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Fixed point theorems for generalized JS-quasi-contractions in complete partial b-metric spaces

Panisa Lohawech, Anchalee Kaewcharoen*

Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand.

Abstract

In this paper, we introduce a concept of generalized JS-quasi-contractions and obtain sufficient conditions for the existence of fixed points of such mappings on p_b-complete partial b-metric spaces. Our results extend the results in the literature. In addition, an example is given to illustrate and support our main result.

Keywords: Fixed point theorems, partial b-metric spaces, generalized JS-quasi-contractions.

2010 MSC: 47H10, 54H25.

1. Introduction and preliminaries

A popular tool in nonlinear analysis is the fixed point theory. The Banach contraction principle [2] is the first important result on fixed points for contractive-type mappings. This principle states that, if (X, d) is a complete metric space and $T : X \to X$ is a Banach contraction (i.e., there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leqslant kd(x, y)$ for all $x, y \in X$), then T has a unique fixed point. According to its importance and simplicity, there are a lot of generalizations of Banach contraction principle and metric spaces. Wardowski [13] suggested the concept of F-contraction and obtained a fixed point result which is a generalization of the Banach contraction principle. Afterward, Hunwisai and Kumam [4] introduced the concept of multivalued fuzzy F-contraction mappings in b-metic spaces and gave some fixed point results. Moreover, the notion of generalized F-Suzuki-contractions as a generalization of the concept of F-contractions was introduced by Piri and Kumam [9].

The notion of b-metric spaces as a generalization of metric spaces was introduced by Bakhtin [1]. In 1993, Czerwik [3] extended results related to the b-metric spaces. Since then, many authors have studied fixed point theorems for single-valued and multi-valued operators in b-metric spaces. Roshana et al. [10] suggested the concept of b-rectangular metric spaces as a generalization of b-metric spaces. In addition, Sookprasert et al. [12] extended results related to the b-rectangular metric spaces. On the other hand,

*Corresponding author
Email addresses: panisa.l@hotmail.com (Panisa Lohawech), anchaleeka@nu.ac.th (Anchalee Kaewcharoen)

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the study of fixed points in partial metric spaces as a generalization of metric spaces was introduced by Matthews [7] in 1994.

After that, Shukla [11] presented the notion of partial b-metric spaces, which is a generalization of partial metric spaces and b-metric spaces as follows:

Definition 1.1 ([11])**.** Let X be a nonempty set and $s \geqslant 1$ be a given real number. A function $p_b : X \times X \to \mathbb{R}_+$ is called a partial b-metric if for all $x, y, z \in X$ the following properties hold:

(p_{b1}) $x = y$ if and only if $p_b(x, x) = p_b(x, y) = p_b(y, y)$;

(p_{b2}) $p_b(x, x) \leqslant p_b(x, y)$;

(p_{b3}) $p_b(x, y) = p_b(y, x)$;

(p_{b4}) $p_b(x, y) \leqslant s[p_b(x, z) + p_b(y, z)] - p_b(z, z)$.

The pair (X, p_b) is called a partial b-metric space.

The class of partial b-metric spaces is larger than the class of partial metric spaces, since a partial metric space is a special case of a partial b-metric space with the coefficient $s = 1$. Also, the class of partial b-metric spaces is larger than the class of b-metric spaces since a b-metric space is a special case of a partial b-metric space with the same coefficient and the self distance $p_b(x, x) = 0$.

The following example shows that a partial b-metric space need not be a partial metric space nor a b-metric space.

Example 1.2 ([11])**.** Let $X = \mathbb{R}_+$ and $q > 1$ be a constant. Define a function $p_b : X \times X \to \mathbb{R}_+$ by

$$p_b(x, y) = [\max\{x, y\}]^q + |x - y|^q \qquad \text{for all } x, y \in X.$$

Then (X, p_b) is a partial b-metric space with the coefficient $s = 2^{q-1} > 1$, but it is neither a partial metric space nor a b-metric space.

Proposition 1.3 ([11])**.** *Let X be a nonempty set, p be a partial metric and d be a b-metric with the coefficient $s \geqslant 1$ on X. Then the function $p_b : X \times X \to \mathbb{R}_+$, defined by $p_b(x, y) = p(x, y) + d(x, y)$ for all $x, y \in X$, is a partial b-metric with the coefficient s.*

Proposition 1.4 ([11])**.** *Let (X, p) be a partial metric space and $q \geqslant 1$. Then (X, p_b) is a partial b-metric space with the coefficient $s = 2^{q-1}$, where $p_b : X \times X \to \mathbb{R}_+$ is defined by $p_b(x, y) = [p(x, y)]^q$.*

Mustafa et al. [8] introduced a modified version of Definition 1.1 in order to get that each partial b-metric p_b generates a b-metric d_{p_b}.

Definition 1.5 ([8])**.** Let X be a nonempty set and $s \geqslant 1$ be a given real number. A function $p_b : X \times X \to \mathbb{R}_+$ is called a partial b-metric if for all $x, y, z \in X$ the following properties hold:

(p_{b1}) $x = y$ if and only if $p_b(x, x) = p_b(x, y) = p_b(y, y)$;

(p_{b2}) $p_b(x, x) \leqslant p_b(x, y)$;

(p_{b3}) $p_b(x, y) = p_b(y, x)$;

($p_{b4'}$) $p_b(x, y) \leqslant s(p_b(x, z) + p_b(y, z) - p_b(z, z)) + (\frac{1-s}{2})(p_b(x, x) + p_b(y, y))$.

The pair (X, p_b) is called a partial b-metric space.

