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Quasi-effective stability for time-dependent nearly integrable Hamiltonian systems

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Abstract

This paper deals with the stability of the orbits for time-dependent nearly integrable Hamiltonian systems. Under the classical non-degeneracy in KAM theory we prove that the considered system possesses quasi-effective stability. Our result generalized the works in [F. Z. Cong, J. L. Hong, H. T. Li, Dyn. Syst. Ser. B, **21** (2016), 67–80] to time-dependent system and gave a connection between KAM theorem and effective stability.

Keywords: Quasi-effective stability, non-degeneracy, time-dependent system.

2010 MSC: 37J25, 37J40.

1. Introduction

Consider the Hamiltonian system

$$\dot{p} = -H_q(p,q,t), \;\; \dot{q} = H_p(p,q,t)$$

with the Hamiltonian

$$H(p,q,t) = h(p) + \epsilon f(p,q,\theta), \;\; \theta = \omega_* t, \tag{1.1}$$

where $(p,q,\theta) \in \mathcal{D} \times \mathbb{T}^n \times \mathbb{T}^m$, $\mathcal{D}$ is some bounded domain in $\mathbb{R}^n$, and $\omega_* \in \mathbb{R}^m$ is a given vector. In 2013 Boenemoura has researched a non-autonomous perturbation of an integrable Hamiltonian system, and he has proved that the system possesses effective stability, if the perturbation depends slowly on time and the integrable Hamiltonian is convex [2].

This paper deals with the similar system which is slow time-dependent nearly integrable Hamiltonian systems. It is different from the work in [2] that we suppose the integrable Hamiltonian to satisfy the classical non-degeneracy by replacing the convexity. We obtain that the system is quasi-effective stable, if the perturbation ϵf is quasi-periodic in time t.

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The definition of quasi-effective stability is proposed in the literatures [3, 4]. This concept is a generalization of effective stability developed by Nekhoroshev [8].

In 1995 Morbidelli and Giorgilli researched a kind of nearly integrable Hamiltonian systems in term of effective stability, and proved the diffusion speed of KAM tori be zero [7]. In this paper we should obtain that under the conditions of KAM theorems, the slow time-dependent nearly integrable Hamiltonian system possesses quasi-effective stability. The above results demonstrate the connections between KAM theory and effective stability.

For the study of the hydrogen atom by applying perturbation theory see [5] and references therein. These works concern with the specific Kepler Hamiltonion with a small perturbation, which represents the external fields. Recently, Fasso et al. [6] used Nekhoroshey theory [1] to discuss the perturbed hydrogen atom. Our results can be applied in this area.

Definition 1.1. System (1.1) is said to be quasi-effective stable if there exist positive constants a, b, c, d and ϵ_0 such that, for any $\epsilon \in (0, \epsilon_0]$, there is an open subset $\mathcal{E}_\epsilon$ of $\mathcal{D}$ suiting the following.

(1) $\mathrm{meas}\mathcal{E}_\epsilon = \mathrm{meas}\mathcal{D} - O(\epsilon^d)$.

(2) For all $(p_0, q_0) \in \mathcal{E}_\epsilon \times \mathbb{T}^n$, the orbit $(p(t), q(t))$ starting from (p_0, q_0) satisfies the estimate

$$|p(t) - p_0| \leqslant c\epsilon^b,$$

provided $|t| \leqslant \exp(c\epsilon^{-a})$.

Here a and b are called stable exponents of the system, $T(\epsilon) = \exp(c\epsilon^{-a})$ stable time, $R(\epsilon) = c\epsilon^b$ stable radius.

It is obvious that the effective stability implies the quasi-effective stability from the above definition. Assume that h, f, and ω_* satisfy the following.

(H1) h and f are real analytic functions defined on $(\mathcal{D} \times \mathbb{T}^n \times \mathbb{T}^m) + \rho$ with respect to variables p, q, and θ, where ρ is a small positive constant, and "$\cdot + \rho$" denotes ρ-neighborhood in the complex space of a given subset.

(H2) ω_* suits the inequalities

$$|\langle l, \omega_* \rangle| > \gamma |l|^{-\tau} \ \text{ for all } l \text{ with } 0 \neq l \in \mathbb{Z}^m,$$

for some positive constants γ and $\tau > n + m$, that is, ω_* satisfies the usual Diophantine condition.

