k \in K_0$, which implies $x_{(k,K_0)} = k \in W_3 \cap W_4 \subseteq \uparrow d_y \cap \downarrow e_z \subseteq \uparrow y \cap \downarrow z$ for all $k \in K_0$. Thus, we conclude that $K_0 = \uparrow y_0 \cap \downarrow z_0 \subseteq W_3 \cap W_4 \subseteq \uparrow d_y \cap \downarrow e_z \subseteq \uparrow y \cap \downarrow z$.

We finally show $\uparrow y_0 \cap \downarrow z_0 \subseteq \Uparrow_S y \cap \Downarrow_S z$. Suppose that D'' is a directed subset of P with $\sup D'' = r \in \uparrow y_0 \cap \downarrow z_0$. Then by Remark 2.3 (4), the net $(x_{d''})_{d''\in D''}$ defined by $x_{d''} = d''$ for all $d'' \in D''$ O_1-converges to r, and thus the net $(x_{d''})_{d''\in D''} \xrightarrow{\mathcal{T}} r$. It follows that there exists $d''_0 \in D''$ such that $x_{d''_0} = d''_0 \in W_3 \cap W_4 \subseteq \uparrow y \cap \downarrow z$, which implies $y \ll_S r$. It is analogous to prove $z \rhd_S r$. Hence, we have $\uparrow y_0 \cap \downarrow z_0 \subseteq \Uparrow_S y \cap \Downarrow_S z$. Therefore P is an S^*-doubly continuous poset. □

Fact 2.15. If the O_1-convergence in a poset P is topological, then P satisfies condition ($\star$).

Proof. Let $(x_j)_{j\in J}$ be a net in P with $x_j \in \uparrow y \cap \downarrow z$ eventually for any $y \in \Downarrow_S x$ and $z \in \Uparrow_S x$. By what has been shown in Fact 2.14, the net $(x_{(k,K)})_{(k,K)\in I_0} \xrightarrow{\mathcal{T}} x$. This means that for every $W \in \mathcal{N}_x$ there exists $(k_1, K_1) = (k_1, \uparrow y_1 \cap \downarrow z_1) \in I_0$, where $y_1 \in \Downarrow_S x$ and $z_1 \in \Uparrow_S x$, such that $x_{(k,K)} = k \in W$ for all $(k, K) \succ (k_1, K_1)$. In particular, $(k, K_1) \succ (k_1, K_1)$ for all $k \in K_1$, i.e., $x_{(k,K_1)} = k \in W$ for all $k \in K_1$, which implies $K_1 = \uparrow y_1 \cap \downarrow z_1 \subseteq W$. By the assumption that $x_j \in \uparrow y_1 \cap \downarrow z_1 \subseteq W$ eventually, it follows that the net $(x_j)_{j\in J} \xrightarrow{\mathcal{T}} x$. Therefore, the net $(x_j)_{j\in J} \xrightarrow{O_1} x$. This shows P satisfies condition ($\star$). □

The combination of Lemma 2.11 and Lemma 2.12 gives the main result of this paper.

Theorem 2.16. *For a poset P the following statements are equivalent:*

(1) *The O_1-convergence in P is topological.*
(2) *P is a B-consistent S^*-doubly continuous poset.*
(3) *For any net $(x_i)_{i\in I}$ in P, $(x_i)_{i\in I} \xrightarrow{O_1} x \in P \iff (x_i)_{i\in I} \xrightarrow{\mathcal{T}_P} x$.*

Proof.

(1)⇒(2): By Lemma 2.12.

(2)⇒(3): By the proof of Lemma 2.11.

(3)⇒(1): It is straightforward. □

Corollary 2.17. *Let P be a B-consistent doubly continuous poset. Then*

(1) *The O_1-convergence in P is topological.*
(2) *For any net $(x_i)_{i\in I}$ in P, $(x_i)_{i\in I} \xrightarrow{O_1} x \in P \iff (x_i)_{i\in I} \xrightarrow{\mathcal{T}_P} x$.*
(3) *For any net $(x_i)_{i\in I}$ in P, $(x_i)_{i\in I} \xrightarrow{O_1} x \in P \iff (x_i)_{i\in I} \xrightarrow{\sigma_P} x$.*

Proof.

(1): By Proposition 1.9 and Theorem 2.16.

(2): By Proposition 1.9 and Theorem 2.16.

(3): By Theorems 2.8 and 2.16. □

3. Special B-consistent S^*-doubly continuous posets

In this section, we will discuss several special B-consistent S^*-doubly continuous posets. We first give a basic property of B-consistent S^*-doubly continuous posets.