Since $s \geqslant 1$, by ($p_{b4'}$), we obtain that

$$p_b(x, y) \leqslant s(p_b(x, z) + p_b(z, y) - p_b(z, z)) \leqslant s(p_b(x, z) + p_b(z, y)) - p_b(z, z).$$

Thus, a partial b-metric in the sense of Definition 1.5 is also a partial b-metric in Definition 1.1.

In a partial b-metric space (X, p_b), if $p_b(x, y) = 0$ implies $p_b(x, x) = p_b(x, y) = p_b(y, y) = 0$, then $x = y$, but if $x = y$, then $p_b(x, y)$ may not be 0. It is clear that every partial metric space is a partial b-metric space with the coefficient $s = 1$ and every b-metric space is a partial b-metric space with the same coefficient and the self distance $p_b(x, x) = 0$, but the converse of these facts may not hold.

The following example shows that a partial b-metric space (Definition 1.5) need not to be a partial metric space nor a b-metric space.

Example 1.6 ([8])**.** Let (X, d) be a metric space and $p_b : X \times X \to \mathbb{R}_+$ defined by

$$p_b(x, y) = d(x, y)^q + a \quad \text{for all } x, y \in X,$$

where $q > 1$ and $a \geqslant 0$. Then p_b is a partial b-metric with $s = 2^{q-1}$, but it is neither a partial metric nor b-metric.

Proposition 1.7 ([8])**.** *Every partial* b*-metric* p_b *defines a* b*-metric* d_{p_b}*, where*

$$d_{p_b}(x, y) = 2p_b(x, y) - p_b(x, x) - p_b(y, y) \quad \textit{for all } x, y \in X.$$

Definition 1.8 ([8])**.** Let $\{x_n\}$ be a sequence in a partial b-metric space (X, p_b).

(i) A sequence $\{x_n\}$ is p_b-convergent to a point $x \in X$ if $\lim_{n\to\infty} p_b(x, x_n) = p_b(x, x)$.

(ii) A sequence $\{x_n\}$ is a p_b-Cauchy sequence if $\lim_{n,m\to\infty} p_b(x_n, x_m)$ exists (and is finite).

(iii) A partial b-metric space (X, p_b) is said to be p_b-complete if every p_b-Cauchy sequence $\{x_n\}$ in X p_b-converges to a point $x \in X$ such that $\lim_{n,m\to\infty} p_b(x_n, x_m) = \lim_{n\to\infty} p_b(x_n, x) = p_b(x, x)$.

Lemma 1.9 ([8])**.**

(1) *A sequence* $\{x_n\}$ *is a* p_b*-Cauchy sequence in a partial* b*-metric space* (X, p_b) *if and only if it is a* b*-Cauchy sequence in the* b*-metric space* (X, d_{p_b})*.*

(2) *A partial* b*-metric space* (X, p_b) *is* p_b*-complete if and only if a* b*-metric space* (X, d_{p_b}) *is* b*-complete. Moreover,* $\lim_{n\to\infty} d_{p_b}(x, x_n) = 0$ *if and only if*

$$\lim_{n\to\infty} p_b(x, x_n) = \lim_{n,m\to\infty} p_b(x_n, x_m) = p_b(x, x).$$

Definition 1.10 ([8])**.** Let (X, p_b) and (X', p'_b) be two partial b-metric spaces and let $f : (X, p_b) \to (X', p'_b)$ be a mapping. Then f is said to be p_b-continuous at a point $a \in X$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$ and $p_b(a, x) < \delta + p_b(a, a)$ imply that $p'_b(f(a), f(x)) < \varepsilon + p'_b(f(a), f(a))$. The mapping f is p_b-continuous on X if it is p_b-continuous at all $a \in X$.

Proposition 1.11 ([8])**.** *Let* (X, p_b) *and* (X', p'_b) *be two partial* b*-metric spaces. Then a mapping* $f : X \to X$ *is* p_b*-continuous at a point* $x \in X$ *if and only if it is* p_b*-sequentially continuous at* x*, that is, whenever* $\{x_n\}$ *is* p_b*-convergent to* x*,* $\{f(x_n)\}$ *is* p'_b*-convergent to* $f(x)$*.*

Recently, the notion of JS-quasi-contractions was introduced by Li and Jiang [6]. They proved some fixed point results for JS-quasi-contractions in complete metric spaces.

Following Hussain et al. [5], Li and Jiang [6] denoted Ψ by the set of all nondecreasing functions $\psi : [0, +\infty) \to [1, +\infty)$ satisfying the following conditions:

(Ψ1) $\psi(t) = 1$ if and only if $t = 0$;

(Ψ2) for each sequence $\{t_n\} \subset (0, +\infty)$, $\lim_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;

(Ψ3) there exist $r \in (0,1)$ and $l \in (0,+\infty]$ such that $\lim_{t\to 0^+} \frac{\psi(t)-1}{t^r} = l$;

(Ψ4) $\psi(t+s) \leqslant \psi(t)\psi(s)$ for all $t,s > 0$.

Li and Jiang [6] set the following symbols:

$\Phi_1 = \{\psi : (0,+\infty) \to (1,+\infty) : \psi$ is a nondecreasing function satisfying (Ψ2) and (Ψ3)$\}$;

$\Phi_2 = \{\psi : (0,+\infty) \to (1,+\infty) : \psi$ is a nondecreasing continuous function$\}$;

$\Phi_3 = \{\psi : [0,+\infty) \to [1,+\infty) : \psi$ is a nondecreasing continuous function satisfying (Ψ1)$\}$;

$\Phi_4 = \{\psi : [0,+\infty) \to [1,+\infty) : \psi$ is a nondecreasing continuous function satisfying (Ψ1) and (Ψ4)$\}$.

