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Controllability and observability of fuzzy matrix discrete dynamical systems

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Abstract

In this paper, sufficient conditions for the controllability of the fuzzy dynamical discrete system with the use of fuzzy rule base are established. Further, a sufficient condition for the fuzzy dynamical discrete system to be observable is constructed. The main advantage of this approach is that the rule base for the initial value can be determined without actually solving the system. Difference inclusions approach is adopted in the construction of these conditions. All the established theories are consolidated and explained with the help of examples.

Keywords: Fuzzy difference equations, fuzzy rule, controllability, observability, discrete dynamical systems.

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1. Introduction

Measurements of data or specified information for an underlying problem may be imprecise or only partially specified. Each and every practical system is endowed with uncertainties. The more the complexity of a system the greater is its uncertainty due to fuzziness. That is, each quantity we want to measure becomes fuzzy valued instead of precise valued. Difference equations describe the evolution of certain phenomena over the course of time. Many of the physical applications may not have the exact information about their deterministic dynamics which is a prerequisite construct of a dynamical system. It is very important to study the controllability and obsarvability of the mathematical models represented by fuzzy difference equations governing the ambiguity in dynamics which is not probabilistic. In general the problem of steering an initial state of a system to a desired final state in R^n become a problem of steering a fuzzy-state to another fuzzy-state in $(E^l)^n$.

The importance of control theory in mathematics and its occurrence in several problems such as mechanics, electromagnetic theory, thermodynamics, and artificial satellites are well known. In general,

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fuzzy systems are classified in to 3 categories, (i) pure fuzzy systems; (ii) T-S fuzzy systems; and (iii) fuzzy logic systems, using fuzzifiers and defuzzifiers. In this paper, we use fuzzy matrix discrete system to describe fuzzy logic system and establish sufficient conditions for controllability and observability of first order fuzzy matrix discrete system S_1modeled by

$$\begin{aligned} T(n+1) &= A(n)T(n) + F(n)U(n), T(0) = T_0, n > 0, \\ Y(n) &= C(n)T(n) + D(n)U(n), \end{aligned} \tag{1.1}$$

where $U(n)$ is an $m \times s$ fuzzy input matrix called fuzzy control and $Y(n)$ is an $n \times n$ fuzzy output matrix. Here $T(n)$, $A(n)$, $F(n)$, $C(n)$, and $D(n)$ are matrices of order $s \times s$, $s \times s$, $s \times m$, $r \times s$, and $r \times m$ whose elements are continuous functions of n on $J = [0, N] \subset R(N > 0)$. Barnett and Cameron [3] studied the problem of controllability and observability for a system of ordinary differential equations in 1985. In 1997, Murty and Anand [11] established necessary and sufficient conditions for controllability and observability of continuous matrix Liapunov systems. Using fuzzy control a complex system can be decomposed into several subsystems according to the expertise of human ability to understand the system and using the human control strategy represented by a simple control law. The popular fuzzy controllers in the literature are Mamdani fuzzy controllers and Takagi-Sugeno (TS) fuzzy controllers. The main difference between them is that the Mamdani fuzzy controllers use fuzzy sets where as the (TS) fuzzy controllers use linear functions, to represent the fuzzy rules. In the works of Takagi [16] and Sugeno [15] a crisp analytical function is used instead of a membership function in a fuzzy model. In recent years many authors [1, 4, 8–10, 17, 18] are studying TS fuzzy controllers, because of to their ability to model real world problems. In 2005, Anand and Murty [2] established conditions for controllability and observability of Liapunov type matrix difference system. In 2008, Murty et al. [13] presented criteria for the existence and uniqueness of solution to Kronecker product initial value problem associated with general first order matrix difference system. In 2009, Murty et al. [12] studied qualitative properties of general first order matrix difference systems. We obtain a unique solution of the system (1.1), when U(n) is a crisp continuous matrix. We use fuzzy matrix discrete system to describe fuzzy logic system and extend some of the results of Ding and Kandel [6, 7] developed for continuous case to that of discrete case by vectorizing the fuzzy matrix discrete system. we obtain sufficient conditions for controllability and observability of the system (1.1) satisfying the initial condition. The fundamental results established in [12, 13] have in-fact motivated us to develop our results on fuzzy matrix discrete dynamical systems.

The paper is organized as follows. Section 2 presents basic definitions and results required to understand the paper. Section 3 is concerned with the formation of fuzzy dynamical discrete systems. Sufficient conditions for the controllability and observability of the fuzzy matrix discrete dynamical system are presented in Sections 4 and 5, respectively. Section 6 presents the numerical example.

2. Preliminaries

Theorem 2.1. *If* $u \in E^s$, *then*

1. $[u]^\alpha \in P_k(((N^+_{N_0}))^{S\times S})$ *for all* $0 \leqslant \alpha \leqslant 1$;
2. $[u]^{\alpha_2} \subset [u]^{\alpha_1}$ *for all* $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant 1$;
3. *If* $\{\alpha_k\}$ *is non decreasing sequence converging to* $\alpha > 0$, *then* $[u]^\alpha = \cap_{k\geqslant 1}[U]^{\alpha_k}$.

Conversely, if $\{A^\alpha : 0 \leqslant \alpha \leqslant 1\}$ *is a family of subsets of* R^S *satisfying* (1)-(3), *then there exists a* $u \in E^S$ *such that* $[u]^\alpha = A^\alpha$*for* $0 < \alpha \leqslant 1$ *and* $[u]^0 = \overline{U_{0\leqslant\alpha\leqslant 1} A^\alpha} \subset A^0$.

Definition 2.2. Let $A \in C^{r\times s}(R^{r\times s})$ and $AB \in C^{p\times q}(R^{p\times q})$. Then kronecker product of of A and B is written as $A \otimes B$ is defined as a partitioned matrix $A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1s}B \\ a_{21}B & a_{22}B & \dots & a_{2s}B \\ \dots & \dots & \dots & \dots \\ a_{r1}B & a_{r2}B & \dots & a_{rs}B \end{bmatrix}$ matrix and is an $rp \times sq$ and is in $C^{rp\times sq}(R^{rp\times sq})$.

The kronecker product has the following properties.

1. $(A \otimes B)^* = A^* \otimes B^*$.
2. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.
3. $(A \otimes B)(C \otimes D) = (AC \otimes BD)$. This rule holds, provided the dimensions of the matrices are such that expressions are defined.
4. $\|(A \otimes B)\| = \|A\|\|B\|$ (where $\|A\| = \max_{i,j}|a_{ij}|$).
5. $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$.

Vectorization of matrix A is denoted by $\mathrm{Vec}(A) = \widehat{A}$ and defined as follows.

