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On stable fixed points under several kinds of strong perturbations

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Abstract

This gives new results on stable fixed points related to several kinds of strong perturbations in references. It is shown that a strong stable set of fixed points has a robust stable property. For a robust stable fixed point set of a correspondence, in its neighborhood, there is a strong stable set for any small perturbation of the correspondence. There exists a robust stable set for a correspondence, if there is at least one fixed point for the correspondence. These generalize the corresponding results in recent references and give an application in the existence of strong stable economy equilibria.

Keywords: Fixed point, essential stable, robust stable, economy equilibrium.

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1. Introduction

Essential stability of fixed points of functions and correspondences is originally from the early seminal research in [7, 13, 14, 19]. Essential fixed points for correspondences attract many attentions for applications [3, 18]. Employing the stability implied by essential fixed points, essential solutions become important concepts in many nonlinear fields and are applied into noncooperative games [8, 26, 31, 34], cooperative games and population games [30, 32], and other equilibrium problems [6, 12, 17]. Hence, there are close relation between fixed points and game equilibria. It is known that one can obtain the existence of Nash equilibria from Kakutani and Brouwer fixed point theorems, the versus is also true, see [35].

Essential stabilities are closely related to the perturbation of mappings. Kohlberg and Mertens, using homeomorphism methods, reveal the essential stable structure of Nash equilibria of a finite game [16], where essential stable equilibria can resist the perturbation of a payoff. Essential stable equilibria, in [13, 23, 32], can resist the perturbation of a mapping. Similar to essential fixed points, an essential mapping has the ability to resist its homotopic perturbations [9], and this is deeply extended to a larger class of mappings in [1, 2].

For strong stable equilibria, the existence condition and discreteness is studied in [15], where a strong stable equilibrium is more stable than an essential equilibrium. Recently, Xiang et al. introduce a strong

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perturbation class of correspondences and obtain strong stable sets for fixed points and Nash equilibira in [27], and further results are obtained in [28]. A strong stable set of fixed points can resist a larger perturbation of correspondences than an essential fixed point set.

Very recently, stability of equilibria is studied further by enlarging the perturbation of a mapping or correspondence to the range including the perturbation of domains, see strongly essential sets of Ky Fan's points by Xiang et al. [29], essential components of fixed points by Song et al. [22], essential equilibria in normal–form games by Carbonell-Nicolau [5]. On the other hand, the existence of stability equilibria for games, in relation to resisting perturbations, is extended from continuous games to discontinuous games (see the results by Scalzo [21] and Carbonell-Nicolau [5]).

On the basis of recent studies related to the strong perturbation of correspondences and games, this paper aims to make a further study on the strong stability of fixed points for correspondences in a normed linear space by considering both the strong perturbation of correspondences and the perturbation of their domains. It obtains that a robust stable set of fixed points exists for a correspondence if the correspondence has at least one fixed point. The existence of stable component under some condition can also be guaranteed. These induce the existence of a strong stable set for economy equilibria.

2. Preliminaries and motivations

Let X be a convex and compact subset of a normed linear space E with norm $\|\cdot\|$. Given a set $A \subset E$, $K(A)$ denotes the collection of all closed and convex subsets of A. A kind of useful correspondence in applications has upper semi–continuity. Denote $U_c(X)$ the collection of correspondences on X as

$$U_c(X) = \{F : X \to 2^X \mid F \text{ is upper semi–continuous with nonempty and closed values}\},$$

and let $U_{co}(X) \subset U_c(X)$ be the collection as the following

$$U_{co}(X) = \{F \in U_c(X) \mid F \text{ has convex values}\}.$$

For each $F \in U_c(X)$, let $\text{fix}(F)$ be the fixed point set of F on X. Clearly, if $\text{fix}(F) \neq \emptyset$, then $\text{fix}(F)$ is closed. A classic metric of any two correspondences F and G in $U_c(X)$ is $\rho_X(F, G)$ with

$$\rho_X(F, G) = \sup_{x \in X} h(F(x), G(x)),$$

where $h(A, B)$ is the Hausdorff distance between the sets A and B which induced by the norm on E. Another usual metric is $\rho_g(F, G)$ such that

$$\rho_g(F, G) = h(gr(F), gr(G)),$$

where $gr(H)$ is the graph of $H \in U_c(X)$. These induce two $\delta\,(\delta > 0)$ neighbors of a correspondence $F \in U_c(X)$, which are given below:

$$N_X(F, \delta) = \{G \in U_c(X) \mid \rho_X(F, G) < \delta\},$$

and

$$N_g(F, \delta) = \{G \in U_c(X) \mid \rho_g(F, G) < \delta\}.$$

For the deep study of strong stability of fixed point set $\text{fix}(F)$ for a correspondence $F \in U_{co}(X)$, in the papers [27] and [28], Xiang et al. introduce an interesting δ neighbor of F as the following:

$$N(F, \delta) = \{G \in U_{co}(X) \mid G(x) \in co(F(x + B_\delta(0)) + B_\delta(0)), \forall x \in X\},$$

where $co(A)$ is the convex hull of the set A, and $B_\delta(0) = \{x \in E : \|x\| < \delta\}$. The papers [27] and [28] prove some strong stable results related to the subsets of $\text{fix}(F)$, and show that a strong stable subset of $\text{fix}(F)$ can eliminate some abnormal fixed points. If we only consider correspondences in $U_{co}(X)$, from the references, it holds that

$$N_X(F,\delta)\cap U_{co}(X)\subseteq N_g(F,\delta)\cap U_{co}(X)\subseteq N(F,\delta),$$

for each $F\in U_{co}(X)$ and each $\delta>0$. A subset S of fix(F), which can resist the strong perturbation of F in the range of $N(F,\delta)$, is more stable than the subsets which can resist the perturbation in $N_g(F,\delta)\cap U_{co}(X)$ or $N_X(F,\delta)\cap U_{co}(X)$. For each $x\in X$, if there is no confusion, denote $co(F(x+B_\delta(0))+B_\delta(0))$ by $coF_{b\delta}(x)$ and $F(x+B_\delta(0))+B_\delta(0)$ by $F_{b\delta}(x)$, respectively.