Proposition 3.1. *Every B-consistent S^*-doubly continuous poset is bounded.*

Proof. Suppose that P is a B-consistent S^*-doubly continuous poset. Let $I = P \cup \{0, 1\}$ and define the order $\leqslant$ on I as follows: $0 \leqslant i \leqslant 1$ for every $i \in I$. Then I is directed since 1 is the largest element in I. For every $x \in P$, let $x_0 = x_1 = x$ and $x_i = i$ for all $i \in P$. Since $x_1 = x \in \uparrow y \cap \downarrow z$ for every $y \in \Downarrow_S x$ and $z \in \Uparrow_S x$, by condition ($\star$), the net $(x_i)_{i\in I} \xrightarrow{O_1} x$. This means that there exist an increasing net $(u_i)_{i\in I}$ and a decreasing net $(v_i)_{i\in I}$ satisfying the conditions (O2) and (O3) in Definition 2.1, which implies that $u_0 \leqslant x_i = i \leqslant v_0$ for all $i \in P$. Thus u_0 is the least element in P and v_0 is the largest element in P. Therefore, P is bounded. □

A *ordinal number* λ, from the order-theoretical point of view, is indeed a linearly ordered set of which every nonempty subset has the least element. It is easy to see that if a subset λ_0 of λ has an upper bound in λ, then $\sup \lambda_0 = \min U(\lambda_0)$, where $U(\lambda_0)$ is the set of all upper bound of λ_0 in λ. One can refer to [2] for a more detailed discussion about ordinal numbers.

Let P and Q be two posets. A mapping $f : P \to Q$ is an *order embedding* if for any $x, y \in P$, $f(x) \leqslant_Q f(y)$ in Q if and only if $x \leqslant_P y$ in P. Similarly, a mapping $g : P \to Q$ is a *dual order embedding* if for any $x, y \in P$, $g(x) \leqslant_Q g(y)$ in Q if and only if $x \geqslant_P y$ in P. We should note that both f and g are injective but need not to be bijective.

Let P and Q be two posets and $Q_1, Q_2 \subseteq Q$. An order embedding $f : P \to Q$ is said to be *cofinal* in Q_1 if $f(P) \subseteq Q_1$, and for any $q_1 \in Q_1$ there exists $p_1 \in P$ such that $f(p_1) \geqslant_Q q_1$. Similarly, a dual order embedding $g : P \to Q$ is said to be *dually cofinal* in Q_2 if $g(P) \subseteq Q_2$, and for any $q_2 \in Q_2$ there exists $p_2 \in P$ such that $g(p_2) \leqslant_Q q_2$.

Definition 3.2. An S^*-doubly continuous poset P is called *locally ordinal embedded* if for every $x \in P$, there exist ordinal numbers λ_x, μ_x, an order embedding $f_x : \lambda_x \to P$ and a dual order embedding $g_x : \mu_x \to P$ such that f_x is cofinal in $\Downarrow_S x$ and g_x is dually cofinal in $\Uparrow_S x$.

Proposition 3.3. *Let P be an S^*-doubly continuous poset. If P is locally ordinal embedded and bounded, then it is B-consistent.*

Proof. Suppose that an S^*-doubly continuous poset P is locally ordinal embedded and bounded. Then for every $x \in P$, there exist ordinal numbers λ_x, μ_x, an order embedding $f_x : \lambda_x \to P$ and a dual order embedding $g_x : \mu_x \to P$ such that f_x is cofinal in $\Downarrow_S x$ and g_x is dually cofinal in $\Uparrow_S x$. Let $(x_i)_{i\in I}$ be a net in P with that $x_i \in \uparrow y \cap \downarrow z$ eventually for every $y \in \Downarrow_S x$ and $z \in \Uparrow_S x$. For every $i \in I$, we define

$$u_i = \begin{cases} x, & \{\lambda \in \lambda_x : (\forall j \geqslant i) x_j \geqslant f_x(\lambda)\} = \lambda, \\ f_x(\sup\{\lambda \in \lambda_x : (\forall j \geqslant i) x_j \geqslant f_x(\lambda)\}), & \emptyset \neq \{\lambda \in \lambda_x : (\forall j \geqslant i) x_j \geqslant f_x(\lambda)\} \subsetneq \lambda_x, \\ \perp, & \{\lambda \in \lambda_x : (\forall j \geqslant i) x_j \geqslant f_x(\lambda)\} = \emptyset, \end{cases}$$

and

$$v_i = \begin{cases} x, & \{\mu \in \mu_x : (\forall j \geqslant i) x_j \leqslant g_x(\mu)\} = \mu_x, \\ g_x(\sup\{\mu \in \mu_x : (\forall j \geqslant i) x_j \leqslant g_x(\mu)\}), & \emptyset \neq \{\mu \in \mu_x : (\forall j \geqslant i) x_j \leqslant g_x(\mu)\} \subsetneq \mu_x, \\ \top, & \{\mu \in \mu_x : (\forall j \geqslant i) x_j \leqslant g_x(\mu)\} = \emptyset, \end{cases}$$

where $\perp$ and $\top$ are the least and largest elements of P, respectively .