(H3) $\omega(p)$ satisfies the classical non-degenerate condition as follows

$$\det\left(\frac{\partial \omega}{\partial p}\right) = n, \ \forall p \in \mathrm{Re}(\mathcal{D} + \rho),$$

where $\omega(p) = \frac{\partial h}{\partial p}(p)$, and $\mathrm{Re}(\mathcal{D} + \rho) = (\mathcal{D} + \rho) \bigcap \mathbb{R}^n$.

Theorem 1.2. *Under assumptions* (H1)-(H3) *system* (1.1) *is quasi-effective stable.*

2. Normal form

From now we prove Theorem 1.2. To this end, we need some preliminaries. Choose

$$\alpha = \kappa = \epsilon^{\frac{1}{2(2\tau+7)}}, \quad L(\kappa) = \left[\frac{1}{3\kappa}(\ln 8 + (n+m)(\ln 2 + \ln(n+m) - 1) - (n+m+1)\ln\kappa)\right] + 1.$$

Here, for any real number r, $[r]$ denotes the integer part of r. Let

$$\mathcal{E}_\epsilon = \{p \in \mathcal{D} : |\langle k, \omega(p)\rangle + \langle l, \omega_*\rangle| > \alpha(|k| + |l|)^{-\tau} \text{ for all } (k, l) \text{ with } 0 < |k| + |l| \leqslant L(\kappa)\},$$

where α is a function in ϵ, $0 < \alpha < \gamma$.

For a given $p_0 \in \mathcal{D}$, define a neighborhood of p_0 as follows

$$\mathcal{O}(p_0, \epsilon) = \{p : p \in \mathcal{D}, |p - p_0| < \sqrt{\epsilon}\}.$$

Let

$$\mathcal{O}(0, 1) = \{P : P \in \mathbb{R}^n, |P| < 1\},$$

and denote

$$M = \max\left\{ \sup_{(y,x,\theta,\varepsilon)\in((\mathcal{D}\times\mathbb{T}^n\times\mathbb{T}^m)+\rho)\times[0,1]} |f(y, x, \theta, \varepsilon)|, \sup_{y\in\mathcal{D}+\rho} |h(y)|, \sup_{y\in\mathcal{D}+\rho} |\omega(y)|, |\omega_*| \right\}.$$

Theorem 2.1. *Assume $\omega(p_0)$ satisfies that, for constants $\alpha > 0$ and $\tau > 0$, $\omega(p_0)$ suits the inequalities*

$$|\langle k, \omega(p_0)\rangle + \langle l, \omega_*\rangle| > \alpha(|k| + |l|)^{-\tau} \tag{2.1}$$

for any $(k, l) \in \mathbb{Z}^n \times \mathbb{Z}^m$ with $0 < |k| + |l| \leqslant L(\kappa)$. Then there is a positive constant ϵ_0 depending on M, n, m, τ and ρ such that, for all ϵ with $0 \leqslant \epsilon \leqslant \epsilon_0$, the following statements hold.

(1) *There exists a transformation Φ_* and a near-identity coordinates transformation Ψ_*, defined on $\mathcal{O}(0, 1) \times \mathbb{T}^n$, to reduce Hamiltonian (1.1) to the form*

$$H_* = N_* + \sqrt{\epsilon} f_*$$

with

$$N_*(Y, \epsilon) = \langle \omega(p_0), Y\rangle + O(\sqrt{\epsilon}),\ \omega_*(Y, \epsilon) = \frac{\partial N_*}{\partial Y}(Y, \epsilon) = \omega(p_0) + O(\sqrt{\epsilon}),\ |f_*| \leqslant M\sqrt{\epsilon}\exp\left(-\frac{\rho \ln 2}{18} \cdot \frac{1}{\kappa}\right).$$

(2) *For all $(p(0), q(0)) \in \mathcal{O}(p_0, \epsilon) \times \mathbb{T}^n$, there is a torus*

$$\hat{p}(t) = p(0),\ \hat{q}(t) = q(0) + \omega_*(p_0, p(0), q(0), \epsilon)t\,(\mathrm{mod}2\pi),\ t \in \mathbb{R}$$

with

$$\omega_*(p_0, p(0), q(0), \epsilon) = \omega(p_0) + O(\sqrt{\epsilon}),$$

such that the orbit $(p(t), q(t))$ starting from $(p(0), q(0))$ of (1.1) to satisfy the estimates

$$|p(t) - \hat{p}(t)| \leqslant 5\kappa^{\tau+4}\sqrt{\epsilon}, \quad |q(t) - \hat{q}(t)| \leqslant 10\left(\frac{8M^2}{\rho^2} + 1\right)\frac{M}{\rho}\kappa^{\tau+1}\sqrt{\epsilon},$$

provided $|t| \leqslant \exp\left(\frac{\rho \ln 2}{54} \cdot \frac{1}{\kappa}\right)$.