Definition 2.3. Let $A = [a_{ij}] \in C^{r\times s}(R^{r\times s})$, we denote $\widehat{A} = \mathrm{Vec}A = [A_{.1}, A_{.2}, \ldots, A_{.s}]^T$, where

$$A_{.j} = [a_{1j}, a_{2j}, \ldots, a_{rj}]^T, (1 \leqslant j \leqslant s).$$

Vectorization has the following properties

1. $\mathrm{Vec}(ATB) = (B^* \otimes A)\mathrm{Vec}T$.
2. If A and B are square matrices of order s, then
   (i) $\mathrm{Vec}(AT) = ((I_s \otimes A)\mathrm{Vec}T$;
   (ii) $\mathrm{Vec}(TB) = ((B^* \otimes I_s)\mathrm{Vec}T$.

Lemma 2.4. *Let $\phi(n)$ be the fundamental matrix for the system*

$$T(n+1) = A(n)T(n), T(0) = I_s. \tag{2.1}$$

Then the matrix $I_s \otimes \phi(n)$ is a fundamental matrix of

$$\widehat{T}(n+1) = G(n)\widehat{T}(n), \widehat{T}(0) = \widehat{T}_0, \tag{2.2}$$

where $G(n) = (I_s \otimes A(n))$ and the solution of (2.2) is $\widehat{T}(n) = (I_s \otimes \phi(n))\widehat{T}_0$.

Theorem 2.5. *Let $\phi(n)$ be the fundamental matrix for the system (2.1). Then the unique solution of the initial value problem*

$$\widehat{T}(n+1) = G(n)\widehat{T}(n) + (I_s \otimes F(n))\widehat{U}(n), \widehat{T}(0) = \widehat{T}_0 \tag{2.3}$$

is given by

$$\widehat{T}(n) = (I_s \otimes \phi(n))\widehat{T}_0 + \sum_{j=0}^{n-1}(I_s \otimes \phi(n-j-1))(I_s \otimes \widehat{F}(j))\widehat{U}(j).$$

3. Formation of fuzzy dynamical discrete systems

Let $u_i(n) \in E^1, n \in J, i = 1, 2, \ldots, s^2$, and define

$$\begin{aligned}\widehat{U}(n) &= (u_1(n), u_2(n), \ldots, u_{s^2}(n)) = u_1(n) \times u_2(n) \times \cdots u_{s^2}(n)\\ &= \{(u_1^{\alpha}(n), u_2^{\alpha}(n), \ldots, u_{s^2}^{\alpha}(n)\} : \alpha \in [0,1]\}\\ &= \{(\widetilde{u}_1(n), \widetilde{u}_2(n), \ldots, \widetilde{u}_{s^2}(n)\} : \widetilde{u}_i(n) \in u_i^{\alpha}(n), \alpha \in [0,1]\},\end{aligned}$$

where $u_i^{\alpha}(n)$ is the α level set of $u_i(n)$. From the above definition of $\widehat{U}(n)$ and Theorem 2.1, it can be easily seen that $\widehat{U}(n) \in E^{s^2}$. We now show that the following system S_2 defined by system (2.3) and the following system

$$\widehat{Y}(n) = ((I_S \otimes C(n)))\widehat{T}(n) + (I_S \otimes D(n))\widehat{U}(n) \tag{3.1}$$

determines a fuzzy system by using the fuzzy control $\widehat{U}(n)$. Assume that $\widehat{U}(n)$ is continuous in E^{s^2}, then the set $\widehat{U}^{\alpha} = u_1(n) \times u_2(n) \times, \dots, \times u_{s^2}(n)$ is a convex and compact set in $(N^+_{n_0})^{s^2}$. For any positive number N, consider the following inclusions

$$\widehat{T}(n+1) \in G(n)\widehat{T}(n) + (I_s \otimes F(n))\widehat{U}^{\alpha}(n), n \in [0, N], \tag{3.2}$$

$$\widehat{T}(0) \in \widehat{T}_0. \tag{3.3}$$

Let $\widehat{T}^{\alpha}$ be the solution of (3.2) satisfying (3.3).

Lemma 3.1. $[\widehat{T}(n)]^{\alpha} \in P_K(N^+_{n_0})^{s^2}$ *for every* $0 \leqslant \alpha \leqslant 1, n \in [0, N]$.

Proof. We can observe that $\widehat{T}^{\alpha}$ is non empty since $\widehat{U}^{\alpha}(n)$ has measurable selection.

By choosing $K = \max_{n\in[0,N]} \parallel \phi(n) \parallel, L = \max_{n\in[0,N]} \parallel I_s \parallel = 1, M_1 = \max\{\parallel u_n \parallel : u(n) \in \widehat{U}^{\alpha}(n), n \in [0, N]\}, M_2 = \max_{n\in[0,N]} \parallel F(n) \parallel$, if for any $\widehat{T} \in \widehat{T}^{\alpha}$, there exists a control $u(n) \in \widehat{U}^{\alpha}(n)$ such that

$$\widehat{T}(n) = (I_s \otimes \phi(n))\widehat{T}_0 + \sum_{j=0}^{n-1} (I_s \otimes \phi(n-j-1))(I_s \otimes F(j))U(j). \tag{3.4}$$

By taking norm on both sides of the equation (3.4), we get $\parallel \widehat{T}(n) \parallel \leqslant KL \parallel \widehat{T}_0 \parallel + KLM_1M_2N$. Hence $\widehat{T}^{\alpha}$ is bounded. For any $n_1, n_2 \in [0, N]$, consider, $\parallel \widehat{T}(n_1) - \widehat{T}(n_2) \parallel \leqslant \parallel I_s \otimes \phi(n_1) - I_s \otimes \phi(n_2) \parallel \parallel \widehat{T}_0 \parallel + KLM_1M_2 \mid n_1 - n_2 \mid + M_1M_2 \sum_{j=0}^{n-1} \parallel I_s \otimes \phi(n_1 - j - 1) - I_s \otimes \phi(n_2 - j - 1) \parallel$. Thus, $\widehat{T^{\alpha}}$ is relatively compact. Let $\widehat{T_k} \in \widehat{T}^{\alpha}$ and $\widehat{T}_k \to \widehat{T}$. For each $\widehat{T}_k$, there is a $u_k \in \widehat{U}^{\alpha}(n)$ such that

$$\widehat{T}_k(n) = (I_s \otimes \phi(n))\widehat{T}_0 + \sum_{j=0}^{n-1} (I_s \otimes \phi(n-j-1))(I_s \otimes F(j))U_k(j). \tag{3.5}$$