They [6] presented the following examples for illustrating the relationship among the above sets.

Example 1.12 ([6])**.** Let $f(t) = e^{te^t}$ for $t \geqslant 0$. Then $f \in \Phi_2 \cap \Phi_3$, but $f \notin \Psi \cup \Phi_1 \cup \Phi_4$ since $\lim_{t\to 0^+} \frac{e^{te^t}-1}{t^r} = 0$ for each $r \in (0,1)$ and $e^{(t+s)e^{(t+s)}} > e^{se^s}e^{te^t}$ for all $s,t > 0$.

Example 1.13 ([6])**.** Let $g(t) = e^{t^a}$ for $t \geqslant 0$, where $a > 0$. When $a \in (0,1)$, $g \in \Psi \cap \Phi_1 \cap \Phi_2 \cap \Phi_3 \cap \Phi_4$. When $a = 1$, $g \in \Phi_2 \cap \Phi_3 \cap \Phi_4$, but $g \notin \Psi \cup \Phi_1$ since $\lim_{t\to 0^+} \frac{e^t-1}{t^r} = 0$ for each $r \in (0,1)$. When $a > 1$, $g \in \Phi_2 \cap \Phi_3$, but $g \notin \Psi \cup \Phi_1 \cup \Phi_4$ since $\lim_{t\to 0^+} \frac{e^{t^a}-1}{t^r} = 0$ for each $r \in (0,1)$ and $e^{(t+s)^a} > e^{t^a}e^{s^a}$ for all $s,t > 0$.

They [6] introduced the concept of JS-quasi-contractions and assure the existence of the fixed point theorems for such mappings in complete metric spaces.

Definition 1.14 ([6])**.** Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be a JS-quasi-contraction if there exist a function $\psi : (0,+\infty) \to (1,+\infty)$ and $\lambda \in (0,1)$ such that

$$\psi(d(Tx,Ty)) \leqslant \psi(M_d(x,y))^\lambda \quad \text{for all } x,y \in X \text{ with } Tx \neq Ty,$$

where $M_d(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$.

Remark 1.15 ([6]). Let $T : X \to X$ and $\psi : [0,+\infty) \to [1,+\infty)$ be such that

$$\psi(d(Tx,Ty)) \leqslant \psi(d(x,y))^{k_1}\psi(d(x,Tx))^{k_2}\psi(d(y,Ty))^{k_3}\psi(\frac{d(x,Ty)+d(y,Tx)}{2})^{2k_4} \quad \text{for all } x,y \in X, \tag{1.1}$$

where k_1,k_2,k_3,k_4 are nonnegative numbers with $k_1+k_2+k_3+2k_4 < 1$. Then T is a JS-quasi-contraction with $\lambda = k_1+k_2+k_3+2k_4$, provided that ($\Psi$2) is satisfied.

Theorem 1.16 ([6])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a JS-quasi-contraction with $\psi \in \Phi_2$. Then T has a unique fixed point in X.*

Theorem 1.17 ([6])**.** *Let (X,d) be a complete metric space and $T : X \to X$. Assume that there exist $\psi \in \Phi_3$ and nonnegative numbers k_1,k_2,k_3,k_4 with $k_1+k_2+k_3+2k_4 < 1$ such that (1.1) is satisfied. Then T has a unique fixed point in X.*

In this paper, we introduce a concept of generalized JS-quasi-contractions and obtain sufficient conditions for the existence of fixed points of such mappings on p_b-complete partial b-metric spaces. Our results extend the results in the literature. In addition, an example is given to illustrate and support our main result.

2. Main result

We now introduce the concept of generalized JS-quasi-contractions on partial b-metric spaces.

Definition 2.1. Let (X, p_b) be a partial b-metric space with the coefficient $s \geqslant 1$. We say that a mapping $T : X \to X$ is a generalized JS-quasi-contraction if there exist a function $\psi : (0, +\infty) \to (1, +\infty)$ and $\lambda \in (0, 1)$ such that

$$\psi(sp_b(Tx, Ty)) \leqslant \psi(M_s(x, y))^\lambda \quad \text{for all } x, y \in X \quad \text{with} \quad Tx \neq Ty, \tag{2.1}$$

where $M_s(x, y) = \max\{p_b(x, y), p_b(x, Tx), p_b(y, Ty), \frac{p_b(x,Ty)+p_b(y,Tx)}{2s}\}$.

The following example shows that a generalized JS-quasi-contraction need not to be p_b-continuous.

Example 2.2. Let $X = [0, +\infty)$ with the partial b-metric $p_b : X \times X \to \mathbb{R}_+$ defined by

$$p_b(x, y) = [\max\{x, y\}]^2$$

for all $x, y \in X$. Obviously, (X, p_b) is a p_b-complete partial b-metric space with $s = 2$. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{2}{3}, & x \in [0, 1), \\ \frac{x-1}{2x}, & \text{otherwise.} \end{cases}$$

We will show that T is a generalized JS-quasi-contraction with $\psi(t) = e^t \in \Phi_2$. In fact, it suffices to show that there exists $\lambda \in (0, 1)$ such that, for all $x, y \in X$ with $Tx \neq Ty$,

$$\frac{2p_b(Tx, Ty)}{M_s(x, y)} \leqslant \lambda.$$

Let $x, y \in X$ with $Tx \neq Ty$. Without loss of generality, we may assume that $x < y$. It follows that $1 \leqslant x < y$. Therefore,

$$p_b(Tx, Ty) = [\max\{\frac{x-1}{2x}, \frac{y-1}{2y}\}]^2 = \frac{y^2 - 2y + 1}{4y^2},$$

and

$$M_s(x, y) = \max\{y^2, x^2, y^2, \frac{[\max\{x, \frac{y-1}{2y}\}]^2 + y^2}{2s}\} = y^2.$$

This implies that

$$\frac{2p_b(Tx, Ty)}{M_s(x, y)} = \frac{y^2 - 2y + 1}{2y^4} \leqslant \frac{1}{32}.$$

This shows that T is a generalized JS-quasi-contraction with $\psi(t) = e^t \in \Phi_2$ and $\lambda \in [\frac{1}{32}, 1)$.