We introduce a coordinate transformation $\Phi_* : (\mathcal{O}(0, 1) \times \mathbb{T}^n) + \rho \to (\mathcal{O}(p_0, \epsilon) \times \mathbb{T}^n) + \rho$,

$$p = p_0 + \sqrt{\epsilon}P,\ q = Q.$$

Thus,

$$\hat{H}(P, Q, t) = \frac{1}{\sqrt{\epsilon}}H(p, q, t) = \frac{1}{\sqrt{\epsilon}}h(p_0) + \langle \omega(p_0), P\rangle + O(\sqrt{\epsilon}P^2) + \sqrt{\epsilon}f(p_0 + \sqrt{\epsilon}P, Q, \theta).$$

Simply write

$$\varepsilon = \sqrt{\epsilon},\quad \omega_0 = \omega(p_0),\quad e_0 = e_0(\varepsilon, p_0) = \frac{1}{\varepsilon}h(p_0),\quad f_0(P, Q, \theta, \varepsilon) = O(\varepsilon P^2) + \varepsilon f(p_0 + \varepsilon P, Q, \theta).$$

Then

$$\hat{H}(P, Q, t) = e_0 + \langle \omega_0, P\rangle + \varepsilon f_0(P, Q, \theta, \varepsilon). \tag{2.2}$$

Rewrite (2.2) as follows

$$H_0(y, x, t) = e_0 + \langle \omega_0, y\rangle + \varepsilon f_0(y, x, \theta, \varepsilon) = N_0(y, \varepsilon) + \varepsilon f_0(y, x, \theta, \varepsilon),$$

defined on $(\mathcal{O}(0, 1) \times \mathbb{T}^n \times \mathbb{T}^m) + \rho$, and $\theta = \omega_* t$.

3. Proof of Theorem 2.1

We prove Theorem 2.1 by employing the inductive method. Here the KAM technique is used. Take rapidly convergent sequences as follows

$$\mathbb{D}_k = (\mathbb{O}(0,1)\times\mathbb{T}^n\times\mathbb{T}^m)+\rho_k = \mathbb{D}_* + \rho_k,\quad \rho_k=\rho-9k\kappa,\ k=0,1,2,\dots,T(\kappa).$$

Here $T(\kappa)$ is an integer depending on κ which is determined bellow, and κ is a parameter depending only on ε.

Assume that we have been constructed a series of symplectic changes $\Phi_{k-1}:\mathbb{D}_k\to\mathbb{D}_{k-1}$, $k=0,1,\dots,i$, such that, under every transformation $\Psi_k=\Phi_0\circ\Phi_1\circ\cdots\circ\Phi_{k-1}$ which is a symplectic change defined on $\mathbb{D}_k$, Hamiltonian (1.1) is reduced into the Hamiltonian

$$H_k(y,x,t)=N_k(y,\varepsilon)+\varepsilon f_k(y,x,\theta,\varepsilon),\ k=0,1,\cdots,i,$$

with the following relations

$$\begin{aligned} N_k(y,\varepsilon)&=N_0(y)+\varepsilon\sum_{j=0}^{k-1}f_{j0}(y,\varepsilon),\ k=1,\dots,i,\\ |f_k|_{\rho_k}&\leqslant\frac{1}{2^k}M,\ k=0,1,\dots,i. \end{aligned}\tag{3.1}$$

Here

$$f_{k0}(y,\varepsilon)=\frac{1}{\mathrm{Vol}\mathbb{T}^{n+m}}\int_{\mathbb{T}^{n+m}}f_k(y,x,\theta,\varepsilon)dxd\theta.$$

Now consider the i^{th} iteration. Introduce a symplectic change $\Phi_i:(Y,X)\to(y,x)$,

$$y=Y+\varepsilon\frac{\partial S_i(Y,x,\theta)}{\partial x},\ X=x+\varepsilon\frac{\partial S_i(Y,x,\theta)}{\partial Y},$$

by using generating function εS_i+Yx, which reduces $H_i(y,x,t)$ into the following