Since $u_k \in \widehat{U}^{\alpha}(n)$ is closed, then there is a sub-sequence $< u_{k_j} >$ of $< u_k >$ converging weakly to $u \in \widehat{U}^{\alpha}(n)$. From Mazur's theorem [5] there exists a sequence of numbers $\lambda_i > 0$, $\Sigma\lambda_i = 1$ such that $\Sigma\lambda_i u_{k_i}$ converges strongly to u. Thus from (3.5) we have

$$\Sigma\lambda_j \widehat{T}_{K_j}(n) = \sum \lambda_j (I_s \otimes \phi(n))\widehat{T}_0 + \sum_{j=0}^{n-1} (I_s \otimes \phi(n-j-1))(I_s \otimes F(j))\Sigma\lambda_i u_{k_i}(j). \tag{3.6}$$

As $l \to \infty$ from equation (3.6) and Fatuou's lemma, it follows that $\widehat{T}(n) = I_s \otimes \phi(n)\widehat{T}_0 + \sum (I_s \otimes \phi(n-j-1))(I_s \otimes F(j))u(j)$. Thus $\widehat{T}(n) \in \widehat{T}^{\alpha}$ and hence $\widehat{T}^{\alpha}$ is closed. Let $\widehat{T}_1, \widehat{T}_2 \in \widehat{T}^{\alpha}$, then there exists $u_1, u_2 \in \widehat{U}^{\alpha}(n)$ such that $\widehat{T}_1(n+1) = G(n)\widehat{T}_1(n) + (I_s \otimes F(n))u_1(n)$, $\widehat{T}_2(n+1) = G(n)\widehat{T}_2(n) + (I_s \otimes F(n))u_2(n)$. Let $\widehat{T}(n) = \lambda\widehat{T}_1(n) + (1-\lambda)\widehat{T}_2(n)$, $0 \leqslant \lambda \leqslant 1$, then $\widehat{T}(n+1) = G(n)[\lambda\widehat{T}_1(n) + (1-\lambda)\widehat{T}_2(n)] + (I_s \otimes F(n))[\lambda u_1(n) + (1-\lambda)u_2(n)]$. Since $\widehat{U}^{\alpha}(n)$ is convex, $\lambda u_1(n) + (1-\lambda)u_2(n) \in \widehat{U}^{\alpha}(n)$, we have $\widehat{T}(n+1) \in G(n)\widehat{T}(n) + (I_s \otimes F(n))\widehat{U}^{\alpha}(n)$, i.e., $\widehat{T} \in \widehat{T}^{\alpha}$. Thus $\widehat{T}^{\alpha}$ is convex. Therefore $\widehat{T}^{\alpha}$ is non empty, compact, and convex in $C[[0, N], (N^+_{n_0})^{s^2}]$. Thus, from Arzela-Ascoli theorem, it follows that $[\widehat{T}(n)]^{\alpha}$ is convex in $(N^+_{n_0})^{s^2}$ for every $n \in [0, N]$. Therefore $[\widehat{T}(n)]^{\alpha} \in P_k((N^+_{n_0})^{s^2})$ for every $0 \leqslant \alpha \leqslant 1, n \in [0, N]$. □

Lemma 3.2. $[\widehat{T}(n)]^{\alpha_2} \subset [\widehat{T}(n)]^{\alpha_1}$ *for all* $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant 1$.

Proof. Let $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant 1$. Since $\widehat{U}^{\alpha_2}(n)$ is contained in $\widehat{U}^{\alpha_1}(n)$, it follows that

$$\widehat{U}^{\alpha_2}(n) = u_1^{\alpha_2}(n) \times u_2^{\alpha_2}(n) \times \cdots \times u_{s^2}^{\alpha_2}(n) \subset u_1^{\alpha_1}(n) \times u_2^{\alpha_1}(n) \times \cdots \times u_{s^2}^{\alpha_1}(n) = \widehat{U}_{\alpha_1}(n).$$

Thus, we have the selection inclusions $S^1_{\widehat{U}^{\alpha_2}}(n) \subset S^1_{\widehat{U}^{\alpha_1}}(n)$ and also the following inclusions:

$$\widehat{T}(n+1) \in G(n)\widehat{T}(n) + (I_s \otimes F(n))\widehat{U}^{\alpha_2}(n) \subset G(n)\widehat{T}(n) + (I_s \otimes F(n))\widehat{U}^{\alpha_1}(n).$$

Consider the following inclusions:

$$\widehat{T}(n+1) \in G(n)\widehat{T}(n) + (I_s \otimes F(n))\widehat{U}^{\alpha_2}(n), n \in [0,N], \tag{3.7}$$
$$\widehat{T}(n+1) \in G(n)\widehat{T}(n) + (I_s \otimes F(n))\widehat{U}^{\alpha_1}(n), n \in [0,N]. \tag{3.8}$$

Let $\widehat{T}^{\alpha_2}$ and $\widehat{T}^{\alpha_1}$ be the solution sets of (3.7) and (3.8), respectively. Clearly the solution of (3.7) satisfies the following inclusion:

$$\begin{aligned}\widehat{T}(n) &\in (I_s \otimes \phi(n))\widehat{T}_0 + \sum_{j=0}^{n-1}((I_s \otimes \phi(n-j-1)))(I_s \otimes F(j))S^1_{\widehat{U}^{\alpha_2}(j+1)}\\ &\subset (I_s \otimes \phi(n))\widehat{T}_0 + \sum_{j=0}^{n-1}((I_s \otimes \phi(n-j-1)))(I_s \otimes F(j))S^1_{\widehat{U}^{\alpha_1}(j+1)}.\end{aligned}$$

Thus $\widehat{T}^{\alpha_2} \subset \widehat{T}^{\alpha_1}$. And hence $\widehat{T}^{\alpha_2}(n) \subset \widehat{T}^{\alpha_1}(n)$. □

Lemma 3.3. *If* $< \alpha_k >$ *is non decreasing sequence converging to* $\alpha > 0$ *then* $\widehat{T}^{\alpha}(n) = \cap_{k\geqslant 1}\widehat{T}^{\alpha_k}(n)$.

Proof. Let $\widehat{U}^{\alpha_k}(n) = u_1^{\alpha_k} \times u_2^{\alpha_k} \times, \dots, \times u_{s^2}^{\alpha_k}, \widehat{U}^{\alpha}(n) = u_1^{\alpha} \times u_2^{\alpha} \times, \dots, u_{s^2}^{\alpha}$, and consider the inclusions

$$\widehat{T}(n+1) \in G(n)\widehat{T}(n) + (I_s \otimes F(n))\widehat{U}^{\alpha_k}(n), \tag{3.9}$$
$$\widehat{T}(n+1) \in G(n)\widehat{T}(n) + (I_s \otimes F(n))\widehat{U}^{\alpha}(n). \tag{3.10}$$