For analysis of fixed points of a correspondence restricted on a subset of X, we write Γ_c, Γ_{co} and Γ'_{co} as

$$\Gamma_c=\{(F,A)\,|\,F\in U_c(X),A\in K(X)\},$$

$$\Gamma_{co}=\{(F,A)\,|\,F\in U_{co}(X),A\in K(X)\},$$

and

$$\Gamma'_{co}=\{(F,A)\in U_{co}(X)\times K(X)\,|\,F(x)\subseteq A,\forall x\in A\}.$$

Clearly, it holds that $\Gamma'_{co}\subset\Gamma_{co}\subset\Gamma_c$. For convenience, we denote each $(F,A)\in\Gamma_c$ as F_A.

A fixed point x of a correspondence $F_A\in\Gamma_c$ means both $x\in F(x)$ and $x\in A$. Furthermore, the set of fixed points of F_A is denoted by fix(F_A). fix(F_A) only considers the fixed points included in $A\subseteq X$. It is found that fix$(F_A)\neq\emptyset$ if $F_A\in\Gamma'_{co}$, while fix(F_A) may be empty if $F_A\in\Gamma_{co}\subset\Gamma_c$.

The paper [22] introduces the following metric ρ_s to analyze the perturbation of correspondences in Γ'_{co} (inwhere it only considers $F_A\in\Gamma'_{co}$ satisfying $A\in K(\text{int}X)$ and $X\subseteq\mathbb{R}^n$). Here, the distance of any two F_A and G_D in Γ'_{co}, is written as $\rho_s(F_A,G_D)$ with

$$\rho_s(F_A,G_D)=\sup_{x\in X}h(F(x),G(x))+h(A,D).$$

The convergence of a sequence $\{F^n\}\subset U_c(X)$ with $F^n\xrightarrow{\rho_X}F$ does not necessarily mean that $F^n_{A_n}\xrightarrow{\rho_s}F_A$ for some $\{A_n\}\subset K(X)$ and $A\in K(X)$. Conversely, it is true. The metric ρ_s also induces a δ neighbor for each $F_A\in\Gamma_{co}$ as

$$N_s(F_A,\delta)=\{G_D\in\Gamma'_{co}\,|\,\rho_s(F_A,G_D)<\delta\}.$$

Inspired by the above the strong perturbation of correspondences and restriction of domains, this paper considers three δ neighbors of a correspondence $F_A\in\Gamma_c$ as:

$$N_c(F_A,\delta)=\{G_D\in\Gamma_c\,|\,G(x)\in F_{b\delta}(x),\forall x\in X;h(A,D)<\delta\},$$
$$N_{co}(F_A,\delta)=\{G_D\in\Gamma_c\,|\,G(x)\in coF_{b\delta}(x),\forall x\in X;h(A,D)<\delta\},$$

and

$$N'_{co}(F_A,\delta)=\{G_D\in\Gamma'_{co}\,|\,G(x)\in coF_{b\delta}(x),\forall x\in X;h(A,D)<\delta\}.$$

Let $F_A\in\Gamma_c$, it is clear that

$$N_s(F_A,\delta)\subset N'_{co}(F_A,\delta)\subset N_{co}(F_A,\delta),\tag{2.1}$$

and

$$N_s(F_A,\delta)\subset N_c(F_A,\delta)\subset N_{co}(F_A,\delta),\tag{2.2}$$

if we do not consider the perturbation of A, we have

$$\mathcal{P}(N_s(F_A,\delta))\subset N_X(F,\delta)\subset\mathcal{P}(N_c(F_A,\delta))\subset\mathcal{P}(N_{co}(F_A,\delta)),\tag{2.3}$$

and

$$N(F,\delta)\subset\mathcal{P}(N'_{co}(F_A,\delta))\subset\mathcal{P}(N_{co}(F_A,\delta)),\tag{2.4}$$

where $\mathcal{P}(Q)$ denotes the projection of Q onto $U_c(X)$ with $Q\subseteq\Gamma_c$.

Definition 2.1. Let $F_A \in \Gamma_c$. A nonempty closed subset S of $\text{fix}(F_A)$ is called a strong stable set with respect to N_c (N_{co} or N'_{co}) if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\text{fix}(G_D) \cap (\text{fix}(F_A) + B_\varepsilon(0)) \neq \emptyset$ for any $G_D \in N_c(F_A, \delta)$ ($N_{co}(F_A, \delta)$ or $N'_{co}(F_A, \delta)$) with $\text{fix}(G_D) \neq \emptyset$. If a strong stable set is a minimal element in the collection of all strong stable sets (ordered by set inclusion) in $\text{fix}(F_A)$, it is called a minimal strong stable set.

Definition 2.2. A strong stable set S of $\text{fix}(F_A)$ for $F_A \in \Gamma_c$ is called a robust stable set with respect to N_c (N_{co} or N'_{co}) if for each $\varepsilon > 0$, there is $\delta > 0$ such that there exists a strong stable set C in $\text{fix}(G_D)$ with $C \subseteq S + B_\varepsilon(0)$ for any $G_D \in N_c(F_A, \delta)$ ($N_{co}(F_A, \delta)$ or $N'_{co}(F_A, \delta)$). If a minimal strong stable set is robust stable, then it is called a minimal robust stable set.

For each $F_A \in \Gamma_c$, it is known that the fixed point set $\text{fix}(F_A)$ can be decomposed as $\text{fix}(F_A) = \cup_{\alpha \in \Lambda} C_\alpha$, where each C_α is a connected component (for short, component) and any two C_α and C_β are disjointed.

Definition 2.3. The set of some components C with $C = \cup_{\beta \in \Lambda'} C_\beta \subseteq \text{fix}(F_A)$ and $\Lambda' \subseteq \Lambda$ for a correspondence $F_A \in \Gamma'_{co}$, is robust stable with respect to N'_{co}, if for each $\varepsilon > 0$, there is $\delta > 0$ such that there exists a component C_γ in $\text{fix}(G_D)$ with $C_\gamma \subseteq C + B_\varepsilon(0)$ for each $G_D \in N'_{co}(F_A, \delta)$.

Remark 2.4. For the above stability concepts, intuitively, a robust stable set (component) has more requirements than that of a strong stable set (component). Here, the robust stability is also related to the structural stability, robustness to bounded rationality equilibria, see [36] for further extension.

Lemma 2.5. *Let* $F : E \to 2^E$ *be an upper semi–continuous correspondence and* $A \subset E$. *For each* $\eta > 0$, *there exists a positive number* r *such that* $F(A + B_r(0)) \subseteq F(A) + B_\eta(0)$.