$$\begin{aligned} H_{i+1}(Y,X,t)&=H_i\circ\Phi_i(Y,X,\theta)+\varepsilon\frac{\partial S_i}{\partial t}\\ &=N_i(Y,\varepsilon)+f_{i0}(Y,\varepsilon)+\varepsilon\left(\left\langle\omega_0,\frac{\partial S_i}{\partial x}\right\rangle+\frac{\partial S_i}{\partial t}+[f_i]_L\right)(Y,x,\theta,\varepsilon)\\ &\quad+\varepsilon(f_i(y,x,\theta,\varepsilon)-f(Y,x,\theta,\varepsilon))+\varepsilon(f_i(Y,x,\theta,\varepsilon)-f_{i0}(Y,\varepsilon)-[f_i]_L(Y,x,\theta,\varepsilon))\\ &\quad+(((N_i(y,\varepsilon)-N_0(y,\varepsilon))-((N_i(Y,\varepsilon)-N_0(Y,\varepsilon)))+\varepsilon\left(\frac{\partial S_i}{\partial t}(Y,X,\theta)-\frac{\partial S_i}{\partial t}(Y,x,\theta)\right), \end{aligned}$$

where

$$[f_i]_L(Y,x,\theta,\varepsilon)=\sum_{0<|k|+|l|\leqslant L}f_{ikl}(Y,\varepsilon)e^{\sqrt{-1}(\langle k,x\rangle+\langle l,\theta\rangle)}$$

for the Fourier series $\sum_{k,l}f_{ikl}(Y,\varepsilon)e^{\sqrt{-1}(\langle k,x\rangle+\langle l,\theta\rangle)}$ of f_i.

Set

$$\begin{aligned} N_{i+1}(y,\varepsilon)&=N_i(y,\varepsilon)+\varepsilon f_{i0}(y,\varepsilon),\\ \widetilde{N_i}(y,\varepsilon)&=N_i(y,\varepsilon)-N_0(y,\varepsilon),\\ f^1_{i+1}&=f_i(y,x,\theta,\varepsilon)-f_i(Y,x,\theta,\varepsilon),\\ f^2_{i+1}&=f_i(Y,x,\theta,\varepsilon)-f_{i0}(Y,\varepsilon)-[f_i]_L(Y,x,\theta,\varepsilon):=(R_Lf_i)(Y,x,\theta,\varepsilon), \end{aligned}\tag{3.2}$$

$$
\begin{aligned}
f_{i+1}^3 &= \frac{1}{\varepsilon}(\widetilde{N_i}(y,\varepsilon) - \widetilde{N_i}(Y,\varepsilon)),\\
f_{i+1}^4 &= \frac{\partial S_i}{\partial t}(Y,X,\theta) - \frac{\partial S_i}{\partial t}(Y,x,\theta),\\
f_{i+1} &= f_{i+1}^1 + f_{i+1}^2 + f_{i+1}^3 + f_{i+1}^4.
\end{aligned}
$$

Choose function S_i to satisfy

$$
\left\langle \omega_0, \frac{\partial S_i}{\partial x} \right\rangle + \frac{\partial S_i}{\partial t} + [f_i]_L = 0. \tag{3.3}
$$

Summing up the above equalities we obtain that under the coordinate transformation Ψ_i, H_0 is changed into H_{i+1},

$$
H_{i+1}(y,x,t) = N_{i+1}(y,\varepsilon) + \varepsilon f_{i+1}(y,x,\theta,\varepsilon)
$$

defined on $\mathbb{D}_{i+1}$.

We prove f_{i+1} to suit inequality (3.1) by replacing i. To this end, we need the lemmas as follows.

Lemma 3.1. *Assume that $f(y,x,\theta)$ is a real analytic function defined on $(\mathbb{O}(0,1)\times\mathbb{T}^n\times\mathbb{T}^m)+\rho$, and $f_0(y)=0$. If the given vectors $\omega\in\mathbb{R}^n$ and $\omega_*\in\mathbb{R}^m$ satisfy Diophantine condition, that is, there are two positive constants γ and τ, so that the following inequalities hold,*

$$
|\langle\omega,k\rangle + \langle\omega_*,l\rangle| > \gamma(|k|+|l|)^{-\tau}
$$

for any $(k,l)\in\mathbb{Z}^n\times\mathbb{Z}^m$ with $0<|k|+|l|\leqslant L$. Then as $\theta=\omega_* t$, the equation

$$
\left\langle \omega, \frac{\partial S}{\partial x} \right\rangle + \frac{\partial S}{\partial t} + [f]_L = 0 \tag{3.4}
$$

has unique real analytic solution $S(y,x,\omega_* t)$ with $S_0(y)=0$. Moreover, for any $0<\sigma<\rho$,

$$
|S|_{\rho-\sigma} \leqslant \frac{M_1}{\gamma\sigma^\tau}|[f]_L|_\rho \tag{3.5}
$$

with some positive constant M_1.