On the other hand, T is not p_b-continuous because there exists a sequence $\{\frac{1}{n+1}\}$ such that

$$\lim_{n\to\infty} p_b(1, x_n) = \lim_{n\to\infty} [\max\{1, x_n\}]^2 = 1 = p_b(1, 1),$$

but

$$\lim_{n\to\infty} p_b(T1, Tx_n) = [\max\{0, \frac{2}{3}\}]^2 = \frac{4}{9} \neq 0 = p_b(T1, T1).$$

The following example shows that a p_b-continuous mapping need not to be a generalized JS-quasi-contraction.

Example 2.3. Let $X = \{0, 1, 2\}$ with the partial b-metric $p_b : X \times X \to \mathbb{R}_+$ defined by

$$p_b(x, y) = |x - y|^2$$

for all $x, y \in X$. Obviously, (X, p_b) is a p_b-complete partial b-metric space with $s = 2$. Define the mapping $T : X \to X$ by $T0 = T1 = 0$ and $T2 = 1$. Then T is p_b-continuous.

We will show that T is not a generalized JS-quasi-contraction with $\psi(t) = e^{te^t} \in \Phi_2$. In fact, it suffices to show that for all $\lambda \in (0, 1)$, there exist $x, y \in X$ with $Tx \neq Ty$ such that

$$\frac{2p_b(Tx, Ty)e^{2p_b(Tx,Ty)-M_s(x,y)}}{M_s(x, y)} > \lambda.$$

Let $\lambda \in (0, 1)$, for $x = 1$ and $y = 2$, we have $p_b(T1, T2) = 1$ and $M_s(1, 2) = 1$. Therefore,

$$\frac{2p_b(T1, T2)e^{2p_b(T1,T2)-M_s(1,2)}}{M_s(1, 2)} = 2(1)e^{2-1} = 2e > \lambda,$$

which implies that T is not a generalized JS-quasi-contraction.

Remark 2.4. As in [6], we obtain the following statements in a partial b-metric space (X, p_b) :

(i) Let $T : X \to X$ and $\lambda \in (0, 1)$ such that

$$sp_b(Tx, Ty) \leqslant \lambda M_s(x, y) \quad \text{for all } x, y \in X.$$

Then T is a generalized JS-quasi-contraction with $\psi(t) = e^t$.

(ii) Let $T : X \to X$ and $\psi : (0, +\infty) \to (1, +\infty)$ be such that

$$\psi(sp_b(Tx, Ty)) \leqslant \psi(p_b(x, y))^\lambda \quad \text{for all } x, y \in X \text{ with } Tx \neq Ty, \tag{2.2}$$

where $\lambda \in (0, 1)$. Then T is a generalized JS-quasi-contraction.

(iii) Let $T : X \to X$ and $\psi : [0, +\infty) \to [1, +\infty)$ be such that

$$\psi(sp_b(Tx, Ty)) \leqslant \psi(p_b(x, y))^{k_1}\psi(p_b(x, Tx))^{k_2}\psi(p_b(y, Ty))^{k_3}\psi(\frac{p_b(x, Ty) + p_b(y, Tx)}{2s})^{2k_4} \tag{2.3}$$

for all $x, y \in X$, where k_1, k_2, k_3, k_4 are nonnegative numbers with $k_1 + k_2 + k_3 + 2k_4 < 1$. Then T is a generalized JS-quasi-contraction with $\lambda = k_1 + k_2 + k_3 + 2k_4$, provided that $(\Psi 1)$ is satisfied.

(iv) Let $T : X \to X$ and $\psi : [0, +\infty) \to [1, +\infty)$ be such that

$$\psi(sp_b(Tx, Ty)) \leqslant \psi(p_b(x, y))^{k_1}\psi(p_b(x, Tx))^{k_2}\psi(p_b(y, Ty))^{k_3}\psi(\frac{p_b(x, Ty) + p_b(y, Tx)}{s})^{k_4} \tag{2.4}$$

for all $x, y \in X$. Suppose that ψ is a nondecreasing function such that $(\Psi 4)$ is satisfied. It follows that

$$\psi(\frac{p_b(x, Ty) + p_b(y, Tx)}{s})^{k_4} \leqslant \psi(\frac{p_b(x, Ty) + p_b(y, Tx)}{2s})^{2k_4} \quad \text{for all } x, y \in X,$$

and so (2.3) holds. Moreover, if $(\Psi 1)$ is satisfied, then it follows from (iii) that T is a generalized JS-quasi-contraction with $\lambda = k_1 + k_2 + k_3 + 2k_4$. Therefore, T is a generalized JS-quasi-contraction with $\psi \in \Phi_4$ or $\psi \in \Psi$.

We now prove the existence of a unique fixed point for a generalized JS-quasi-contraction.