Proof. Since F is upper semi–continuous on E, for each $\eta > 0$ and each $x \in X$, there is an open neighborhood $O(x)$ of x in E such that $F(O(x)) \subseteq F(x) + B_\eta(0)$. Denote $\cup_{x \in A} O(x)$ by U, hence, we have

$$F(U) \subseteq F(A) + B_\eta(0).$$

Since the open $U \supset A$ and A is compact, there exists a number $r > 0$ with $A + B_r(0) \subset U$. Therefore, $F(A + B_r(0)) \subseteq F(A) + B_\eta(0)$. □

Lemma 2.6. *Let* $A, B \subseteq E$. *For a number* $\eta > 0$, *if* $A \subseteq B + B_\eta(0)$, *then it holds that* $\text{co}(A) \subseteq \text{co}(B) + B_\eta(0)$.

Proof. Take a point $x \in \text{co}(A)$. There are two points $a_1, a_2 \in A$ and a number $\alpha \in [0, 1]$ such that $\alpha a_1 + (1-\alpha)a_2 = x$. Since the points $a_1, a_2 \in A$ and $A \subseteq B + B_\eta(0)$, we have that $a_1, a_2 \in B + B_\eta(0)$. Then, there exist two points $r_1, r_2 \in B_\eta(0)$ and two points $b_1, b_2 \in B$ such that $a_1 = b_1 + r_1$ and $a_2 = b_2 + r_2$. Hence, $x = \alpha a_1 + (1-\alpha)a_2 = \alpha b_1 + (1-\alpha)b_2 + \alpha r_1 + (1-\alpha)r_2 \in \text{co}(B) + \text{co}(B_\eta(0))$. Therefore, $x \in \text{co}(B) + B_\eta(0)$, because $B_\eta(0)$ is convex. We obtain that $\text{co}(A) \subseteq \text{co}(B) + B_\eta(0)$. □

Lemma 2.7. *If* $F : X \to 2^E$ *is upper semi–continuous, then* $P : X \to 2^E$ *is upper semi–continuous, where*

$$P(x) = \text{co}(F(x + \bar{B}_{\frac{\delta}{2}}(0)) + \bar{B}_{\frac{\delta}{2}}(0)), \quad \forall x \in X, \ \text{and} \ \ \delta > 0.$$

Proof. For each $x \in X$ and each $\eta > 0$, from the upper semi–continuity of F and Lemma 2.5, we have a number r with $0 < r < \frac{\delta}{6}$ such that,

$$F(x + \bar{B}_{\frac{\delta}{2}}(0) + B_r(0)) \subseteq F(x + \bar{B}_{\frac{\delta}{2}}(0)) + B_\eta(0).$$

Then,

$$F(x + \bar{B}_{\frac{\delta}{2}}(0) + B_r(0)) + \bar{B}_{\frac{\delta}{2}}(0) \subseteq F(x + \bar{B}_{\frac{\delta}{2}}(0)) + \bar{B}_{\frac{\delta}{2}}(0) + B_\eta(0).$$

By Lemma 2.6, we have

$$\text{co}(F(x + \bar{B}_{\frac{\delta}{2}}(0) + B_r(0)) + \bar{B}_{\frac{\delta}{2}}(0)) \subseteq \text{co}(F(x + \bar{B}_{\frac{\delta}{2}}(0)) + \bar{B}_{\frac{\delta}{2}}(0)) + B_\eta(0).$$

That is,

$$P(x+B_r(0)) \subseteq P(x)+B_\eta(0). \qquad \square$$

The following lemma is well known. We will use it to prove a new result.

Lemma 2.8. *Let* A *be a nonempty bounded subset of* E *and* $b \notin \text{int}A$. *Let* $a \in \bar{A}$, *where* $\bar{A}$ *is the closure of* A. *Then there exists* $s \in L(a,b) \cap \partial A$ *and* $\|b-s\| \leqslant \|a-b\|$, *where* ∂A *is the boundary of* A *and*

$$L(a,b) = \{ta + (1-t)b : t \in [0,1]\}.$$

Lemma 2.9. *Let* A, D *and* C *be non–empty, closed, bounded convex sets in* E *with* $D \subset \text{int}(A)$ *and* $C \subseteq A$. *If* $\max_{x\in\partial A} d(x,\partial C) < \min_{x\in\partial A} d(x,\partial D)$, *then* $D \subseteq C$, *where* $d(z,V) = \min_{v\in V}\|z-v\|$.

Proof. Suppose that $D \not\subseteq C$, then, there is a point $z \in D$ with $z \notin C$. It can be found a point $c \in C$ such that $\|z-c\| = d(z,C)$, because C is closed and convex. It can be guaranteed that $c \in \partial C$. If not, it holds that $c \in \text{int}C$ (the interior of C). From Lemma 2.8, there is a point $c' \in L(c,z) \cap \partial C$. Let $c' = \lambda_1 c + (1-\lambda_1)z$. Clearly, $\lambda_1 > 0$, then, there is a number λ_2 with $0 < \lambda_2 < \lambda_1$ such that $c'' = \lambda_2 c + (1-\lambda_2)z \in \text{int}C \cap L(c,z)$, due to the fact that $c \in \text{int}C$. Hence, we have $\|z-c'\| < \|z-c\| = \|z-c'\| + \|c'-c''\| + \|c''-c\|$, a contradiction to $\|z-c\| = d(z,C)$. Therefore, $c \in \partial C$.

Since the sets A and D are bounded, there exists a number $\alpha > 0$ such that $b = c + \alpha(z-c) \notin A$. By Lemma 2.8, there are two points a and z' with $a \in L(c,b) \cap \partial A$ and $z' \in L(c,b) \cap \partial D$ such that

$$\|b-a\| \leqslant \|b-z'\| \leqslant \|b-z\| \leqslant \|b-c\|.$$

Then,

$$\|b-z'\| - \|b-a\| \leqslant \|b-z\| - \|b-a\| \leqslant \|b-c\| - \|b-a\|.$$

Hence, $\|a-z'\| \leqslant \|a-z\| \leqslant \|a-c\|$. Thus, we have

$$\min_{x\in\partial A} d(x,\partial D) \leqslant \|a-z'\| \leqslant \|a-z\| \leqslant \|a-c\| \leqslant \max_{x\in\partial A} d(x,\partial C),$$

which contradicts to the condition $\max_{x\in\partial A} d(x,\partial C) < \min_{x\in\partial A} d(x,\partial D)$. Therefore, we obtain that $D \subseteq C$. $\square$

Lemma 2.10 ([25]). *If* A *and* C *are non–empty, closed, bounded convex sets in* E, $h(A,C) = h(\partial A, \partial C)$.