Proof. Expand f and S as Fourier series as

$$
f(y,x,\theta) = \sum_{(k,l)\in\mathbb{Z}^{n+m}\setminus\{(0,0)\}} f_{kl}(y)e^{\sqrt{-1}(\langle k,x\rangle+\langle l,\theta\rangle)}, \quad S(y,x,\theta) = \sum_{(k,l)\in\mathbb{Z}^{n+m}\setminus\{(0,0)\}} S_{kl}(y)e^{\sqrt{-1}(\langle k,x\rangle+\langle l,\theta\rangle)},
$$

respectively. Substituting the above expressions into the equation (3.4), and equaling the terms of the same order in k and l we obtain the unique real analytic solution

$$
S(y,x,\omega_* t) = \sum_{(k,l)\in\mathbb{Z}^{n+m},0<|k|+|l|\leqslant L} \frac{f_{kl}(y)}{\sqrt{-1}(\langle\omega,k\rangle+\langle\omega_*,l\rangle)} e^{\sqrt{-1}(\langle k,x\rangle+\langle l,\theta\rangle)}
$$

with $S_0(y)=0$. For the details of proof of the inequality (3.5), see [9, 10]. □

Lemma 3.2 ([1]). *Assume $l(q)$ be real analytic in $\mathbb{T}^n+\delta$. Then, as $0<2\sigma_0<\nu$ and $\sigma_0+\nu<\delta<1$, on $\mathbb{T}^n+(\delta-\sigma_0-\nu)$ one has*

$$
\|R_L l\| < \left(\frac{2n}{e}\right)^n \frac{\|l\|}{\sigma_0^{n+1}} e^{-L\nu}.
$$

Lemma 3.3 ([1]). *Let $\mathbb{U}$ be a bounded domain in $\mathbb{R}^s$, and $\Phi:\mathbb{U}\to\mathbb{R}^s$ a continuous mapping. If for any $x\in\mathbb{U}$, there exists a small constant $\kappa>0$ such that $|\Phi(x)-x|<\kappa$, then $\mathbb{U}-\kappa\subset\Phi(\mathbb{U})$.*

We begin to estimate the perturbations. By taking $\sigma_0 = \kappa$ and $\nu = 3\kappa$ in Lemma 3.2, we obtain that, on $\mathbb{D}_* + (\rho_i - 4\kappa)$,

$$|R_L f_i|_{\rho_i-4\kappa} < \left(\frac{2(n+m)}{e}\right)^{n+m} \frac{e^{-3L\kappa}}{\kappa^{n+m+1}} |f_i|_{\rho_i}.$$

Hence, from the choice of L,

$$|R_L f_i|_{\rho_i-4\kappa} < \frac{1}{8}|f_i|_{\rho_i}. \tag{3.6}$$

By the definition of f_{i0},

$$|f_{i0}|_{\rho_i} \leqslant |f_i|_{\rho_i}. \tag{3.7}$$

On the basis of (3.2), (3.6), and (3.7), we obtain

$$|[f_i]|_{\rho_i-4\kappa} \leqslant |f_i|_{\rho_i-4\kappa} + |f_{i0}|_{\rho_i-4\kappa} + |R_L f_i|_{\rho_i-4\kappa} \leqslant 3|f_i|_{\rho_i}. \tag{3.8}$$

Notice that the function S_i is determined by (3.3) and p_0 satisfies (2.1). By Lemma 3.1, we obtain

$$|S_i|_{\rho_i-5\kappa} \leqslant \frac{M_1}{\alpha\kappa^{\tau}} |[f_i]|_{\rho_i-4\kappa},$$

which implies

$$\max\left\{ \left|\frac{\partial S_i}{\partial x}\right|_{\rho_i-6\kappa}, \left|\frac{\partial S_i}{\partial y}\right|_{\rho_i-6\kappa} \right\} \leqslant \frac{M_1}{\alpha\kappa^{\tau+1}} |[f_i]|_{\rho_i-4\kappa} \tag{3.9}$$

from Cauchy's formula. Take parameter κ satisfies the inequality

$$0 < \kappa < \min\left\{ \frac{1}{3M_1M+1}, \left(\frac{3}{4}\right)^{\frac{1}{\tau+3}}, \left(\frac{1}{8M_1M+1}\right)^{\frac{1}{\tau+4}}, \left(\frac{1}{3M_1M+1}\right)^{\frac{1}{\tau+4}}, \gamma \right\}. \tag{3.10}$$