It is tedious but straightforward to check that the nets $(u_i)_{i\in I}, (v_i)_{i\in I}$ are respectively increasing and decreasing, and satisfy the conditions (O2) and (O3) in Definition 2.1. Thus the net $(x_i)_{i\in I} \xrightarrow{O_1} x$. By Definition 2.10, P is B-consistent. □

Definition 3.4. We say that an S^*-doubly continuous poset P is *locally countable* if $|\Downarrow_S x|, |\Uparrow_S x| \leqslant \omega_0$ for every $x \in P$, where $|\Downarrow_S x|$ and $|\Uparrow_S x|$ denote the cardinalities of the sets $\Downarrow_S x$ and $\Uparrow_S x$, respectively, and ω_0 is the first infinite cardinal number.

Proposition 3.5. *Let P be an S^*-doubly continuous poset. If P is locally countable, then P is locally ordinal embedded.*

Proof. For every $x \in P$, we consider the following two cases:

(Case 1.) If $x \in \Downarrow_S x$, let $\lambda_x = 1$ and define $f_x : 1 \to P$ by $f_x(0) = x$. Now we can easily see that $f_x : 1 \to P$ is an order embedding and is cofinal in $\Downarrow_S x$.

(Case 2.) If $x \notin \Downarrow_S x$, then $\Downarrow_S x$ is directed and $|\Downarrow_S x| = \omega_0$ since P is S^*-doubly continuous and locally countable. Without loss of generality, we assume $\Downarrow_S x = \{y_0, y_1, y_2, y_3, \ldots\}$. Let $\lambda_x = \omega_0$ and define $f_x : \omega_0 \to P$ by inductive approach with respect to ω_0:

(Step 1.) Let $f_x(0) = y_0$.

(Step 2.) Suppose that all of $f_x(0), f_x(1), f_x(2), \ldots, f_x(n)$ have been defined. Then we can take a $y^{n+1} \in \Downarrow_S x$ such that $y^{n+1} > y_i$ and $y^{n+1} > f_x(i)$ for all $i \leqslant n$. Now let $f_x(n+1) = y^{n+1}$.

It is clear that $f_x : \omega_0 \to P$ is an order embedding which is cofinal in $\Downarrow_S x$.

In sum, we have proved that there exist an ordinal numbers λ_x and an order embedding $f_x : \lambda_x \to P$ such that f_x is cofinal in $\Downarrow_S x$. It can be similarly showed that there exist an ordinal numbers μ_x and a dual order embedding $g_x : \mu_x \to P$ such that g_x is dually cofinal in $\Uparrow_S x$. Hence P is locally ordinal embedded. □

Definition 3.6. An S^*-doubly continuous poset P is said to be *weakly locally countable* if for every $x \in P$ there always exist a countable directed subset D_x of $\Downarrow_S x$ and a countable filtered subset F_x of $\Uparrow_S x$ such that $\sup D_x = x = \inf F_x$.

Proposition 3.7. *Let* P *be an* S^**-doubly continuous poset. If* P *is weakly locally countable, then* P *is locally ordinal embedded.*

Proof. Suppose that an S^*-doubly continuous poset P is weakly locally countable. Then for every $x \in P$, there exist a countable directed subset D_x of $\Downarrow_S x$ and a countable filtered subset F_x of $\Uparrow_S x$ such that $\sup D_x = x = \inf F_x$. By the technique of induction used in the proof of Proposition 3.5, we can get ordinal numbers λ_x, μ_x, an order embedding $f_x : \lambda_x \to P$ and a dual order embedding $g_x : \mu_x \to P$ such that f_x is cofinal in D_x and g_x is dually cofinal in F_x. Furthermore, One can easily check by the definitions of $\ll_S$ and $\rhd_S$ that f_x is also cofinal in $\Downarrow_S x$ and g_x is dually cofinal in $\Uparrow_S x$. Hence P is locally ordinal embedded. □

Theorem 3.8. *If* P *is an* S^**-doubly continuous, locally ordinal embedded and bounded poset, then the* O_1*-convergence in* P *is topological.*

