Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 12 (2019), 740–744
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

Quasi-arithmetic F-convex functions and their characterization

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Abstract

In this paper we demonstrate that the results presented in the paper [B. Bin-Mohsin, M. A. Noor, K. I. Noor, S. Iftikhar, J. Nonlinear. Sci. Appl., **11** (2018),1070–1076] are not true in general. Moreover, we give some new notions, which could be applied in type problems as in [B. Bin-Mohsin, M. A. Noor, K. I. Noor, S. Iftikhar, J. Nonlinear. Sci. Appl., **11** (2018), 1070–1076].

Keywords: F-convex function, quasi-arithmetic F-convex function, quadratic function, quasi-arithmetic quadratic function.

2010 MSC: 26A51, 39B62, 15A63.

1. Introduction

In the introduction of the paper [2], the authors present the following definitions.

Let $(X, \|\cdot\|)$ be a real normed space, and I be a harmonic convex subset of X.

Definition 1.1. A set $I \subseteq \mathbb{R} \setminus \{0\}$ is said to be a *harmonic set* if

$$\frac{xy}{tx + (1-t)y} \in I, \quad \forall x, y \in I, t \in [0, 1].$$

Definition 1.2. A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be a *strongly general harmonic convex function* if there exists a non-negative function $F : X \setminus \{0\} \to \mathbb{R}$, such that

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leqslant (1-t)f(x) + tf(y) - t(1-t)F\left(\frac{xy}{x-y}\right), \quad \forall x, y \in I, t \in [0, 1]. \tag{1.1}$$

If (1.1) is assumed only for $t = \frac{1}{2}$, we have

$$f\left(\frac{2xy}{x+y}\right) \leqslant \frac{f(x) + f(y)}{2} - \frac{1}{4}F\left(\frac{xy}{x-y}\right),$$

which is called a *strongly general harmonic* J*-convex function*.

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doi: 10.22436/jnsa.012.11.05

Received: 2018-11-02 Revised: 2019-06-15 Accepted: 2019-06-19

Let us now focus on these definitions. First of all, notice that the set I has to be a subset of $\mathbb{R}$, and not a subset of a real normed space X. In the light of the first definition, it is worth to observe that a subset of $\mathbb{R}$ is a harmonic set (in my opinion it would be better to write "harmonic convex set") if and only if it is only an interval of positive numbers or an interval of negative numbers.

We can not accept Definition 1.2. Namely, the function F is defined on a normed space without the zero vector, not on subset of the real line; but in (1.1) we take real numbers x, y. From the context of the main results of the work in question, we conclude that $X = \mathbb{R}$, but it also does not save Definitions 1.2, because we do not know what happens when x equals y.

Suppose that the deficiencies of Definition 1.2 are removed and let us get to the main results in [2]. The main results in [2] are based on the following two lemmas.

Lemma 1.3 ([2, Lemma 2.1]). *Let $F : I = [a, b] \subseteq X \setminus \{0\} \to [0, \infty)$ be an even function. If the function F is a strongly general harmonic convex function, then*

$$F\left(\frac{x}{1-t}\right) = (1-t)^2 F(x), \quad \forall t \in (0,1), x \in I.$$

Lemma 1.4 ([2, Lemma 2.2]). *Let $F : I = [a, b] \subseteq X \setminus \{0\} \to [0, \infty)$ be an even function. If the function F is a strongly general harmonic J-convex function, then $F(x) = 4F(2x)$ for all $x \in I$.*

The authors seem not to have noticed that the theses of these lemmas make sense only when an interval I is unbounded and contains only numbers with the same sign. In the light of the above, and the definition o strongly general harmonic convex functions (strongly general harmonic J-convex functions, resp.) the assumption that F is an even function is incomprehensible.

Moreover, Lemmas 1.3 and 1.4 are not true also if we forget the above comments. A counterexample will be presented in the next section, together with some general results inspired by the paper [2].

2. Main result

2.1. A counterexample

Let $F : (0, \infty) \to \mathbb{R}$ be the function defined by the formula $F(x) = \frac{1}{|x|^3}$. Consider the inequality

$$F\left(\frac{2xy}{x+y}\right) \leqslant \frac{F(x) + F(y)}{2} - \frac{1}{4} F\left(\frac{xy}{x-y}\right) \tag{2.1}$$

for all positive and different x and y (for the same x and y we may think about the limit "$\lim_{(x,y)\to(a,a)}$" in both sides). Inequality (2.1) can be rewritten in the following equivalent form

$$F\left(\left(\frac{\frac{1}{x}+\frac{1}{y}}{2}\right)^{-1}\right) \leqslant \frac{F\left(\left(\frac{1}{x}\right)^{-1}\right) + F\left(\left(\frac{1}{y}\right)^{-1}\right)}{2} - \frac{1}{4} F\left(\left(\frac{1}{x} - \frac{1}{y}\right)^{-1}\right),$$

which means that for the function $G(x) = |x|^3$ the inequality (2.1) is equivalent to the inequality (2.2).

$$G\left(\frac{x+y}{2}\right) \leqslant \frac{G(x) + G(y)}{2} - \frac{1}{4} G(x-y) \tag{2.2}$$

for all positive real numbers x, y.