Thus, by applying (3.9) and (3.8), we derive

$$\max\left\{ \varepsilon\left|\frac{\partial S_i}{\partial x}\right|_{\rho_i-6\kappa}, \varepsilon\left|\frac{\partial S_i}{\partial y}\right|_{\rho_i-6\kappa} \right\} \leqslant \frac{3\varepsilon M_1}{\alpha\kappa^{\tau+1}} |f_i|_{\rho_i-4\kappa} < \frac{1}{2^i}\kappa^{\tau+4} < \kappa.$$

Now we estimate the coordinate transformations. By the definition, Φ_i is real analytic and satisfies $\mathbb{D}_* + (\rho_i - 8\kappa) \subset \Phi(\mathbb{D}_* + (\rho_* - 6\kappa))$ from Lemma 3.3, that is, new and old coordinates satisfy the following estimate,

$$|(Y, X) - (y(Y, X), x(Y, X))|_{\rho_i-8\kappa} < \max\left\{ \varepsilon\left|\frac{\partial S_i}{\partial x}\right|_{\rho_i-6\kappa}, \varepsilon\left|\frac{\partial S_i}{\partial y}\right|_{\rho_i-6\kappa} \right\} < \frac{1}{2^i}\kappa^{\tau+4} < \kappa. \tag{3.11}$$

In the perturbations f^1_{i+1}, f^2_{i+1}, and f^3_{i+1}, y and x must be expressed in term of Y and X by the coordinate transformation Φ_i. By the mean value theorem, Cauchy's formula, (3.10), and (3.11), we have

$$|f^1_{i+1}|_{\rho_i-8\kappa} \leqslant \left|\frac{\partial f_i}{\partial y}\right|_{\rho_i-6\kappa} |y(Y, X, \theta, \varepsilon) - Y|_{\rho_i-8\kappa} \leqslant \frac{1}{6\kappa}|f_i|_{\rho_i}\kappa^{\tau+4} < \frac{1}{8}|f_i|_{\rho_1}. \tag{3.12}$$

By employing (3.6) and (3.11),

$$|f^2_{i+1}|_{\rho_i-8\kappa} = |R_L f_i(Y, x(Y, X, \theta, \varepsilon), \theta, \varepsilon)|_{\rho_i-8\kappa} \leqslant |R_L f_i|_{\rho_i-6\kappa} < \frac{1}{8}|f_i|_{\rho_i}. \tag{3.13}$$

Inductively, we estimate f^3_{i+1} and f^4_{i+1} as follows,

$$|f^3_{i+1}|_{\rho_i-8\kappa} \leqslant \sum_{j=0}^{i-1} |f_{j0}(y(Y, X, \theta, \varepsilon), \varepsilon) - f_{j0}(Y, \varepsilon)|_{\rho_i-8\kappa}$$

$$\begin{aligned}&\leqslant \sum_{j=0}^{i-1}\left|\frac{\partial f_{j0}}{\partial y}\right|_{\rho_i-6\kappa}|y(Y,X,\theta,\varepsilon)-Y|_{\rho_i-8\kappa}\\&\leqslant \frac{1}{6\kappa}|f_{j0}|_{\rho_j}\cdot\varepsilon\left|\frac{\partial S_i}{\partial x}\right|_{\rho_i-6\kappa}\leqslant\frac{M}{3\kappa}\cdot\frac{3M_1\varepsilon}{\alpha\kappa^{\tau+1}}|f_i|_{\rho_i-4\kappa}\leqslant MM_1\kappa^{\tau+3}|f_i|_{\rho_i}<\frac{1}{8}|f_i|_{\rho_i},\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}|f_{i+1}^4|_{\rho_i-9\kappa}&\leqslant|\omega_*|\left|\frac{\partial^2 S_i}{\partial\theta\partial x}\right|_{\rho_i-7\kappa}|X-x(Y,X,\theta,\varepsilon)|_{\rho_i-9\kappa}\\&\leqslant M\kappa^{\tau+4}\cdot\frac{1}{\kappa}\left|\frac{\partial S_i}{\partial x}\right|_{\rho_i-6\kappa}\leqslant M\kappa^{\tau+3}\cdot\frac{3M_1}{\alpha\kappa^{\tau+1}}|f_i|_{\rho_i-4\kappa}\leqslant 3MM_1\kappa|f_i|_{\rho_i}\leqslant\frac{1}{8}|f_i|_{\rho_i}.\end{aligned}\tag{3.15}$$