Proof. Straightforwardly from Theorem 2.16 and Proposition 3.3. □

Corollary 3.9. *Let* P *be a poset. Then*

(1) *If* P *is* S^**-doubly continuous, bounded and locally countable, then the* O_1*-convergence in* P *is topological.*
(2) *If* P *is a countable poset, i.e.,* $|P| \leqslant \omega_0$*, then the* O_1*-convergence in* P *is topological if and only if* P *is* S^**-doubly continuous and bounded.*
(3) *If* P *is a finite poset, i.e.,* $|P| < \omega_0$*, then the* O_1*-convergence in* P *is topological if and only if* P *is bounded.*

Proof.

(1): Straightforwardly follows from Proposition 3.5 and Theorem 3.8.

(2): Let P be a countable poset. If the O_1-convergence in P is topological. Then, by Theorem 2.16, P is a B-consistent S^*-doubly continuous poset. It follows from Proposition 3.1 that P is bounded. Conversely, suppose that P is S^*-doubly continuous and bounded. Since P is a countable poset, it is locally countable. By Proposition 3.5, P is locally ordinal embedded. Thus we can conclude by Theorem 3.8 that the O_1-convergence in P is topological.

(3): It is straightforwardly from (2) if we notice the fact that every finite poset is a doubly continuous poset and hence an S^*-doubly continuous poset. □

Definition 3.10. An S^*-doubly continuous poset P is said to be *locally complete* if for any $x \in P$, $A \subseteq \Downarrow_S x$ and $B \subseteq \Uparrow_S x$, both of $\sup A$ and $\inf B$ in P exist.

Proposition 3.11. *If* P *be an* S^**-doubly continuous, locally complete and bounded poset, then* P *is B-consistent.*

Proof. Suppose P is an S^*-doubly continuous, locally complete and bounded poset. Let $(x_i)_{i\in I}$ be a net in P with that $x_i \in \uparrow y \cap \downarrow z$ eventually for every $y \in \Downarrow_S x$ and $z \in \Uparrow_S x$. For every $i \in I$ we define

$$u_i = \begin{cases} \sup\{y' \in \Downarrow_S x : (\forall j \geqslant i) x_j \geqslant y'\}, & \{y' \in \Downarrow_S x : (\forall j \geqslant i) x_j \geqslant y'\} \neq \emptyset, \\ \bot, & \{y' \in \Downarrow_S x : (\forall j \geqslant i) x_j \geqslant y'\} = \emptyset, \end{cases}$$

and dually define

$$v_i = \begin{cases} \inf\{z' \in \Uparrow_S x : (\forall j \geqslant i) x_j \leqslant z'\}, & \{z' \in \Uparrow_S x : (\forall j \geqslant i) x_j \leqslant z'\} \neq \emptyset, \\ \top, & \{z' \in \Uparrow_S x : (\forall j \geqslant i) x_j \leqslant z'\} = \emptyset, \end{cases}$$

where, $\bot$ and $\top$ are the least element and largest element of P, respectively.

One can straightforwardly verifies that the increasing net $(u_i)_{i\in I}$ and decreasing net $(v_i)_{i\in I}$ satisfy the conditions (O2) and (O3) in Definition 2.1. Hence the net $(x_i)_{i\in I} \xrightarrow{O_1} x$. Thus, P is B-consistent. □

Theorem 3.12. *If* P *is an* S^**-doubly continuous, locally complete and bounded poset, then the* O_1*-convergence in* P *is topological.*

Proof. Straightforwardly from Theorem 2.16 and Proposition 3.11. □

Corollary 3.13. *Let* P *be a poset.*

(1) *If* P *is a complete lattice, then the* O_1*-convergence in* P *is topological if and only if* P *is* S^**-doubly continuous.*

(2) *If* P *is a completely distributive lattice, then the* O_1*-convergence in* P *is topological.*

(3) *If* P *is a complete chain, then the* O_1*-convergence in* P *is topological.*

Proof.

(1) Let P be a complete lattice. If the O_1-convergence in P is topological, then P is S^*-doubly continuous by Theorem 2.16. Conversely, suppose P is an S^*-doubly continuous poset. Since P is a complete lattice, it is locally complete and bounded. By Theorem 3.12, the O_1-convergence in P is topological.

(2) Noticing that every completely distributive lattice is doubly continuous, and hence is S^*-doubly continuous, by (1), the O_1-convergence in P is topological.

(3) Noticing that every complete chain is completely distributive, by (2), the O_1-convergence in P is topological. □

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