Let $\widehat{T}^{\alpha_k}$ and $\widehat{T}^{\alpha}$ be the solution sets of (3.9) and (3.10), respectively. Since $u_i(n)$ is a fuzzy set and from Theorem 2.1, we have

$$u_i^{\alpha} = \cap_{k\geqslant 1} u_i^{\alpha_k},$$

we consider

$$\widehat{U}^{\alpha}(n) = u_1^{\alpha} \times u_2^{\alpha} \times, \dots, \times u_{s^2}^{\alpha} = \cap_{k\geqslant 1} u_1^{\alpha_k} \times \cap_{k\geqslant 1} u_2^{\alpha_k} \times, \dots, \cap_{k\geqslant 1} u_{s^2}^{\alpha_k} = \cap_{k\geqslant 1}\widehat{U}^{\alpha_k}(n)$$

and then $S^1_{\widehat{U}^{\alpha}(n)} = S^1_{\cap_{k\geqslant 1}\widehat{U}^{\alpha_k}(n)}$. Therefore

$$\begin{aligned}\widehat{T}(n+1) \in G(n)\widehat{T}(n) + (I_s \otimes F(n))\widehat{U}^{\alpha}(n) &= G(n)\widehat{T}(n) + (I_s \otimes F(n)) \cap_{k\geqslant 1} \widehat{U}^{\alpha_k}(n)\\ &\subset G(n)\widehat{T}(n) + (I_s \otimes F(n))\widehat{U}^{\alpha_k}(N), k = 1,2,3,\dots.\end{aligned}$$

Thus we have $\widehat{T}^{\alpha} \subset \widehat{T}^{\alpha_k}, k = 1,2,3,\dots$, which implies that

$$\widehat{T}^{\alpha} \subset \cap_{k\geqslant 1}\widehat{T}^{\alpha_k}. \tag{3.11}$$

Let $\widehat{T}$ be the solution set of the inclusion (3.9) for $k \geqslant 1$. Then

$$\widehat{T}(n) \in (I_s \otimes \phi(n))\widehat{T}_0 + \sum_{j=0}^{n-1}(I_s \otimes \phi(n-j-1))(I_s \otimes F(j))S^1_{\widehat{U}^{\alpha k}(n)}.$$

It follows that $\widehat{T}(n) \in (I_s \otimes \phi(n))\widehat{T}_0 + \cap_{k\geqslant 1}\sum_{j=0}^{n-1}(I_s \otimes \phi(n-j-1))(I_s \otimes F(j))S^1_{\widehat{U}^{\alpha k}(n)} \subset (I_s \otimes \phi(n))\widehat{T}_0 + \sum_{j=0}^{n-1}(I_s \otimes \phi(n-j-1))(I_s \otimes F(j))S^I_{\cap_{k\geqslant 1}\widehat{U}^{\alpha_k}} = (I_s \otimes \phi(n))\widehat{T}_0 + \sum_{j=0}^{n-1}(I_s \otimes \phi(n-j-1))(I_s \otimes F(j))S^1_{\widehat{U}^{\alpha_k}}$. This implies that $\widehat{T} \in \widehat{T}^{\alpha}$. Therefore

$$\cap_{k\geqslant 1}\widehat{T}^{\alpha_k} \subset \widehat{T}^{\alpha}. \tag{3.12}$$

From (3.11) and (3.12) we have $\widehat{T}^{\alpha} = \cap_{k\geqslant 1}\widehat{T}^{\alpha_k}$ and hence, $\widehat{T}^{\alpha}(n) = \cap_{k\geqslant 1}\widehat{T}^{\alpha k}(n)$. □

Theorem 3.4. *The system (2.3) and (3.1) is a fuzzy dynamical discrete system, and it can be expressed as*

$$\widehat{T}(n+1) = G(n)\widehat{T}(n) + (I_s \otimes F(n))\widehat{U}(n), \widehat{T}(0) = \{\widehat{T}_0\}, \tag{3.13}$$

$$\widehat{Y}(n) = (I_s \otimes C(n))\widehat{T}(n) + (I_s \otimes D(n))\widehat{U}(n). \tag{3.14}$$

The solution set of fuzzy dynamical system (3.13)-(3.14) is given by

$$\widehat{T}(n) \in (I_s \otimes \phi(n))\widehat{T}_0 + \sum_{j=0}^{n-1}(I_s \otimes \phi(n-j-1))(I_s \otimes F(j))\widehat{U}(j). \tag{3.15}$$

Proof. Proof follows from the Lemmas 3.1, 3.2, 3.3 and Theorem 2.1 since there exists $\widehat{T}(n) \in E^{s^2}$ on $[0, N]$ such that $\widehat{T}^{\alpha}(n)$ is a solution set to the difference inclusions (3.2) and (3.3). □

Corollary 3.5. *If the input is in the form* $\widehat{U}(n) = \widetilde{u}_1(n) \times \widetilde{u}_2(n) \times \cdots \times \widetilde{u}_i(n) \times \cdots \times \widetilde{u}_{S^2}(n)$, *where* $\widetilde{u}_k(n) \in R^1, k \neq i$ *are crisp numbers, then the* i^{th} *component of the solution set of* (2.3) *is a fuzzy set in* E^1.

Definition 3.6. The fuzzy system given by equations (3.13)-(3.14) is said to be completely controllable if for any initial state $\widehat{T}(0) = \widehat{T}_0$ and any given final state $\widehat{T}_f$ there exists a finite time $n_1 > 0$ and a control $\widehat{U}(n), 0 \leqslant n \leqslant n_1$ such that $\widehat{T}(n_1) = \widehat{T}_f$.

Definition 3.7. The fuzzy system given by equations (3.13)-(3.14) is said to be completely observable over the interval $[0, N]$ if the knowledge of rule base of input $\widehat{U}$ and output $\widehat{Y}$ over $[0, N]$ suffices to determine a rule base of initial state $\widehat{T}_0$.

Let $u_i^l, y_i^l, i = 1, 2, \ldots, s^2, l = 1, 2, \ldots, m$, be fuzzy sets in E^1. We assume that the rule base for the input and output is given by

$$\begin{aligned} R^l: &\text{ If } \widetilde{u}_1(n) \text{ is in } u_1^l(n), \widetilde{u}_2(n) \text{ is in } u_2^l(n), \ldots, \widetilde{u}_{S^2}(n) \text{ is in } u_{S^2}^l(n), \\ &\text{then } \widetilde{y}_1(n) \text{ is in } y_1^l(n), \widetilde{y}_2(n) \text{ is in } y_2^l(n), \ldots, \widetilde{y}_{s^2}(n) \text{ is in } y_{s^2}^l(n), l = 1, 2, \ldots, m, \end{aligned} \tag{3.16}$$

and the output can be expressed as a function of input by the equation (3.1).

Definition 3.8. Let $x, y \in E^{s^2}$ and $x = x_1 \times x_2 \times \cdots \times x_{s^2}$, $y = y_1 \times y_2 \times \cdots \times y_{s^2}$, $x_i, y_i \in E^l, i = 1, 2, \ldots, s^2, l = 1, 2, \ldots, m$.