Theorem 2.5. *Let* (X, p_b) *be a* p_b*-complete partial* b*-metric space with the coefficient* $s \geqslant 1$*. Let* $T : X \to X$ *be a generalized JS-quasi-contraction with* $\psi \in \Phi_2$ *and be* p_b*-continuous. Then* T *has a unique fixed point in* X*.*

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. If there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1}$ then x_n is a fixed point of T and the proof is finished. So we may assume that for every $n \in \mathbb{N}$,

$$x_n \neq x_{n+1}. \tag{2.5}$$

From (2.1), (2.5), and ψ is nondecreasing, we have

$$\psi(p_b(x_n, x_{n+1})) \leqslant \psi(sp_b(x_n, x_{n+1})) \leqslant \psi(M_s(x_{n-1}, x_n))^\lambda$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned}
M_s(x_{n-1}, x_n) &= \max\{p_b(x_{n-1}, x_n), p_b(x_{n-1}, Tx_{n-1}), p_b(x_n, Tx_n), \frac{p_b(x_{n-1}, Tx_n) + p_b(x_n, Tx_{n-1})}{2s}\} \\
&= \max\{p_b(x_{n-1}, x_n), p_b(x_{n-1}, x_n), p_b(x_n, x_{n+1}), \frac{p_b(x_{n-1}, x_{n+1}) + p_b(x_n, x_n)}{2s}\} \\
&\leqslant \max\{p_b(x_{n-1}, x_n), p_b(x_n, x_{n+1}) \\
&\quad , \frac{s(p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1})) - p_b(x_n, x_n) + p_b(x_n, x_n)}{2s}\} \\
&= \max\{p_b(x_{n-1}, x_n), p_b(x_n, x_{n+1}), \frac{p_b(x_{n-1}, x_n) + p_b(x_n, x_{n+1})}{2}\} \\
&= \max\{p_b(x_{n-1}, x_n), p_b(x_n, x_{n+1})\}.
\end{aligned}$$

This implies that

$$\psi(p_b(x_n, x_{n+1})) \leqslant \psi(sp_b(x_n, x_{n+1})) \leqslant \psi(\max\{p_b(x_{n-1}, x_n), p_b(x_n, x_{n+1})\})^\lambda \tag{2.6}$$

for all $n \in \mathbb{N}$. If there exists some $n \in \mathbb{N}$ such that $p_b(x_n, x_{n+1}) > p_b(x_{n-1}, x_n)$, then

$$\psi(p_b(x_n, x_{n+1})) \leqslant \psi(p_b(x_n, x_{n+1}))^\lambda < \psi(p_b(x_n, x_{n+1})),$$

which is a contradiction. It follows that

$$p_b(x_n, x_{n+1}) \leqslant p_b(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$. So the sequence $\{p_b(x_n, x_{n+1})\}$ is a nonincreasing sequence of real numbers which is bounded from below and thus there exists $\alpha \geqslant 0$ such that

$$\lim_{n\to\infty} p_b(x_n, x_{n+1}) = \alpha \quad \text{and} \quad p_b(x_n, x_{n+1}) \geqslant \alpha. \tag{2.7}$$

Suppose that $\alpha > 0$. From (2.6), (2.7), and ψ being nondecreasing, we obtain that

$$1 < \psi(\alpha) \leqslant \psi(p_b(x_n, x_{n+1})) \leqslant \psi(p_b(x_{n-1}, x_n))^\lambda \leqslant \cdots \leqslant \psi(p_b(x_0, x_1))^{\lambda^n} \tag{2.8}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.8), we have $1 < \psi(\alpha) \leqslant 1$, which is a contradiction. Thus $\alpha = 0$ and this yields

$$\lim_{n\to\infty} p_b(x_n, x_{n+1}) = 0. \tag{2.9}$$

Now, we show that $\{x_n\}$ is a p_b-Cauchy sequence in (X, p_b) which is equivalent to show that $\{x_n\}$ is a b-Cauchy sequence in (X, d_{p_b}). Suppose not, that is, there exist $\varepsilon > 0$ and two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that n_k is the smallest index with $n_k > m_k > k$ for which

$$d_{p_b}(x_{m_k}, x_{n_k}) \geqslant \varepsilon \tag{2.10}$$

and

$$d_{p_b}(x_{m_k}, x_{n_k-1}) < \varepsilon. \tag{2.11}$$

This implies that

$$\varepsilon \leqslant d_{p_b}(x_{m_k}, x_{n_k}) \leqslant s d_{p_b}(x_{m_k}, x_{n_k-1}) + s d_{p_b}(x_{n_k-1}, x_{n_k}) < s\varepsilon + s d_{p_b}(x_{n_k-1}, x_{n_k}). \tag{2.12}$$

Taking the upper limit as $k \to \infty$ in (2.11), we get that

$$\frac{\varepsilon}{s} \leqslant \liminf_{k\to\infty} d_{p_b}(x_{m_k}, x_{n_k-1}) \leqslant \limsup_{k\to\infty} d_{p_b}(x_{m_k}, x_{n_k-1}) \leqslant \varepsilon. \tag{2.13}$$

It follows from (2.12) that,

$$\varepsilon \leqslant \limsup_{k\to\infty} d_{p_b}(x_{m_k}, x_{n_k}) \leqslant s\varepsilon. \tag{2.14}$$

By using the triangular inequality, we have

$$\begin{aligned} d_{p_b}(x_{m_k+1}, x_{n_k}) &\leqslant s d_{p_b}(x_{m_k+1}, x_{m_k}) + s d_{p_b}(x_{m_k}, x_{n_k}) \\ &\leqslant s d_{p_b}(x_{m_k+1}, x_{m_k}) + s^2 d_{p_b}(x_{m_k}, x_{n_k-1}) + s^2 d_{p_b}(x_{n_k-1}, x_{n_k}) \\ &\leqslant s d_{p_b}(x_{m_k+1}, x_{m_k}) + s^2\varepsilon + s^2 d_{p_b}(x_{n_k-1}, x_{n_k}). \end{aligned}$$