3. Stable sets of fixed points under strong perturbations

Theorem 3.1. *For each* $F_A \in \Gamma_c$ *and each* $\varepsilon > 0$, *if the fixed point set* $\text{fix}(F_A) \neq \emptyset$, *then, there is* $\delta > 0$ *such that* $\text{fix}(G_D) \subseteq \text{fix}(F_A) + B_\varepsilon(0)$ *for any* $G_D \in N_c(F_A,\delta)$.

Proof. Since $\text{fix}(F_A) \neq \emptyset$, we have that the set $\text{fix}(F_A) + B_\varepsilon(0)$ is well defined. By way of contradiction, we assume that there is a number $\bar{\varepsilon} > 0$, a sequence $\{\delta_n\}$ with $\delta_n \to 0\,(n \to \infty)$ and a sequence $\{G^n_{D_n}\} \subset N_c(F_A,\delta_n)$ but there exists a point $x_n \in \text{fix}(G^n_{D_n})$ with $x_n \notin \text{fix}(F_A) + B_{\bar{\varepsilon}}(0)$ for each $n = 1, 2, \cdots$.

Due to the compactness of X, there is a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$ $(k \to \infty)$. From $\{G^n_{D_n}\} \subset N_c(F_A,\delta_n)$, we know that $h(D_n, A) < \delta_n$. Since $x_n \in \text{fix}(G^n_{D_n})$, we have $x_n \in D_n$. Hence, the distance of the point x_0 and the set A, $d(x_0,A)$, is as the following

$$d(x_0,A) \leqslant d(x_0,x_{n_k}) + d(x_{n_k},D_{n_k}) + h(D_{n_k},A) \to 0.$$

Then $x_0 \in A$, because A is closed.

By the upper semi–continuity of F_A, for any $\varepsilon > 0$, it can be found a number r with $0 < r < \frac{\varepsilon}{2}$ such that $F(B_r(x_0)) \subseteq F(x_0) + B_{\frac{\varepsilon}{2}}(0)$. Then as k is large enough, it holds that

$$F(B_{\delta_{n_k}}(x_0)) + B_{\delta_{n_k}}(0) \subseteq F(B_r(x_0)) + B_{\frac{\varepsilon}{2}}(0) \subseteq F(x_0) + B_\varepsilon(0). \tag{3.1}$$

Note that $x_n \in \text{fix}(G^n_{D_n})$ and $\{G^n_{D_n}\} \subset N_c(F_A, \delta_n)$, we have

$$x_n \in G^n(x_n) \subseteq F(B_{\delta_n}(x_n)) + B_{\delta_n}(0). \tag{3.2}$$

Therefore, as long as k is large enough, we obtain that $x_{n_k} \in F(x_0) + B_\varepsilon(0)$. Since ε is arbitrary, it gets that $x_0 \in F(x_0)$. Combing the result that $x_0 \in A$, it can be asserted that $x_0 \in \text{fix}(F_A)$, which contradicts with the assumption $x_{n_k} \notin \text{fix}(F_A) + B_{\bar{\varepsilon}}(0)$ for each $k = 1, 2, \cdots$. The proof is completed. □

Theorem 3.2. *For each $F_A \in \Gamma_{co}$ and each $\varepsilon > 0$, if the fixed point set $\text{fix}(F_A) \neq \emptyset$, then, there is $\delta > 0$ such that $\text{fix}(G_D) \subseteq \text{fix}(F_A) + B_\varepsilon(0)$ for any $G_D \in N_{co}(F_A, \delta)$.*

Proof. Follow the whole proof of Theorem 3.1, corresponding to the inclusion relation (3.1), it needs to prove

$$\text{co}(F(B_{\delta_{n_k}}(x_0)) + B_{\delta_{n_k}}(0)) \subseteq F(x_0) + B_\varepsilon(0).$$

In fact, from the fact (3.1) and Lemma 2.6, we have

$$\text{co}(F(B_{\delta_{n_k}}(x_0)) + B_{\delta_{n_k}}(0)) \subseteq \text{co}(F(x_0)) + B_\varepsilon(0).$$

Note that $F_A \in \Gamma_{co}$, F has convex values. Then, $\text{co}(F(x_0)) = F(x_0)$.

Corresponding to the inclusion relation (3.2), from $G^n_{D_n} \in N_{co}(F_A, \delta_n)$, we have

$$x_n \in G^n(x_n) \subseteq \text{co}(F(B_{\delta_n}(x_n)) + B_{\delta_n}(0)).$$

The other part follows the proof of Theorem 3.1. □

Remark 3.3. It can be observed from Theorem 3.1 and Theorem 3.2 that, for a correspondence

$$F_A \in \Gamma_c\ (\Gamma_{co}),$$

if $\text{fix}(F_A) \neq \emptyset$, then $\text{fix}(F_A)$ itself is strong stable with respect to N_c (N_{co}). Furthermore, for a correspondence $F_A \in \Gamma'_{co}$, then $\text{fix}(F_A)$ itself is strong stable with respect to N'_{co}, hence, a strong stable set does exist in this situation.

Theorem 3.4. *For a correspondence $F_A \in \Gamma_c$ (Γ_{co}), if $\text{fix}(F_A) \neq \emptyset$, then $\text{fix}(F_A)$ is robust stable with respect to N_c (N_{co}). For a correspondence $F_A \in \Gamma'_{co}$, $\text{fix}(F_A)$ is robust stable with respect to N'_{co}*

Proof. For the case that $F_A \in \Gamma_c$, from Theorem 3.1, for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$\text{fix}(G_D) \subset \text{fix}(F_A) + B_\varepsilon(0), \quad \forall\, G_D \in N_c(F_A, \delta).$$

By Remark 3.3, the set $\text{fix}(F_A)$ is strong stable, and note that $\text{fix}(G_D)$ is also strong stable. Then, $\text{fix}(F_A)$ is robust stable. Similarly, for the case that $F_A \in \Gamma_{co}$ or $F_A \in \Gamma'_{co}$, the result follows from Theorem 3.2 and Remark 3.3. □

Remark 3.5. Theorem 3.4 shows that the robust stable set exists if the correspondence has at least one fixed point. The following results further show the size of robust stable sets can be reduced.