Now, instead of proving the inequality (2.1), we prove the inequality (2.2), which is easier. From the inequality (2.2) we obtain

$$\left|\frac{x+y}{2}\right|^3 \leqslant \frac{|x|^3 + |y|^3}{2} - \frac{1}{4} |x-y|^3. \tag{2.3}$$

Due to the symmetry of elements x, y in the inequality (2.3), it will suffice to prove this inequality only for $y \leqslant x$. And for $y \leqslant x$ it becomes the following equivalent inequality

$$0 \leqslant x^3 + 3x^2y - 9xy^2 + 5y^3. \tag{2.4}$$

To prove the inequality (2.4), fix an arbitrary positive number x_0 and consider the function

$$h(y) = x_0^3 + 3x_0^2y - 9x_0y^2 + 5y^3$$

for $y \leqslant x_0$. Calculating the derivative we have

$$h'(y) = 15(y - x_0)(y - \frac{x_0}{5}).$$

And it means that

$$h(y) \geqslant \min\left\{\lim_{y\to 0} h(y), h(x_0)\right\} = \min\left\{x_0^3, 0\right\} = 0.$$

The inequality (2.4) is valid true.

In conclusion, the function $F(x) = \frac{1}{|x|^3}$ is a strongly general harmonic J-convex function and $F(x) = 8F(2x)$. Thus Lemmas 1.3 and 1.4 are not true.

2.2. Quasi-arithmetic F-convex and F-midconvex functions

In what follows, X is a real vector space, $F : X \to \mathbb{R}$ is a fixed function, and D stands for a convex subset of X.

Recall (see [1]) that a function $f : D \to \mathbb{R}$ is called F-*convex* if

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) - t(1-t)F(x-y)$$

for all $x, y \in D$ and $t \in (0, 1)$, and we say that f is F-*midconvex* if

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} - \frac{1}{4}F(x-y)$$

for all $x, y \in D$.

Now we define quasi-arithmetic F-convex functions and quasi-arithmetic F-midconvex functions, which will generalize the concept of strongly general harmonic convex functions, strongly general harmonic J-convex functions (in some sense), and also F-convex and F-midconvex functions, respectively.

Definition 2.1. Let $\varphi : D \to X$ be an injective function such that the set $\phi(D)$ is a linear subspace of X. A function $f : D \to \mathbb{R}$ will be called a *quasi-arithmetic* F-*convex function* if

$$f\left(\varphi^{-1}\left(t\varphi(x) + (1-t)\varphi(y)\right)\right) \leqslant tf(x) + (1-t)f(y) - t(1-t)F\left(\varphi^{-1}\left(\varphi(x) - \varphi(y)\right)\right) \tag{2.5}$$

for all $x, y \in D$ and $t \in (0, 1)$.

If a function f satisfies condition (2.5) with the zero function F, i.e.,

$$f\left(\varphi^{-1}\left(t\varphi(x) + (1-t)\varphi(y)\right)\right) \leqslant tf(x) + (1-t)f(y)$$

for all $x, y \in D$ and $t \in (0, 1)$, then f will be called a *quasi-arithmetic convex function*.

Definition 2.2. Let $\varphi : D \to X$ be an injective function such that the set $\phi(D)$ is a linear subspace of X. A function $f : D \to \mathbb{R}$ will be called a *quasi-arithmetic* F-*midconvex function* if

$$f\left(\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)\right) \leqslant \frac{f(x)+f(y)}{2} - \frac{1}{4}F\left(\varphi^{-1}\left(\varphi(x) - \varphi(y)\right)\right) \tag{2.6}$$

for all $x, y \in D$.

If a function f satisfies condition (2.6) with the zero function F, i.e.,

$$f\left(\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)\right) \leqslant \frac{f(x)+f(y)}{2}$$

for all $x, y \in D$, then f will be called a *quasi-arithmetic midconvex function*.

It is easy to observe that for certain specific functions φ these definitions are reduced to the mentioned notions, namely:

Remark 2.3. If we take $\varphi = id$, then it is reduced to the definitions of F-convex and F-midconvex functions, respectively.

Remark 2.4. If we take $X = \mathbb{R}$ and $\varphi = \frac{1}{id}$ defined for positive or negative numbers, then the inequalities (2.5) and (2.6) become the inequalities proposed for strongly general harmonic convex functions and strongly general harmonic J-convex functions, respectively.