Taking the upper limit as $k \to \infty$ in above inequality, we obtain that

$$\limsup_{k\to\infty} d_{p_b}(x_{m_k+1}, x_{n_k}) \leqslant s^2\varepsilon.$$

Further,

$$d_{p_b}(x_{m_k+1}, x_{n_k-1}) \leqslant s d_{p_b}(x_{m_k+1}, x_{m_k}) + s d_{p_b}(x_{m_k}, x_{n_k-1}) \leqslant s d_{p_b}(x_{m_k+1}, x_{m_k}) + s\varepsilon,$$

and hence

$$\limsup_{k\to\infty} d_{p_b}(x_{m_k+1}, x_{n_k-1}) \leqslant s\varepsilon. \tag{2.15}$$

By Proposition 1.7 and (2.9), we deduce that

$$\begin{aligned} \limsup_{k\to\infty} d_{p_b}(x_{m_k}, x_{n_k-1}) &= \limsup_{k\to\infty} (2p_b(x_{m_k}, x_{n_k-1}) - p_b(x_{m_k}, x_{m_k}) - p_b(x_{n_k-1}, x_{n_k-1})) \\ &= 2\limsup_{k\to\infty} p_b(x_{m_k}, x_{n_k-1}). \end{aligned} \tag{2.16}$$

Also, by (2.13) and (2.16), we get that

$$\frac{\varepsilon}{2s} \leqslant \liminf_{k\to\infty} p_b(x_{m_k}, x_{n_k-1}) \leqslant \limsup_{k\to\infty} p_b(x_{m_k}, x_{n_k-1}) \leqslant \frac{\varepsilon}{2}. \tag{2.17}$$

In analogy to (2.16), by (2.12), (2.14), and (2.15), we can prove that

$$\limsup_{k\to\infty} p_b(x_{m_k}, x_{n_k}) \leqslant \frac{s\varepsilon}{2}, \tag{2.18}$$

$$\frac{\varepsilon}{2s} \leqslant \limsup_{k\to\infty} p_b(x_{m_k+1}, x_{n_k}),$$

$$\limsup_{k\to\infty} p_b(x_{m_k+1}, x_{n_k-1}) \leqslant \frac{s\varepsilon}{2}. \tag{2.19}$$

By (2.17), (2.18), and (2.19), we obtain that

$$\limsup_{k\to\infty} M_s(x_{m_k}, x_{n_k-1}) = \max\{\limsup_{k\to\infty} p_b(x_{m_k}, x_{n_k-1}), \limsup_{k\to\infty} p_b(x_{m_k}, Tx_{m_k}), \limsup_{k\to\infty} p_b(x_{n_k-1}, Tx_{n_k-1})$$

$$
\begin{aligned}
&, \limsup_{k\to\infty} \frac{p_b(x_{m_k}, Tx_{n_k-1}) + p_b(x_{n_k-1}, Tx_{m_k})}{2s}\} \\
&\leqslant \max\{\limsup_{k\to\infty} p_b(x_{m_k}, x_{n_k-1}), 0, 0, \limsup_{k\to\infty} \frac{p_b(x_{m_k}, x_{n_k}) + p_b(x_{n_k-1}, x_{m_k+1})}{2s}\} \\
&= \max\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\} = \frac{\varepsilon}{2}.
\end{aligned}
$$

We claim that $x_{m_k+1} \neq x_{n_k}$. If $x_{m_k+1} = x_{n_k}$, then $d_{p_b}(x_{m_k+1}, x_{n_k}) = 0$. From (2.10) and Proposition 1.7, we have

$$
\begin{aligned}
\varepsilon \leqslant d_{p_b}(x_{m_k}, x_{n_k}) &\leqslant sd_{p_b}(x_{m_k}, x_{m_k+1}) + sd_{p_b}(x_{m_k+1}, x_{n_k}) \\
&= sd_{p_b}(x_{m_k}, x_{m_k+1}) \\
&= s(2p_b(x_{m_k}, x_{m_k+1}) - p_b(x_{m_k}, x_{m_k}) - p_b(x_{m_k+1}, x_{m_k+1})) \\
&\leqslant 2sp_b(x_{m_k}, x_{m_k+1}).
\end{aligned}
$$

Letting $k \to \infty$ and using (2.9), we deduce that

$$
\frac{\varepsilon}{2s} \leqslant \lim_{k\to\infty} p_b(x_{m_k}, x_{m_k+1}) = 0,
$$

which is a contradiction. It follows from (2.1) that

$$
\begin{aligned}
\psi(\frac{\varepsilon}{2}) \leqslant \psi(s \limsup_{k\to\infty} p_b(x_{m_k+1}, x_{n_k})) &= \limsup_{k\to\infty} \psi(sp_b(x_{m_k+1}, x_{n_k})) \\
&\leqslant \limsup_{k\to\infty} \psi(M_s(x_{m_k}, x_{n_k-1}))^\lambda \leqslant \psi(\frac{\varepsilon}{2})^\lambda < \psi(\frac{\varepsilon}{2}),
\end{aligned}
$$

which is a contradiction. Thus $\{x_n\}$ is a b-Cauchy in b-metric space (X, d_{p_b}). Since (X, p_b) is p_b-complete, then (X, d_{p_b}) is a b-complete b-metric space. So, there exists $z \in X$ such that $\lim_{n\to\infty} d_{p_b}(x_n, z) = 0$. By Lemma 1.9, we get that

$$
\lim_{n\to\infty} p_b(z, x_n) = p_b(z, z). \tag{2.20}
$$

By Proposition 1.7, (2.9), (2.20), and condition (p_{b2}), we have

$$
\lim_{n\to\infty} p_b(z, x_n) = \lim_{n\to\infty} p_b(x_n, x_n) = 0. \tag{2.21}
$$