In the above inequalities we used the relations (3.9), (3.10), and (3.11), and the mean value theorem and Cauchy's formula. On the basis of (3.12), (3.13), (3.14), and (3.15), finally, we derive the estimate of new perturbation as follows,

$$|f_{i+1}|_{\rho_i-9\kappa}\leqslant\frac{1}{2}|f_i|_{\rho_i}.$$

Choose

$$T(\kappa)=\left[\frac{\rho}{18\kappa}\right].$$

Take $\Psi_*=\Phi_0\circ\Phi_1\circ\cdots\circ\Phi_{T(\kappa)-1}$. Then, from the choice of $T(\kappa)$, $\mathbb{D}_{T(\kappa)-1}\supseteq\mathbb{D}+\frac{\rho}{2}$. This shows the change Ψ_* maps $\mathbb{D}+\frac{\rho}{2}$ into $\mathbb{D}_0$, and the Hamiltonian $H_0(y,x,\theta)$ is changed into $H_{T(\kappa)}(Y,X,\theta)$ which satisfies

$$H_{T(\kappa)}(Y,X,\theta)=N_{T(\kappa)}(Y,\varepsilon)+\varepsilon f_{T(\kappa)}(Y,X,\theta,\varepsilon)$$

with

$$|\widetilde{N}_{T(\kappa)}|_{\frac{\rho}{2}}\leqslant\varepsilon\sum_{j=0}^{T(\kappa)-1}|f_j|_{\rho_j}<2\varepsilon M<2M,\quad |f_{T(\kappa)}|_{\rho_{T(\kappa)}}\leqslant Me^{-\frac{\rho\ln 2}{18}\cdot\frac{1}{\kappa}}.\tag{3.16}$$

We consider the orbit $\{(Y(t),X(t))\}$ of the Hamiltonian system

$$\begin{cases}\dot{Y}=-\varepsilon\dfrac{\partial f_{T(\kappa)}}{\partial X}(Y,X,\theta,\varepsilon),\\ \dot{X}=\dfrac{\partial N_{T(\kappa)}}{\partial Y}(Y,\varepsilon)+\varepsilon\dfrac{\partial f_{T(\kappa)}}{\partial Y}(Y,X,\theta,\varepsilon).\end{cases}$$

Here $(Y(0),X(0))\in\mathbb{D}_*$. Notice that, on $\mathbb{D}_*$,

$$\max\left\{\left|\frac{\partial f_{T(\kappa)}}{\partial X}\right|,\left|\frac{\partial f_{T(\kappa)}}{\partial Y}\right|\right\}\leqslant\frac{2}{\rho}|f_{T(\kappa)}|_{\rho_{T(\kappa)}},\qquad\left|\frac{\partial^2 N_{T(\kappa)}}{\partial Y^2}\right|=\left|\frac{\partial^2\widetilde{N}_{T(\kappa)}}{\partial Y^2}\right|\leqslant\frac{4}{\rho^2}|\widetilde{N}_{T(\kappa)}|_{\rho_{T(\kappa)}}.$$

Consequently, from (3.16), as $|t|\leqslant e^{\frac{\rho\ln 2}{54}\cdot\frac{1}{\kappa}}$,

$$|Y(t)-Y(0)|\leqslant\varepsilon|t|\left|\frac{\partial f_{T(\kappa)}}{\partial X}\right|_{\mathbb{D}_*}\leqslant\frac{2\varepsilon}{\rho}|t||f_{T(\kappa)}|_{\rho_{T(\kappa)}}\leqslant\frac{2}{\rho}M\varepsilon e^{-\frac{\rho\ln 2}{27}\cdot\frac{1}{\kappa}},$$

$$\begin{aligned}|X(t)-X(0)-\omega_*(Y(0),\varepsilon)t|&\leqslant\int_0^{|t|}\left|\frac{\partial N_{T(\kappa)}}{\partial Y}(Y(t),\varepsilon)-\frac{\partial N_{T(\kappa)}}{\partial Y}(Y(0),\varepsilon)\right|dt\\&\quad+\varepsilon\int_0^{|t|}\left|\frac{\partial f_{T(\kappa)}}{\partial Y}(Y(t),X(t),\theta(t),\varepsilon)\right|dt\\&\leqslant|t|\left|\frac{\partial^2\widetilde{N}_{T(\kappa)}}{\partial Y^2}\right||Y(t)-Y(0)|+\frac{2\varepsilon}{\rho}|t||f_{T(\kappa)}|_{\rho_{T(\kappa)}}\end{aligned}$$

$$\leqslant \frac{4}{\rho^2}|t||\widetilde{N}_{T(\kappa)}|_{T(\kappa)}|Y(t)-Y(0)|+\frac{2}{\rho}M\varepsilon e^{-\frac{\rho\ln 2}{27}\cdot\frac{1}{\kappa}}$$
$$\leqslant 10\left(\frac{8M^2}{\rho^2}+1\right)\frac{M}{\rho}\varepsilon\kappa^{\tau+4}.$$