If $y = z + x$, then $z = y - x$, which is defined by $[z]^{\alpha} = [y - x]^{\alpha} = [y]^{\alpha} - [x]^{\alpha} = \begin{bmatrix} [y_1]^{\alpha} - [x_1]^{\alpha} \\ \cdots \\ [y_{s^2}]^{\alpha} - [x_{s^2}]^{\alpha}. \end{bmatrix}$.

If $y = w - x$, then $w = y + x$, which is defined by $[w]^{\alpha} = [y + x]^{\alpha} = [y]^{\alpha} + [x]^{\alpha} = \begin{bmatrix} [y_1]^{\alpha} + [x_1]^{\alpha} \\ \cdots \\ [y_{s^2}]^{\alpha} + [x_{s^2}]^{\alpha} \end{bmatrix}$.

Definition 3.9. Let $C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s^2} \\ c_{21} & c_{22} & \cdots & c_{2s^2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{s^21} & c_{s^22} & \cdots & c_{s^2s^2} \end{bmatrix}$ be an $s^2 \times s^2$ matrix, $p = p_1 \times p_2 \times \cdots \times p_{s^2}$, let $p_i \in E^1, i = 1, 2, \ldots, s^2, l = 1, 2, \ldots, m$, be a fuzzy set in E^{s^2}, and let $[p_i]^{\alpha}$ be α level sets of p_i, define the product Cp of C and p as

$$[Cp]^{\alpha} = C[p]^{\alpha} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s^2} \\ c_{21} & c_{22} & \cdots & c_{2s^2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{s^21} & c_{s^22} & \cdots & c_{s^2s^2} \end{bmatrix} \begin{bmatrix} [p_1]^{\alpha} \\ [p_2]^{\alpha} \\ \cdots \\ [p_{s^2}]^{\alpha} \end{bmatrix} = \begin{bmatrix} c_{11}[p_1]^{\alpha} + \cdots + c_{1s^2}[p_{s^2}]^{\alpha} \\ c_{21}[p_1]^{\alpha} + \cdots + c_{2s^2}[p_{s^2}]^{\alpha} \\ \cdots \\ c_{s^21}[p_1]^{\alpha} + \cdots + c_{s^2s^2}[p_{s^2}]^{\alpha} \end{bmatrix}.$$

4. Controllability of fuzzy discrete dynamical systems

Theorem 4.1. *The fuzzy system (3.13)-(3.14) is completely controllable if the $s^2 \times s^2$ symmetric controllable matrix*

$$W(0,N) = \sum_{j=0}^{N-1} (I_s \otimes (N-j-1))(I_s \otimes F(j))(I_s \otimes F(j))^*(I_s \otimes \phi(N-j-1))^* \tag{4.1}$$

(where $*$ indicates conjugate transpose (Hermitian transpose) of the matrix) is non singular. Furthermore, the fuzzy control $\widehat{U}(n)$ which transfers the state of the system from $\widehat{T}(0) = \widehat{T}_0$ to a fuzzy state

$$\widehat{T}(N) = \widehat{T}_f = (t_{f_1}, t_{f_2}, \dots, t_{f_{s^2}}) \tag{4.2}$$

can be modified by the following fuzzy rule base:

$$R: \text{If } \widetilde{t_1} \textit{ is in } t_{f_1}, t_{f_2}, \dots, \widetilde{t}_{f_{s^2}} \textit{ is in } t_{f_{s^2}}, \quad \textit{then } \widetilde{u_1} \textit{ is in } u_1 \cdots \widetilde{u_{s^2}} \textit{ is in } u_{s^2}, \tag{4.3}$$

where

$$\begin{aligned}(\widetilde{u}_1(n), \widetilde{u}_2(n), \dots, \widetilde{u}_{s^2}(n)) = \frac{1}{N}((I_s \otimes F(n))^{-1}(I_s \otimes (\phi(N-j-1)^{-1}))(\widetilde{t_1}(N), \widetilde{t_2}(N), \dots, t_{f_i}, \dots, \widetilde{t}_{s^2}(N)) \\ - (I_s \otimes F(n))^*(I_s \otimes \phi(N-j-1)^*)W^{-1}(0,N)(I_s \otimes \phi(N)))\widehat{T}(0), i = 1, 2, \dots, s^2.\end{aligned}$$

Proof. Suppose that the symmetric controllability matrix W(0,N) is nonsingular. Therefore $W^{-1}(0,N)$ exists. By multiplying equation (4.1) on both sides by $W^{-1}(0,N)(I_s \times \phi(N))\widehat{T}_0$, we get

$$\begin{aligned}(I_s \otimes \phi(N))\widehat{T}_0 = \sum_{j=0}^{N-1} (I_s \otimes \phi(N-j-1))(I_s \otimes F(j)) \\ \times (I_s \otimes F(n))^*(I_s \otimes \phi(N-j-1)^*)W^{-1}(0,N)(I_S \otimes \phi(N))\widehat{T}_0.\end{aligned} \tag{4.4}$$

Now our problem is to find the control $\widehat{U}(n)$ such that

$$\widehat{T}(N) = \widehat{T}_f = (I_s \otimes \phi(N))\widehat{T}_0 + \sum_{j=0}^{N-1} (I_s \otimes \phi(N-j-1))(I_s \otimes F(j))\widehat{U}(j). \tag{4.5}$$

Since $\widehat{T}$ is fuzzy $\widehat{U}(n)$ must be fuzzy, otherwise the left side of equation (4.5) cannot be equal to the crisp right side. Now $\widehat{T}_f$ can be written as

$$\widehat{T}_f = \frac{1}{N}\sum_{j=0}^{N-1} \widehat{T}_f = \frac{1}{N}\sum_{j=0}^{N-1} (I_s \otimes \phi(N-j-1))(I_s \otimes F(j)) \times (I_s \otimes F(n))^{-1}(I_s \otimes \phi(N-j-1))^{-1}\widehat{T}_f. \tag{4.6}$$

From (4.5) and (4.6) we have

$$\begin{aligned}&\frac{1}{N}\sum_{j=0}^{N-1} (I_s \otimes \phi(N-j-1))(I_s \otimes F(j)) \times (I_s \otimes F(n))^{-1}(I_s \otimes \phi(N-j-1))^{-1}\widehat{T}_f \\ &= (I_s \otimes \phi(N))\widehat{T}_0 + \sum_{j=0}^{N-1} (I_s \otimes \phi(N-j-1))(I_s \otimes F(j)))\widehat{U}(j).\end{aligned}$$