The following analysis considers the perturbation in Γ'_{co}, which restricts the perturbation of X in itself.

Theorem 3.6. *Let $F_X \in \Gamma'_{co}$. If the fixed point set $\text{fix}(F_X)$ satisfies that $\text{fix}(F_X) \subset \text{int}(X)$, then, there is a robust stable component in $\text{fix}(F_X)$ with respect to N'_{co}.*

Proof. Firstly, it is clear that $\text{fix}(F_X) \neq \emptyset$. If there is only one component in $\text{fix}(F_X)$, then the result follows from Theorem 3.4. If not, take an arbitrary component C_{α_1} from $\text{fix}(F_X) = \cup_{\alpha \in \Lambda} C_\alpha$. Denote $\text{fix}(F_A) \backslash C_{\alpha_1}$ by C_1. Since $\text{fix}(F_X)$ is strong stable, for any $\varepsilon > 0$, there is $\delta > 0$ such that for each $S_A \in \Gamma_c$ with $S_A \in N'_{co}(F_X, \delta)$, it holds that $\text{fix}(S_A) \subset \text{fix}(F_X) + B_\varepsilon(0)$.

Since $fix(F_X) \subset intX$, there exists a special positive number ε with $(C_{\alpha_1} + \bar{B}_\varepsilon(0)) \cap (C_1 + \bar{B}_\varepsilon(0)) = \emptyset$ and $C_{\alpha_1} + \bar{B}_\varepsilon(0), C_1 + \bar{B}_\varepsilon(0) \subset intX$, where $\bar{B}_\varepsilon(0)$ is the closure of the open ball $B_\varepsilon(0)$. Denote the boundaries of $C_{\alpha_1} + \bar{B}_\varepsilon(0)$ and $C_1 + \bar{B}_\varepsilon(0)$ by $\partial(\bar{B}_\varepsilon(C_{\alpha_1}))$ and $\partial(\bar{B}_\varepsilon(C_1))$, respectively. Let $\delta_{\alpha_1} = \min_{x \in \partial X} d(x, \partial(\bar{B}_\varepsilon(C_{\alpha_1})))$ and $\delta_1 = \min_{x \in \partial X} d(x, \partial(\bar{B}_\varepsilon(C_1)))$.

Assume that C_{α_1} and C_1 are both not roust stable. There exists $G_V \in N'_{co}(F_X, \min\{\frac{\delta}{3}, \delta_{\alpha_1}\})$ such that each component C^G in $fix(G_V)$ satisfies that $C^G \not\subset C_{\alpha_1} + B_\varepsilon(0)$. Because any component in $fix(G_V)$ is connected, which cannot be separated by two disjoint open sets $C_{\alpha_1} + B_\varepsilon(0)$ and $C_1 + B_\varepsilon(0)$. Then, by Theorem 3.2 and Remark 3.3, we have $fix(G_V) \subset C_1 + B_\varepsilon(0)$. Similarly, there exists $H_D \in N'_{co}(F_X, \min\{\frac{\delta}{3}, \delta_1\})$ such that $fix(H_D) \subset C_{\alpha_1} + B_\varepsilon(0)$.

Since $G_V \in N'_{co}(F_X, \min\{\frac{\delta}{3}, \delta_{\alpha_1}\})$, we have $h(X, V) < \delta_{\alpha_1}$. From Lemma 2.10, it holds that $h(X, V) = h(\partial X, \partial V)$. Noting that $V \in K(X)$, then,

$$h(\partial X, \partial V) = \max_{x \in \partial X} d(x, \partial V) < \min_{x \in \partial X} d(x, \partial(\bar{B}_\varepsilon(C_{\alpha_1}))) = \delta_{\alpha_1},$$

by Lemma 2.9, we have $C_{\alpha_1} + \bar{B}_\varepsilon(0) \subseteq V$. Similarly, it holds that $C_1 + \bar{B}_\varepsilon(0) \subseteq D$.

We define a correspondence T with $(T, X) \in U_{co}(X) \times K(X)$ as the following:

$$T(x) = \begin{cases} G(x), \ x \in C_{\alpha_1} + B_\varepsilon(0), \\ H(x), \ x \in C_1 + B_\varepsilon(0), \\ co(F(x + \bar{B}_{\frac{\delta}{2}}(0)) + \bar{B}_{\frac{\delta}{2}}(0)) \cap X, \text{ elsewhere.} \end{cases}$$

It will check that $(T, X) \in \Gamma'_{co}$. Firstly, Since $G, H, F \in \Gamma'_{co}$, for each $x \in X$, we have $G(x)$ and $H(x)$ are convex and closed sets, and it is also closed and convex for the set $co(F(x + \bar{B}_{\frac{\delta}{2}}(0)) + \bar{B}_{\frac{\delta}{2}}(0))$. Then, $T(x)$ is closed and convex for each $x \in X$. Secondly, it is also clear that $T(x) \subseteq X$ for each $x \in X$.

Next, we need to prove the upper semi–continuity of T on X. Since G is upper semi–continuous on X, then, for each $x \in C_{\alpha_1} + B_\varepsilon(0)$ and each $\eta > 0$, there is $r > 0$ such that $x + B_r(0) \subseteq C_{\alpha_1} + B_\varepsilon(0)$ and $G(x + B_r(0)) \subseteq G(x) + B_\eta(0)$, that is, $T(x + B_r(0)) \subseteq T(x) + B_\eta(0)$. Hence, T is upper semi–continuous on $C_{\alpha_1} + B_\varepsilon(0)$. Similarly, it can be checked that T is also upper semi–continuous on $C_1 + B_\varepsilon(0)$.