Suppose that $z \neq Tz$ implies that $p_b(z, Tz) > 0$ and $d_{p_b}(z, Tz) > 0$. It follows from (2.9) and (2.21) that there exists a positive integer n_0 such that

$$
p_b(x_n, z) \leqslant \frac{p_b(z, Tz)}{2} \quad \text{and} \quad p_b(x_n, x_{n+1}) \leqslant \frac{p_b(z, Tz)}{2}
$$

for all $n \geqslant n_0$. This implies that

$$
\begin{aligned}
M_s(x_n, z) &= \max\{p_b(x_n, z), p_b(x_n, x_{n+1}), p_b(z, Tz), \frac{p_b(x_n, Tz) + p_b(z, x_{n+1})}{2s}\} \\
&\leqslant \max\{\frac{p_b(z, Tz)}{2}, \frac{p_b(z, Tz)}{2}, p_b(z, Tz), p_b(z, Tz)\} = p_b(z, Tz)
\end{aligned} \tag{2.22}
$$

for all $n \geqslant n_0$. Since T is p_b-continuous and (2.20), we obtain that

$$
\lim_{n\to\infty} p_b(x_{n+1}, Tz) = p_b(Tz, Tz). \tag{2.23}
$$

By the triangle inequality, we deduce that

$$
p_b(z, Tz) \leqslant sp_b(z, x_{n+1}) + sp_b(x_{n+1}, Tz)
$$

for all $n \in \mathbb{N}$. So by taking limit as $n \to \infty$ and using (2.23), we have

$$p_b(z, Tz) \leqslant s \lim_{n\to\infty} p_b(z, x_{n+1}) + s \lim_{n\to\infty} p_b(x_{n+1}, Tz) = sp_b(Tz, Tz). \tag{2.24}$$

If there are infinitely many $n \in \mathbb{N}$ such that $x_{n+1} = Tz$, then $d_{p_b}(x_{n+1}, Tz) = 0$. This implies that

$$d_{p_b}(z, Tz) \leqslant sd_{p_b}(z, x_{n+1}) + sd_{p_b}(x_{n+1}, Tz) = sd_{p_b}(z, x_{n+1}).$$

Letting $n \to \infty$, we get that $d_{p_b}(z, Tz) \leqslant s \lim_{n\to\infty} d_{p_b}(z, x_{n+1}) = 0$, which is a contradiction. This implies that there exists $n_1 \in \mathbb{N}$ such that $x_{n+1} \neq Tz$ for all $n \geqslant n_1$. Choose $N = \max\{n_0, n_1\}$. Thus, by (2.1) and (2.22), for each $n \geqslant N$, we get that

$$\psi(sp_b(x_{n+1}, Tz)) \leqslant \psi(M_s(x_n, z))^\lambda \leqslant \psi(p_b(z, Tz))^\lambda.$$

Letting $n \to \infty$ in this inequality, using the continuity of ψ, (2.23), and (2.24), we obtain that

$$\psi(p_b(z, Tz)) \leqslant \psi(sp_b(Tz, Tz)) = \lim_{n\to\infty} \psi(sp_b(x_{n+1}, Tz)) \leqslant \psi(p_b(z, Tz))^\lambda < \psi(p_b(z, Tz)),$$

which is a contradiction. Hence $Tz = z$. Thus z is a fixed point of T. Let x be another fixed point of T with $x \neq z$. It follows from (2.1) that

$$\begin{aligned}
\psi(p_b(x, z)) &\leqslant \psi(sp_b(Tx, Tz)) \\
&\leqslant p_b(M_s(x, z))^\lambda \\
&= \psi(\max\{p_b(x, z), p_b(x, Tx), p_b(z, Tz), \frac{p_b(x, Tz) + p_b(z, Tx)}{2s}\})^\lambda \\
&= \psi(\max\{p_b(x, z), p_b(x, x), p_b(z, z), \frac{p_b(x, z) + p_b(z, x)}{2s}\})^\lambda \\
&= \psi(\max\{p_b(x, z), p_b(x, x), p_b(z, z), \frac{p_b(x, z)}{s}\})^\lambda \\
&\leqslant \psi(\max\{p_b(x, z), p_b(x, z), p_b(x, z), \frac{p_b(x, z)}{s}\})^\lambda \\
&= \psi(p_b(x, z))^\lambda \\
&< \psi(p_b(x, z)),
\end{aligned}$$

which is a contradiction. So $x = z$. Hence T has a unique fixed point. □

We illustrate the following example for supporting our result.

Example 2.6. Let $X = \{0, 1, 2\}$ with the partial b-metric $p_b : X \times X \to \mathbb{R}_+$ defined by

$$p_b(x, y) = [\max\{x, y\}]^2$$

for all $x, y \in X$. Obviously, (X, p_b) is a p_b-complete partial b-metric space with $s = 2$, but it is not a metric on X. To see this, let $x = y = 2$ then $p_b(2, 2) = [\max\{2, 2\}]^2 = 4 \neq 0$. Define the mapping $T : X \to X$ by $T0 = T1 = 0$ and $T2 = 1$.