From (4.4) and (4.6) it follows that

$$\frac{1}{N}\sum_{j=0}^{N-1} (I_s \otimes \phi(N-j-1))(I_s \otimes F(j))(I_s \otimes F(j))^{-1}(I_s \otimes \phi(N-j-1))^{-1}\widehat{T}_f$$

$$
\begin{aligned}
&= \sum_{j=0}^{N-1} (I_s \otimes \phi(N-j-1))(I_s \otimes F(j))(I_s \otimes F(n))^*(I_s \otimes \phi(N-j-1))^* \\
&\quad \times W^{-1}(0,N)(I_s \otimes \phi(N))\widehat{T}_0 + \sum_{j=0}^{N-1} (I_s \otimes \phi(N-j-1))(I_s \otimes F(j))\widehat{U}(j),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\sum_{j=0}^{N-1} (I_s \otimes \phi(N-j-1))(I_s \otimes F(j))\widehat{U}(j) \\
&\quad = \sum_{j=0}^{N-1} (I_s \otimes \phi(N-j-1))(I_s \otimes F(j))\{\frac{1}{N}(I_s \otimes F(j))^{-1}(I_s \otimes \phi(N-j-1))^{-1}\widehat{T}_f \\
&\quad\quad - (I_s \otimes F(j))^*(I_s \otimes \phi(N-j-1))^* W^{-1}(0,N)(I_s \otimes \phi(N))\widehat{T}_0\}.
\end{aligned}
$$

Now $\widehat{U}(N)$ can be expressed as

$$
\begin{aligned}
\widehat{U}(N) = \frac{1}{N}\{&(I_s \otimes F(j))^{-1}((I_s \otimes \phi(N-j-1))^{-1}\widehat{T}_f) - (I_s \otimes F(j))^* \\
&\times ((I_s \otimes \phi(N-j-1))^*\}W^{-1}(0,N)(I_s \otimes \phi(N))\widehat{T}_0.
\end{aligned} \tag{4.7}
$$

Now are the following two possible cases for (4.7).

Case(i). When $\widehat{T}(N) = \widehat{T}_f = (\widetilde{t}_1(N), \widetilde{t}_2(N), \dots, t_{s^2}(N))$ is a crisp point, equation (4.7) gives corresponding control $\widehat{U}(n)$ and is given by $\widehat{U}(n) = (\widetilde{u}_1, \widetilde{u}_2, \dots, \widetilde{u}_{s^2})$.

Case(ii). When $\widehat{T}(N) = (\widetilde{t}_1(N), \widetilde{t}_2(N), \dots, t_{f_i}, \dots, \widetilde{t}_{s^2}(N))$, equation (4.7) gives the corresponding control $\widehat{U}(n)$ and is given by $\widehat{U}(n) = (\widetilde{u}_1, \widetilde{u}_2, \dots, u_i, \dots, \widetilde{u}_{s^2})$ in which the component of $\widehat{U}(n)$ is a fuzzy set in E^1.

Clearly $\widetilde{u}_i(n)$ is in $u_i(n)$, $\mu_{t_{f_i}}(\widetilde{t}_i(N))$ gives the grade of the membership of $\widetilde{t}_i(N)$ in t_{f_i}. Hence fuzzy rule base for the control $\widehat{U}$ given by equations (4.2) and (4.3) follows. □

Note 4.2. The converse of the above theorem need not be true. Since fuzzy rule base cannot imply the non singularity of the controllability matrix $W(0,N)$ given by (4.1), it follows that the condition in the above theorem is only sufficient condition but not necessary.

5. Observability of fuzzy dynamical discrete systems

Theorem 5.1. *Assume that the fuzzy rule base (3.16) holds, then the fuzzy system (3.13)-(3.14) is completely observable over the interval* $[0,N]$ *and is non singular. Furthermore if*

$$
\widehat{T}_0 = (\widetilde{t}_0^1, \widetilde{t}_0^2, \dots, \widetilde{t}_0^{s^2}),
$$

then one has the following rule base for the initial value $\widehat{T}_0$,

$$
\begin{aligned}
R^l:\ &\textit{If } \widetilde{u}_1(N) \in u_1^l(N), \dots, \widetilde{u_{s^2}}(N) \in u_{s^2}^l(N) \textit{ and } \widetilde{y_1}(N) \in y_1^l(N), \dots, \widetilde{y_{S^2}}(N) \in y_{s^2}^l(N) \\
&\textit{then} \quad \widetilde{t}_0^1 \textit{ is in } t_0^l(1), \dots, \widetilde{t}_0^{s^2}(n) \textit{ is in } t_0^l(S^2), l = 1,2,\dots,m,
\end{aligned} \tag{5.1}
$$

where

$$
\begin{aligned}
t_0^l(i) = (I_S \otimes C(N)(I_s \otimes \phi(N))^{-1}\{&V_i^l(N) - (I_S \otimes D(N))\widehat{U}(N) \\
&- (I_S \otimes C(N)) \sum_{j=0}^{N-1} (I_S \otimes \phi(N-j-1))(I_S \otimes F(j))H_i^l(j)\},
\end{aligned} \tag{5.2}
$$

$$\begin{aligned}\widehat{T}_0 = ((I_S \otimes C(N))((I_S \otimes \phi(N))^{-1}\{\widetilde{Y}(n) - (I_S \otimes D(N))\widetilde{U}(N)\\ - (I_s \otimes C(N)) \times \sum_{j=0}^{N-1} (I_S \otimes \phi(N-j-1))(I_s \otimes F(j))\widetilde{U}(j),\end{aligned} \tag{5.3}$$

$$H_i^l(n) = \widetilde{u}_1(n) \times \widetilde{u}_2(n)\times, \ldots, u_i^l(n) \cdots \times \widetilde{u}_{S^2}(n), \tag{5.4}$$
$$V_i^l(n) = \widetilde{y}_1(n) \times \widetilde{y}_2(n)\times, \ldots, y_i^l(n) \cdots \times \widetilde{y}_{S^2}(n), \tag{5.5}$$

where $i = 1, 2, .., s^2; l = 1, 2, \ldots, m$.