For each $x \notin (C_{\alpha_1} + B_\varepsilon(0)) \cup (C_1 + B_\varepsilon(0))$, it needs to check the upper semi-continuity of T. Let $P : X \to 2^X$ such that $P(z) = co(F(z + \bar{B}_{\frac{\delta}{2}}(0)) + \bar{B}_{\frac{\delta}{2}}(0)), \forall z \in X$. Let $P'(z) = P(z) \cap X, \forall z \in X$. Since $F(z) \subseteq X$ for each $z \in X$, we know that $P'(z) \neq \emptyset$, hence, $P' : X \to 2^X$ is well defined and P' is upper semi–continuous. From Lemma 2.7 and the upper semi–continuity of P', for each $\eta > 0$, there is a number r with $0 < r < \frac{\delta}{6}$ such that,

$$co(F(x + B_r(0) + \bar{B}_{\frac{\delta}{2}}(0)) + \bar{B}_{\frac{\delta}{2}}(0)) \cap X \subseteq co(F(x + \bar{B}_{\frac{\delta}{2}}(0)) + \bar{B}_{\frac{\delta}{2}}(0)) \cap X + B_\eta(0). \tag{3.3}$$

It aims to show that $T(x + B_r(0)) \subset T(x) + B_\eta(0)$.

(a) Let y be a point in the set $x + B_r(0)$. If $y \in (x + B_r(0)) \cap (C_{\alpha_1} + B_\varepsilon(0))$, because of the fact that $G_V \in N'_{co}(F_X, \frac{\delta}{3})$, we have

$$\begin{aligned} T(y) = G(y) &\in co(F(y + B_{\frac{\delta}{3}}(0)) + B_{\frac{\delta}{3}}(0)) \\ &\subseteq co(F(y + B_r(0) + \bar{B}_{\frac{\delta}{3}}(0)) + B_{\frac{\delta}{2}}(0)) \\ &\subseteq co(F(y + \bar{B}_{\frac{\delta}{2}}(0)) + \bar{B}_{\frac{\delta}{2}}(0)). \end{aligned}$$

Note that $G(y) \subseteq X$, it holds that

$$T(y) = G(y) \subseteq co(F(y + \bar{B}_{\frac{\delta}{2}}(0)) + \bar{B}_{\frac{\delta}{2}}(0)) \cap X. \tag{3.4}$$

(b) Similarly, when $y \in (x + B_r(0)) \cap (C_1 + B_\varepsilon(0))$, then

$$T(y) = H(y) \in co(F(y + \bar{B}_{\frac{\delta}{2}}(0)) + \bar{B}_{\frac{\delta}{2}}(0)) \cap X. \tag{3.5}$$

(c) If $y \in x + B_r(0)$ and $y \notin (C_{\alpha_1} + B_\varepsilon(0)) \cup (C_1 + B_\varepsilon(0))$, then

$$T(y) = co(F(y + \bar{B}_{\frac{\delta}{2}}(0)) + \bar{B}_{\frac{\delta}{2}}(0)) \cap X. \tag{3.6}$$

Thus, we obtain that $T(x + B_r(0)) \subseteq co(F(x + B_r(0) + \bar{B}_{\frac{\delta}{2}}(0)) + \bar{B}_{\frac{\delta}{2}}(0)) \cap X$ from (3.4), (3.5) and (3.6). Noting the fact (3.3), we have

$$\begin{aligned} T(x + B_r(0)) &\subseteq co(F(x + \bar{B}_{\frac{\delta}{2}}(0))) + \bar{B}_{\frac{\delta}{2}}(0)) \cap X + B_\eta(0) \\ &= T(x) + B_\eta(0). \end{aligned}$$

Therefore, T is also upper semi–continuous for each $x \notin (C_{\alpha_1} + B_\varepsilon(0)) \cup (C_1 + B_\varepsilon(0))$, hence, T is upper semi–continuous on X. We obtain that $T_X \in \Gamma'_{co}$, then, $fix(T_X) \neq \emptyset$.

In addition, it is easy to check that for each $x \in X$, $T(x) \in co(F(x + B_\delta(0)) + B_\delta(0))$, that is, $T_X \in N_{co}(F_X, \delta)$. By Theorem 3.4, we know that

$$fix(T_X) \subseteq (C_{\alpha_1} + B_\varepsilon(0)) \cup (C_1 + B_\varepsilon(0)). \tag{3.7}$$

For a point $x \in fix(T_X)$, we have $x \in T(x)$ and $x \in X$. If the point x satisfies that $x \in C_{\alpha_1} + B_\varepsilon(0) \subseteq V$, then, $x \in G(x)$. Hence, it is true that $x \in fix(G_V)$. Noting that $fix(G_V) \subset C_1 + B_\varepsilon(0)$, it holds that $x \in C_1 + B_\varepsilon(0)$, which is a contradiction to $x \in C_{\alpha_1} + B_\varepsilon(0)$. Similarly, if it is the case that $x \in C_1 + B_\varepsilon(0)$, then $x \in H(x) = T(x)$, and it will deduce a contradiction between $x \in C_{\alpha_1} + B_\varepsilon(0)$ and $x \in C_1 + B_\varepsilon(0)$. Thus, we get that there is no fixed point of T_X which belongs to $(C_{\alpha_1} + B_\varepsilon(0)) \cup (C_1 + B_\varepsilon(0))$, a contradiction to the fact (3.7).

Therefore, we assert that it is false that C_{α_1} and C_1 are both not robust stable. If C_{α_1} is robust stable, the result is obtained. If not, the set C_1 is robust stable, then, we can repeat the above steps by taking a component C_{α_2} from C_1, and denote $C_1 \backslash C_{\alpha_2}$ by C_2. Follow the repeat process, if the total number of components is finite, it will stops at a robust stable component. If the number is infinite, it will lead to two situations:

a) a component C_{α_i} with $i \geqslant 1$ is robust stable;

b) there is a sequence of robust stable sets $\{C_i\}_{i=1}^{\infty}$ with $C_j \subset C_k$ for any $j > k$, where each C_i consists of some components of $fix(F_X)$, furthermore, it is intersection $C_0 = \cap_{i=1}^{\infty} C_i$ is a robust stable component.

The proof is completed. □

Theorem 3.7.

(a) *Let $F_A \in \Gamma_c$ (Γ_{co}) with $fix(F_A) \neq \emptyset$. There exists a minimal robust stable set in $fix(F_A)$ with respect to N_c (N_{co}).*

(b) *For each $F_X \in \Gamma'_{co}$, there exists a minimal robust stable set C in $fix(F_X)$ with respect to N'_{co}. If a minimal robust stable set $C \subseteq fix(F_X)$ satisfies that $C \subset intX$, then C is connected.*

Proof.