Lemma 2.5. *Let $\varphi : D \to X$ be an injective function such that the set $\phi(D)$ is a linear subspace of X. A function $f : D \to \mathbb{R}$ is a quasi-arithmetic F-convex function if and only if the function $g = f \circ \varphi^{-1}$ is G-convex on $\phi(D)$, with $G = F \circ \varphi^{-1}$.*

Proof. It is enough to observe that for $u = \varphi(x)$ and $v = \varphi(y)$ the inequality

$$f\left(\varphi^{-1}\left(t\varphi(x)+(1-t)\varphi(y)\right)\right) \leqslant tf(x)+(1-t)f(y)-t(1-t)F\left(\varphi^{-1}\left(\varphi(x)-\varphi(y)\right)\right), \quad x, y \in D,$$

is equivalent to the inequality

$$g\left(tu+(1-t)v\right) \leqslant tg(u)+(1-t)g(v)-t(1-t)G\left(u-v\right), \quad u, v \in \varphi(D),$$

which ends the proof. □

Corollary 2.6. *A function $f : D \to \mathbb{R}$ is a quasi-arithmetic convex function if and only if the function $g = f \circ \varphi^{-1}$ is convex on $\varphi(D)$.*

Similarly, the next lemma and corollary hold true.

Lemma 2.7. *Let $\varphi : D \to X$ be an injective function such that the set $\phi(D)$ is a linear subspace of X. A function $f : D \to \mathbb{R}$ is a quasi-arithmetic F-midconvex function if and only if the function $g = f \circ \varphi^{-1}$ is G-midconvex on $\varphi(D)$, with $G = F \circ \varphi^{-1}$.*

Corollary 2.8. *A function $f : D \to \mathbb{R}$ is a quasi-arithmetic midconvex function if and only if the function $g = f \circ \varphi^{-1}$ is midconvex on $\varphi(D)$.*

Definition 2.9. Let $\varphi : D \to X$ be an injective function such that the set $\phi(D)$ is a linear subspace of X. A function $F : D \to \mathbb{R}$ will be called a *quasi-arithmetic quadratic function* if

$$F\left(\varphi^{-1}\left(\varphi(x)+\varphi(y)\right)\right)+F\left(\varphi^{-1}\left(\varphi(x)-\varphi(y)\right)\right)=2F(x)+2F(y) \tag{2.7}$$

for all $x, y \in D$.

Remark 2.10. If we take $\varphi = id$, then the equation (2.7) reduces to the equation of quadratic functions.

Remark 2.11. If we take $X = \mathbb{R}$ and $\varphi = \frac{1}{id}$ defined for positive or negative numbers, then the equation (2.9) becomes the equation proposed in [2] for quadratic harmonic functions.

Lemma 2.12. *Let $\varphi : D \to X$ be an injective function such that the set $\varphi(D)$ is a linear subspace of X. A function $F : D \to \mathbb{R}$ is a quasi-arithmetic quadratic function if and only if the function $G = F \circ \varphi^{-1}$ is a quadratic function on $\varphi(D)$.*

Proof. Observe that for $u = \varphi(x)$ and $v = \varphi(y)$ the equation

$$F\left(\varphi^{-1}\left(\varphi(x)+\varphi(y)\right)\right) + F\left(\varphi^{-1}\left(\varphi(x)-\varphi(y)\right)\right) = 2F(x) + 2F(y), \quad x, y \in D,$$

is equivalent to

$$G(u+v) + G(u-v) = 2G(u) + 2G(v), \quad u, v \in \varphi(D).$$

The proof is finished. □

Theorem 2.13. *Let $\varphi : X \to X$ be a bijective function and $F : X \to [0, \infty)$ be a fixed even function. The following conditions are equivalent.*

1. *For all functions $f : D \to \mathbb{R}$, f is quasi-arithmetic F-midconvex if and only if the function $h = f - F$ is quasi-arithmetic midconvex.*
2. *The function F is a quasi-arithmetic F-midconvex function.*
3. *The function F is a quasi-arithmetic quadratic function.*

Proof. Adopting $g = f \circ \varphi^{-1}$, $G = F \circ \varphi^{-1}$ and taking into consideration Lemma 2.7 and Lemma 2.12, we conclude that Theorem 2.13 is equivalent to the theorem obtained in [1] (Theorem 1, p. 1290), with functions g and G; which ends the proof. □

Remark 2.14. We can replace the first condition in Theorem 2.13 with the following:

1. A function $f : X \to \mathbb{R}$ is quasi-arithmetic F-midconvex if and only if the function $h = f - F$ is quasi-arithmetic midconvex.

Similarly, using lemmas 2.5, 2.12, and [1, Theorem 2] we derive the following theorem.

Theorem 2.15. *Let $\varphi : X \to X$ be a bijective function and $F : X \to [0, \infty)$ be a fixed even function. The following conditions are equivalent.*

1. *For all functions $f : D \to \mathbb{R}$, f is quasi-arithmetic F-convex if and only if the function $h = f - F$ is quasi-arithmetic convex.*
2. *The function F is a quasi-arithmetic F-convex function.*
3. *The function F is a quasi-arithmetic F-affine function (i.e., F satisfies (2.5) with "=" instead of "$\leqslant$").*

Remark 2.16. We can replace the first condition in Theorem 2.15 with the following:

1. A function $f : X \to \mathbb{R}$ is quasi-arithmetic F-convex if and only if the function $h = f - F$ is quasi-arithmetic convex.

Remark 2.17. From the third condition of Theorem 2.15 we conclude that F is a quasi-arithmetic quadratic function and the function $F \circ \varphi^{-1}$ is homogeneous of degree 2.

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