Proof. Consider the case when $l = 1$. Let

$$\widetilde{u}(n) = (\widetilde{u}_1(n), \widetilde{u}_2(n), \ldots, \widetilde{u}_{s^2}(n)), \qquad \widetilde{y}(n) = (\widetilde{y}_1(n), \widetilde{y}_2(n), \ldots, \widetilde{y}_{s^2}(n)).$$

Let $\mu_{u_i^1(n)}(\widetilde{u}_i(n))$ be the grade of the membership of $\widetilde{u}_i(n)$ in $u_i^1(n)$, and let $\mu_{y_i^1(n)}(\widetilde{y}_i(n))$ be the grade of membership of $\widetilde{Y}_i(n)$ in $y_i^1(n)$. Since $(I_s \otimes C(N))(I_s \otimes \phi(N))$ is non singular and from (3.15) we have

$$\begin{aligned}\widehat{T}_0 = [(I_s \otimes C(N))(I_s \otimes \phi(N))]^{-1}\{\widetilde{y}(N) - (I_s \otimes D(N))\widetilde{u}(N)\\ - (I_s \otimes C(N))\} \sum_{j=0}^{N-1} (I_s \otimes \phi(N-j-1))(I_s \otimes F(j)\widetilde{u}(j)).\end{aligned}$$

When the input and output are both fuzzy sets it follows from equation (3.1) that

$$(I_s \otimes C(N))\widehat{T}(n) = \widehat{Y}(n) - (I_s \otimes D(N))\widetilde{u}(N)$$

is a fuzzy set. From equation (3.15), we get

$$(I_s \otimes C(N))(I_s \otimes \phi(N))\widehat{T}_0 + \sum_{j=0}^{N-1} (I_s \otimes \phi(n-j-1))(I_s \otimes F(j))\widehat{U}(j) = \widehat{Y}(n) - ((I_s \otimes D(N)))\widehat{U}(n)$$

using Definition 3.8. It follows that

$$\begin{aligned}(I_s \otimes C(N))(I_s \otimes \phi(N))\widehat{T}_0 = \{\widehat{Y}(n) - (I_s \otimes D(N)))\widehat{U}(n)\\ - (I_s \otimes C(N))\} \sum_{j=0}^{N-1} (I_s \otimes \phi(n-j-1))(I_s \otimes F(j))\widehat{U}(j).\end{aligned}$$

Since $(I_s \otimes C(N))(I_s \otimes \phi(N))$ is nonsingular, we have

$$\begin{aligned}\widehat{T}_0 = [(I_s \otimes C(N))(I_s \otimes \phi(N))]^{-1}\{\widehat{Y}(N) - ((I_s \otimes D(N))\widehat{U}(n))\\ - (I_s \otimes C(N)) \times \sum_{j=0}^{N-1} (I_s \otimes \phi(N-j-1))(I_s \otimes F(j))\widehat{U}(j)\}.\end{aligned}$$

Now, the initial value $\widehat{T}_0$ should be a fuzzy set but not a crisp value. The following assumptions will enable us to determine each component of $\widehat{T}_0$

$$\begin{aligned}H_i^1(n) &= \widetilde{u}_1(n) \times u_i(n+1) \times \cdots \times \widetilde{u}_{s^2}(n),\\ V_i^1(n) &= \widetilde{y}_1(n) \times y_i(n+1) \times \cdots \times \widetilde{y}_{s^2}(n), \text{ where } i = 1, 2, \ldots, s^2.\end{aligned}$$

From the Corollary 3.5, we know that the i^{th} component of the set

$$(I_s \otimes \phi(N))\widehat{T}_0 + \sum_{j=0}^{n-1}(I_s \otimes \phi(n-j-1))(I_s \otimes F(j))H_i^1(j).$$

is a fuzzy set in E^1. From the fact that the product of a square matrix of size s^2 and column vector whose elements are α-level sets defined on fuzzy set in E^{s^2} is again a fuzzy set in E^{s^2} [6], it follows that the product

$$(I_s \otimes C(N)) \times \sum_{j=0}^{n-1}(I_s \otimes \phi(n-j-1))(I_s \otimes F(j))H_i^1(j)$$

is a fuzzy set in E^{s^2}. Hence $\widehat{T}_o$ is a fuzzy set in E^{s^2} and the i^{th} component of it denoted by $t_0^1(i)$ is a fuzzy set in E^1. The grade of membership of $\widetilde{t}_0^i$ in $t_0^1(i)$ is defined by $\mu_{t_0^1(i)}\widetilde{t}_0^i = \min\{\mu_{u_i^1(n)}\widetilde{u}_i(n), \mu_{y_i^1}(n)\widetilde{y}_i(n)\}$. Now the initial value is determined by using the equations (5.1)-(5.5). In general, computation of $t_0^1(i)$ is very difficult, but to solve the real value problem the following approximation is chosen. Now we take the point $(\widetilde{t}_0^i, \mu_{t_0^1}(i)(\widetilde{t_0}^i))$ and the zero level set $[t_0^1(i)]^0$ to determine a triangle as the new fuzzy set $t_0^1(i)$. We can use the center average defuzzifier

$$\widetilde{t}_0^i = \frac{\sum_{l=1}^{m}(\widetilde{t}_0^i)^l \mu_{t_0^1(i)}(\widetilde{t}_0^i)^l}{\sum_{l=1}^{m}\mu_{t_0^1(i)}(\widetilde{t}_0^i)^l} \tag{5.6}$$

to determine the initial value $\widehat{T}_0 = (\widetilde{t}_0^1, \widetilde{t}_0^2, \dots, \widetilde{t}_0^{s^2})$. To obtain more accurate value for the initial state, more rule bases may be provided. □

Example 5.2. Consider the fuzzy system (3.13) satisfying (3.14) with $A(n) = \begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}$, $F(n) = \begin{bmatrix}2^n & 0\\ 0 & 3^n\end{bmatrix}$, $C(n) = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, and $D(n) = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$, $N = 2$, $T(0) = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$. Let $\widehat{t_f} = (t_{f_1}, t_{f_2}, t_{f_3}, t_{f_4})$ in E^4, where $[\widehat{T}_f]^\alpha = ([t_{f_1}]^\alpha, [t_{f_2}]^\alpha, [t_{f_3}]^\alpha, [t_{f_4}]^\alpha)^T$ $=[[\alpha-1, 1-\alpha], [\alpha-1, 1-\alpha], [0.1(\alpha-1), 0.1(1-\alpha)], [0.1(\alpha-1), 0.1(1-\alpha)]^T$. We select the points $\widetilde{t_{f_1}} = 0.5, \widetilde{t_{f_2}} = 0.25, \widetilde{t_{f_3}} = 0.05,$ and $\widetilde{t_{f_4}} = 0.025$, which are in $t_{f_1}, t_{f_2}, t_{f_3}$, and t_{f_4} with $0.5, 0.75, 0.5$ and 0.75 as respective membership grades. The fundamental matrix of system (2.3) is $\phi(n) = \begin{bmatrix}1^n & 0\\ 0 & (-2)^n\end{bmatrix}$. By using the equation (4.1) of Theorem 4.1, we get $2^2 \times 2^2$ symmetric controllable matrix $W(0,2) = \begin{bmatrix}5 & 0 & 0 & 0\\ 0 & 37 & 0 & 0\\ 0 & 0 & 5 & 0\\ 0 & 0 & 0 & 37\end{bmatrix}$. We can easily observe that symmetric controllability matrix $W(0,2)$ is non-singular. Thus, from Theorem 4.1 the α level fuzzy control $\widehat{U}(n)$ are given by

$$\widehat{U}^\alpha(n) = \begin{bmatrix}(-1)^{-n-1}(2)^{-n-3}[\alpha-1, 1-\alpha]\\ \frac{1}{5}[\alpha-1, 1-\alpha]\\ (-1)^{-n-1}(2)^{-n-3}[0.1(\alpha-1), 0.1(1-\alpha)]\\ \frac{1}{5}[0.1(\alpha-1), 0.1(1-\alpha)].\end{bmatrix} - \begin{bmatrix}\frac{(-2)^{-n-1}}{37}\\ \frac{1}{5}\\ \frac{(-2)^{-n-1}}{37}\\ \frac{1}{5}\end{bmatrix}.$$