(a) From Theorem 3.4, $fix(F_A)$ itself is a robust stable set. In the collection of all robust stable subsets in $fix(F_A)$, ordered by set inclusion, every decreasing chain consisting of robust stable sets has its intersection as a lower bound. Then, there exists a minimal robust stable set C in $fix(F_A)$ by the Zorn's Lemma.

(b) For each $F_X \in \Gamma'_{co}$, we have that $fix(F_X) \neq \emptyset$. Then, from the part (a), there is a minimal robust stable set $C \subseteq fix(G_X)$ with respect to $N'_{co} \subseteq N_{co}$. By using C instead of $fix(F_X)$ in the proof of Theorem 3.6, if C consists of two disjoint sets C_{α_1} and C_1, then one of them is robust stable as the proof of Theorem 3.6, which contradicts to the fact that C is a minimal element in the collection of robust stable subsets in $fix(G_X)$. Therefore, C is connected. □

For a correspondence $F_A \in \Gamma_c$ with $\text{fix}(F_A) \neq \emptyset$, noting the relations (2.1) and (2.2), any strong stable set in $\text{fix}(F_A)$ with respect to N_{co} is also strong stable with respect to N_c, N'_{co} and N_s.

Let $F_X \in \Gamma_c$ with $\text{fix}(F_X) \neq \emptyset$. If we do not consider the perturbation of X, similar to the Definition 2.1, we can define the strong stable set (minimal stable set) with respect to $\mathcal{P}(N_{co}), N_X$, N_g or N. Then, in this special case of no perturbation of X, from the expressions (2.3) and (2.4), a strong stable set in $\text{fix}(F_X)$ with respect to $\mathcal{P}(N_{co})$ is also strong stable with respect to N_X and N.

Theorem 3.8. *Let $F \in U_c$ and fix$(F) \neq \emptyset$ on X.*

(a) *If a set C with $C \subseteq \text{fix}(F)$ is strong stable with respect to $\mathcal{P}(N_{co})$, then C is strong stable with respect to N_X and N.*

(b) *If C is minimal strong stable with respect to $\mathcal{P}(N_{co})$, then there exists a minimal strong stable set D such that $D \subseteq C$ with respect to N_X and N.*

Proof. From the relations (2.3) and (2.4), the part (a) is true. For the part (b), since C is minimal strong stable with respect to $\mathcal{P}(N_{co})$, by the part (a), we get that C is also strong stable with respect to N_X and N. For the collection of all strong stable sets (ordered by set inclusion) with respect to N_X or N in C, each deceasing chain has a lower bound, which is the intersection of the chain. By the Zorn's lemma, there is a minimal element (strong stable set) D in C. □

Theorem 3.9. *Let $F_A \in \Gamma_c$ with $\text{fix}(F_A) \neq \emptyset$. If a set $C \subseteq \text{fix}(F_A)$ is strong stable with respect to N_c, then C is a robust stable set.*

Proof. Since C is strong stable, for each $\varepsilon > 0$, there exists a number $\delta > 0$ such that $\text{fix}(G_D) \cap (C + B_{\frac{\varepsilon}{2}}(0)) \neq \emptyset$ for any $G_D \in N_c(F_A, \delta)$ with $\text{fix}(G_D) \neq \emptyset$. Take an arbitrary $G_D \in N_c(F_A, \frac{\delta}{2})$ and $\text{fix}(G_D) \neq \emptyset$. Let $S = \text{fix}(G_D) \cap (C + B_\varepsilon(0)) \subseteq \text{fix}(G_D)$. Next, we show that S is a strong stable set of G_D.

If we assume that S is not strong stable, then there is a number $\varepsilon' > 0$ and a sequence $\{\delta_n\}$ with $\delta_n < \frac{\delta}{2}$ and $\delta_n \to 0\, (n \to \infty)$, and for each $n = 1, 2, \cdots$, there exists $H^n_{D_n} \in N_c(G_D, \delta_n) \in N_c(G_D, \frac{\delta}{2})$ with $\text{fix}(H^n_{D_n}) \neq \emptyset$ but $\text{fix}(H^n_{D_n}) \cap (S + B_{\varepsilon'}(0)) = \emptyset$. From $H^n_{D_n} \in N_c(G_D, \delta_n)$, we have

$$H^n(x) \subseteq G(x + B_{\delta_n}(0)) + B_{\delta_n}(0) \text{ for each } x \in X.$$

Since $G_D \in N_c(F_A, \frac{\delta}{2})$, we have $G(x) \subseteq F(x + B_{\frac{\delta}{2}}(0)) + B_{\frac{\delta}{2}}(0)$ for each $x \in X$. Then,

$$\begin{aligned} G(x + B_{\delta_n}(0)) + B_{\delta_n}(0) &\subseteq F(x + B_{\frac{\delta}{2}}(0) + B_{\delta_n}(0)) + B_{\frac{\delta}{2}}(0) + B_{\delta_n}(0) \\ &\subseteq F(x + B_\delta(0)) + B_\delta(0). \end{aligned}$$

Thus, we get that $H^n(x) \subseteq F(x + B_\delta(0) + B_\delta(0))$. In addition, since $G_D \in N_c(F_A, \frac{\delta}{2})$ and $H^n_{D_n} \in N_c(G_D, \delta_n)$, we have that $h(D_n, D) < \delta_n < \frac{\delta}{2}$ and $h(D, A) < \frac{\delta}{2}$, hence, $h(D_n, A) < \delta$. Therefore, $H^n_{D_n} \in N_c(F_A, \delta)$. So, we can find that $x_n \in H^n(x_n)$ and $x_n \in D_n$ such that $x_n \in C + B_{\frac{\varepsilon}{2}}(0)$. By the compactness of X, without loss of generality, assume that $x_n \to x_0\, (n \to \infty)$. Combing that $h(D_n, D) < \delta_n$ and $x_n \in D_n$, it can be obtained that $x_0 \in D$. In addition, we have $x_0 \in C + B_\varepsilon(0)$ from the fact that $x_n \in C + B_{\frac{\varepsilon}{2}}(0)$. Using the upper semi–continuity of G_D, for each $\eta > 0$, as n is large enough, we have that

$$x_n \in H^n(x_n) \subseteq G(x_n + B_{\delta_n}(0)) + B_{\delta_n}(0) \subseteq G(x_0) + B_\eta(0).$$

From the arbitrariness of η, it follows that $x_0 \in G_(x_0)$. Then, $x_0 \in \text{fix}(G_D)$, further, $x_0 \in S$. Hence, as n tends to infinity, it holds that $x_n \in S + B_{\varepsilon'}(0)$ which contradicts to $\text{fix}(H^n_{D_n}) \cap (S + B_{\varepsilon'}(0)) = \emptyset$. □

Remark 3.10. From the Remark 2.4 and Theorem 3.9, a robust stable set with respect to N_c is equivalent to a strong stable set with respect to N_c, however, the definition of a robust stable set is more meaningful than that of a strong stable set. This reveals that a strong stable set (component) of fixed points for a given correspondence admits a strong stable set (component) in it is neighborhood, rather than a fixed point for any perturbation which is near the correspondence.