Let $(y,x)=\Psi_*(Y,X)$. By (3.11), we have

$$|(y,x)-(Y,X)|=|\Psi_*(Y,X)-(Y,X)|\leqslant\sum_{j=0}^{T(\kappa)-1}\frac{1}{2^i}\kappa^{\tau+4}<2\kappa^{\tau+4}. \tag{3.17}$$

Therefore, as $|t|\leqslant e^{\frac{\rho\ln 2}{54}\cdot\frac{1}{\kappa}}$,

$$|y(t)-y(0)|\leqslant|Y(t)-y(t)|+|Y(0)-y(0)|+|Y(t)-Y(0)|\leqslant 4\kappa^{\tau+4}+\frac{2}{\rho}M\varepsilon e^{-\frac{\rho\ln 2}{27}\cdot\frac{1}{\kappa}}\leqslant 5\kappa^{\tau+4}, \tag{3.18}$$

provided ϵ is sufficiently small. Express $H_{T(\kappa)}=H_*$, $N_{T(\kappa)}=N_*$, and $f_{T(\kappa)}=f_*$. On the basis of (3.17), (3.18), and the definition of Ψ_*, we complete the proof of Theorem 2.1.

4. Proof of Theorem 1.2

In this section we give the proof of Theorem 1.2. By Theorem 2.1, we need only to prove the estimate of the measure in Theorem 1.2. To this end, we define some subsets as follows:

$$\Omega=\omega(\mathcal{D}),\qquad \Omega_{k,l}=\left\{\omega\in\Omega:|\langle k,\omega\rangle+\langle l,\omega_*\rangle|\leqslant\alpha(|k|+|l|)^{-\tau}\right\},$$
$$\widetilde{\Omega}_\epsilon=\bigcup_{0<|k|+|l|\leqslant L(\kappa),k\neq 0}\Omega_{k,l},\qquad \Omega_{*\epsilon}=\Omega-\widetilde{\Omega}_\epsilon.$$

Thus, $\mathcal{E}_\epsilon=\omega^{-1}(\Omega_{*\epsilon})$ and it is an open subset because that mapping $\omega:p\to\omega(p)$ is homeomorphic. We follow Arnold's idea in [1]. From the arithmetical lemma in [1], one has

$$\mathrm{meas}(\Omega_{k,l})\leqslant 2\alpha(|k|+|l|)^{-\tau}Cn\,\mathrm{meas}\Omega$$

for any $k\in\mathbb{Z}_+^n\backslash\{0\}$ and $l\in\mathbb{Z}_+^m$, where C is a constant depending only on Ω. This leads to

$$\mathrm{meas}\widetilde{\Omega}_\epsilon\leqslant\sum_{j=1}^{L(\kappa)}\sum_{|k|+|l|=j}\mathrm{meas}\Omega_{k,l}\leqslant\sum_{j=1}^{L(\kappa)}2^{n+m+1}j^{n+m-1}\alpha j^{-\tau}Cn\,\mathrm{meas}\Omega$$
$$\leqslant 2^{n+m+1}Cn\,\mathrm{meas}\Omega\,\alpha\sum_{j=1}^{\infty}\frac{1}{j^{\tau-(n+m)+1}}=O(\alpha).$$

Consequently,

$$\mathrm{meas}\Omega_{*\epsilon}=\mathrm{meas}\Omega-O(\alpha),$$

and

$$\mathrm{meas}\mathcal{E}_\epsilon=\int_{\mathcal{E}_\epsilon}dp=\left|\int_{\Omega_{*\epsilon}}\left(\det\left(\frac{\partial\omega}{\partial p}\right)\right)^{-1}d\omega\right|=\mathrm{meas}\mathcal{D}-O(\alpha).$$

This completes the proof of Theorem 1.2.

Remark 4.1. Under the Rüssmann's non-degeneracy, we also can obtain the same conclusion as in Theorem 1.2. This need to apply the technique in [3, 4].

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