We will show that T is a generalized JS-quasi-contraction with $\psi(t) = e^{te^t}$. In fact, it suffices to prove that there exists $\lambda \in (0, 1)$ such that, for all $x, y \in X$ with $Tx \neq Ty$,

$$\frac{2p_b(Tx, Ty)e^{2p_b(Tx,Ty)-M_s(x,y)}}{M_s(x, y)} \leqslant \lambda.$$

Let $x, y \in X$ with $Tx \neq Ty$. Therefore, $x = 0, y = 2$ or $x = 1, y = 2$. For both cases, we get $p_b(T0, T2) = p_b(T1, T2) = 1$ and $M_s(0, 2) = M_s(1, 2) = 4$. This implies that

$$\frac{2p_b(T0, T2)e^{2p_b(T0,T2)-M_s(0,2)}}{M_s(0, 2)} = \frac{2p_b(T1, T2)e^{2p_b(T1,T2)-M_s(1,2)}}{M_s(1, 2)} = \frac{e^{-2}}{2}.$$

This shows that T is a generalized JS-quasi-contraction with $\psi(t) = e^{te^t}$ and $\lambda \in [\frac{e^{-2}}{2}, 1)$. By Example 1.12, we know that $e^{te^t} \in \Phi_2$. Therefore, the conclusion immediately follows from Theorem 2.5 to obtain that T has a unique fixed point which is $x = 0$.

Theorem 2.7. *Let (X, p_b) be a p_b-complete partial b-metric space with the coefficient $s \geqslant 1$. Let $T : X \to X$ be a p_b-continuous mapping. Assume that there exist a function $\psi \in \Phi_3$ and nonnegative numbers k_1, k_2, k_3, k_4 with $k_1 + k_2 + k_3 + 2k_4 < 1$ such that (2.3) is satisfied. Then T has a unique fixed point in X.*

Proof. From Remark 2.4 (iii), we get that T is a generalized JS-quasi-contraction with $\lambda = k_1 + k_2 + k_3 + 2k_4$. In case of $0 < \lambda < 1$, by Theorem 2.5, the proof is completed. In case of $\lambda = 0$, by (2.1) we have

$$\psi(sp_b(Tx, Ty)) \leqslant \psi(M_s(x, y))^0 = 1 \quad \text{for all } x, y \in X.$$

Further, by (Ψ1) we deduce that $p_b(Tx, Ty) = 0$ for all $x, y \in X$. Thus, for $y = Tx$, we have $p_b(Tx, T(Tx)) = 0$. It follows that $y = Tx$ is a fixed point of T. Let z be another fixed point of T. Then

$$p_b(y, z) = p_b(Ty, Tz) = 0.$$

Therefore, $y = z$ and so T has a unique fixed point. □

Corollary 2.8. *Let (X, p_b) be a p_b-complete partial b-metric space with the coefficient $s \geqslant 1$ and $T : X \to X$ be a p_b-continuous mapping. Assume that there exist $\psi \in \Phi_2$ and nonnegative real numbers k_1, k_2, k_3, k_4 with $k_1 + k_2 + k_3 + 2k_4 < 1$ such that (2.2) is satisfied. Then T has a unique fixed point in X.*

Corollary 2.9. *Let (X, p_b) be a p_b-complete partial b-metric space with the coefficient $s \geqslant 1$ and $T : X \to X$ be a p_b-continuous mapping. Assume that there exist $\psi \in \Phi_4$ and nonnegative real numbers k_1, k_2, k_3, k_4 with $k_1 + k_2 + k_3 + 2k_4 < 1$ such that (2.4) is satisfied. Then T has a unique fixed point in X.*

Corollary 2.10. *Let (X, p_b) be a p_b-complete partial b-metric space with the coefficient $s \geqslant 1$, and $T : X \to X$ be p_b-continuous. Assume that there exist $a > 0$ and nonnegative numbers k_1, k_2, k_3, k_4 with $k_1 + k_2 + k_3 + 2k_4 < 1$ such that*

$$(sp_b(Tx, Ty))^a \leqslant k_1 p_b(x, y)^a + k_2 p_b(x, Tx)^a + k_3 p_b(y, Ty)^a + 2k_4 (\frac{p_b(x, Ty) + p_b(y, Tx)}{2s})^a \tag{2.25}$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Proof. From Example 1.13, we have $\psi(t) = e^{t^a} \in \Phi_3$, and so (2.3) immediately follows from (2.25). Thus, by Theorem 2.7, T has a unique fixed point. □

Corollary 2.11. *Let (X, p_b) be a p_b-complete partial b-metric space with the coefficient $s \geqslant 1$, and $T : X \to X$ be p_b-continuous. Assume that there exist nonnegative numbers k_1, k_2, k_3, k_4 with $k_1 + k_2 + k_3 + 2k_4 < 1$ such that*

$$(sp_b(Tx, Ty))^a \leqslant k_1 p_b(x, y)^a + k_2 p_b(x, Tx)^a + k_3 p_b(y, Ty)^a + k_4 (\frac{p_b(x, Ty) + p_b(y, Tx)}{s})^a \tag{2.26}$$

for all $x, y \in X$, where $a = \frac{1}{n}$. Then T has a unique fixed point in X.

Proof. For each $a \in (0, 1]$, we obtain that

$$(\frac{p_b(x, Ty) + p_b(y, Tx)}{s})^a \leqslant 2(\frac{p_b(x, Ty) + p_b(y, Tx)}{2s})^a.$$

Then (2.26) implies (2.25). Thus, Corollary 2.11 immediately follows from Corollary 2.10. This implies that T has a unique fixed point. □

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