4. An application to stable sets of economy equilibria

This gives an application in the stability of equilibria of Abstract Economy. For the convenience of applications, we rewrite some signs in order to show the stability of equilibria of abstract economy.

Let I be an index set. For each $i \in I$, X_i is a nonempty, compact and convex subset in a normed linear space E_i. Let $X = \times_{i\in I} X_i \subset E$ with $E = \times_{i\in I} E_i$ and $X_{-i} = X \setminus X_i$. For each $i \in I$, the correspondence $\mathcal{R}_i : X \to 2^{X_i}$ has nonempty, compact and convex values, and the function $\mathcal{U}_i : X \to \mathbb{R}$ is continuous and quasiconcave in $x_i \in X_i$. Then $\Omega = \{X_i, \mathcal{R}_i, \mathcal{U}_i, i \in I\}$ is an abstract economy, and it aims to find an equilibrium $\bar{x} \in X$ such that,

$$\mathcal{U}_i(\bar{x}) = \mathcal{U}_i(\bar{x}_i, \bar{x}_{-i}) = \sup_{z_i\in\mathcal{R}_i(\bar{x})} \mathcal{U}_i(z_i, \bar{x}_{-i}), \quad \forall i \in I.$$

For each $x \in X$ and $i \in I$, let

$$F_i(x) = \{y_i \in X_i : \mathcal{U}_i(y_i, x_{-i}) = \sup_{z_i\in\mathcal{R}_i(x)} \mathcal{U}_i(z_i, x_{-i})\} \subseteq \mathcal{R}_i(x).$$

Denote F such that $F(x) = \times_{i\in I} F_i(x)$, for each $x \in X$. Then, F is a correspondence on X. From [24, Theorem 4.12], F has upper semi–continuous, compact and convex values. Each equilibria of such abstract economy Ω is equivalent to a fixed point of F on X, and $\text{fix}(F) \neq \emptyset$.

Clearly, each $\mathcal{R}$ and $\mathcal{U}$ corresponds to a correspondence F, then we can define U_{co} as

$$U'_{co}(X) = \{F : X \to 2^X \mid \mathcal{R} \text{ and } \mathcal{U} \text{ satisfy the above conditions}\}.$$

The corresponding Γ''_{co} and N''_{co} are given below

$$\Gamma''_{co} = \{(F, A) \in U'_{co}(X) \times K(X) \mid F(x) \subseteq A, \forall x \in A\},$$

and for a correspondence $F_A \in \Gamma''_{co}$,

$$N''_{co}(F_A, \delta) = \{G_D \in \Gamma''_{co} \mid G(x) \in \text{co}F_{b\delta}(x), \forall x \in X; h(A, D) < \delta\}.$$

Theorem 4.1. *For each $F_X \in \Gamma''_{co}$, there exists a minimal robust stable set C with $C \subseteq \text{fix}(F_X)$ with respect to N''_{co}. Further, if C is a subset of $\text{int}(X)$, then C is connected.*

Proof. It can be found that $U'_{co} \subseteq U_{co}$, $\Gamma''_{co} \subseteq \Gamma'_{co}$, $N''_{co}(F_X, \delta) \subseteq N'_{co}(F_X, \delta)$ and $\text{fix}(F_X) \neq \emptyset$, $\forall F_X \in \Gamma''_{co}$. Then, the result follows from Theorem 3.7. □

It should be pointed that a usual kind of perturbation (shaking, like that of perfect equilibria in normal–form games) of $(F, X) \in U_{co}(X) \times K(X)$ restricts the response $F_i(x)$ for a point $x \in X$ in the range of $K(\text{int}(X_i))$ due to agents' bounded rationality and decision errors. Based on this point, we write Γ_{int} as

$$\Gamma_{\text{int}} = \{(F, A) \in U'_{co}(X) \times K(\text{int}(X)) \mid F(x) \subseteq A, \forall x \in A\}.$$

Naturally, it holds that $\Gamma_{\text{int}} \subseteq \Gamma'_{co}$. Theorem 4.1 means that a minimal robust stable set C with respect to N'_{co} is not only robust to shaking in Γ_{int}, but also robust to more perturbation in Γ'_{co}. It should also be pointed that the abstract economy is a generalization of a normal–form noncooperation game. In a normal–form game, it is known that the connectedness and minimality of a stable set fall into the famous axiomatic requirements in [16], and the requirements are analyzed deeply in [10, 11].

5. Conclusion

This obtains the strong stability of fixed points for two kinds correspondences in a normed linear space:

(1) those correspondences with upper semi–continuities and closed values;

(2) correspondences with upper semi–continuities, closed and convex values.

A conception of robust stable set in the fixed point set of a correspondence is introduced. The existence of a robust stable set is guaranteed for the second kind of correspondence. And a roust stable set exists if there exists at least one fixed point for the first kind of correspondence.

The results, in relation to the existence of a minimal strong stable set in fixed point sets for correspondences, generalize the corresponding results in [28] and [27] from N neighbors to N_{co} (N_c, N'_{co}) neighbors, from the above second kind of correspondence to the first kind of correspondence, and from no perturbation of the domain X to the perturbation set $K(X)$. The results also generalize the corresponding results from several aspects in [22] such that, the defined space is generalized from an Euclidean space to a normed linear space; the perturbation of a correspondence is enlarged from an N_X neighbor to a strong perturbed N_{co} (N_c, N'_{co}) neighbor; the perturbation of the domain X is extended from $K(\text{int}(X))$ to $K(X)$.

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