The α-level sets of fuzzy input $\widehat{U}(n)$ and fuzzy output $\widehat{Y}(n)$ by rule base 1 and rule base 2 given as follows.
Rule Base 1:

$$[\widehat{U}^{(1)}]^\alpha = \begin{bmatrix}[0, -0.75(\alpha-1)]\\ [0.75(\alpha-1)+1, 1]\\ [0, -0.5(\alpha-1)]\\ [0.5(\alpha-1)+1, 1]\end{bmatrix} [\widehat{Y}^{(1)}]^\alpha = \begin{bmatrix}[0, -2(\alpha+1)]\\ [0.5\alpha+2.5, 3]\\ [0, -1.5(\alpha-1)]\\ [0.5(\alpha-1)+3, 3]\end{bmatrix}.$$

Rule Base 2:

$$[\widehat{U}^{(2)}]^{\alpha} = \begin{bmatrix} [0, -0.8(\alpha-1)] \\ [0.8\alpha+0.2, 1] \\ [0, -0.5(\alpha-1)] \\ [0.5\alpha+0.5, 1] \end{bmatrix} [\widehat{Y}^{(2)}]^{\alpha} = \begin{bmatrix} [0, -1.5(\alpha-1)] \\ [\alpha+1, 2] \\ [0, -2.5(\alpha-1)] \\ [(2\alpha+1), 3] \end{bmatrix}.$$

From rule base 1, select

$$\widetilde{u^1} = (\widetilde{u_1}, \widetilde{u_2}, \widetilde{u_3}, \widetilde{u_4}) = (0.5, 0.85, 0.4, 0.75).$$

The grades of the membership of $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3,$ and $\widetilde{u}_4$ are $\frac{1}{3}, 0.8, 0.2,$ and $\frac{1}{2}$, respectively. Also

$$\widetilde{y^1} = (\widetilde{y^1}, \widetilde{y^2}, \widetilde{y^3}, \widetilde{y^4}) = (1, 2.8, 0.5, 2.9).$$

The grades of the membership of $\widetilde{y^1}, \widetilde{y^2}, \widetilde{y^3}$, and $\widetilde{y^4}$ are $\frac{1}{2}, 0.6, \frac{2}{3}$, and 0.8, respectively. From rule base 2, we select

$$\widetilde{u}^2 = (\widetilde{u_1}, \widetilde{u_2}, \widetilde{u_3}, \widetilde{u_4}) = (0.5, 0.8, 0.25, 0.75),$$

the grades of the membership of $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3,$ and $\widetilde{u}_4$, respectively are $\frac{1}{3}$, 0.8, 0.2, and $\frac{1}{2}$, respectively. Also

$$\widetilde{y^2} = (\widetilde{y^1}, \widetilde{y^2}, \widetilde{y^3}, \widetilde{y^4}) = (1, 1.75, 2, 1.5),$$

the grades of the membership of $\widetilde{y^1}, \widetilde{y^2}, \widetilde{y^3}$, and $\widetilde{y^4}$ are $\frac{1}{3}$, $\frac{3}{4}$, 0.2, and 0.25, respectively. From rule base 1 and equation (5.3) we have $\widehat{T}_0 = \begin{bmatrix} [-0.175] \\ [-15.15] \\ [0.025] \\ [-13.75] \end{bmatrix}$. From rule base 1 and equation (5.2) we have $t_0^1(1) = \begin{bmatrix} [-0.3625+1.3125\alpha, 0.7] \\ [-16.15, -2\alpha-14.15] \\ [0.025] \\ [-13.75] \end{bmatrix}$. When $\alpha = 0$, we observed that $\widetilde{t}_0^1 = -0.175$ belongs to the interval $[-0.3625, 0.7]$. We choose its membership grade in t_0^1 as

$$\mu_{t_0^1}(1) = \min\{\mu_{u_1^1}(\widetilde{u}_1(n)), \mu_{y_1^1}(n)(\widetilde{y}_1(n))\} = \min(\frac{1}{3}, \frac{1}{2}) = \frac{1}{3}.$$

$t_0^1(2) = \begin{bmatrix} [0.125\alpha-0.25, -0.125] \\ [-18, -3.75-14.15\alpha] \\ [-13.75] \\ [0.1] \end{bmatrix}$. When $\alpha = 0$ we observed that $\widetilde{t}_0^2 = -15.15$ belongs to the interval $[-18, -3.75]$. We choose its membership grade in $t_0^1(2)$ as $\mu_{t_0^1(2)}\widetilde{t}_0^1 = \min(0.8, 0.6) = 0.6$.

$t_0^1(3) = \begin{bmatrix} [-1.75] \\ [-15.15] \\ [0.8725\alpha-0.015, 0.725] \\ [-14.25, -1.5\alpha-12.75] \end{bmatrix}$. When $\alpha = 0$, we observed that $\widetilde{t}_0^3 = -0.025$ belongs to the interval $[-0.015, 0.725]$. We choose its membership grade in $t_0^1(3)$ as $\mu_{t_0^1}(3)\widetilde{t}_0^1 = \min(0.2, \frac{2}{3}) = 0.2$.

$t_0^1(4) = \begin{bmatrix} [-1.75] \\ [-15.15] \\ [0.125\alpha-0.75, 0.05] \\ [-14.25, -1.5\alpha-12.75] \end{bmatrix}$. When $\alpha = 0$, we observed that $\widetilde{t}_0^4 = -13.75$ belongs to the interval $[-14.25, -12.75]$. We choose its membership grade in $t_0^1(4)$ as $\mu_{t_0^1}(4)\widetilde{t}_0^1 = \min(\frac{1}{2}, 0.8) = \frac{1}{2}$.

Similarly for rule base 2 using equation (5.3) we get $\widehat{T}_0 = \begin{bmatrix} [-0.4375] \\ [-14.2] \\ [0.0625] \\ [-12.25] \end{bmatrix}$. By using rule base 2 and equation (5.2) we get

$$t_0^2(1) = \begin{bmatrix} [-0.9625 + 1.4\alpha, 0.4375] \\ [-15.2, -1.5\alpha - 13.7] \\ [-0.0625] \\ [-12.25] \end{bmatrix}, \quad \mu_{t_0^2}(1)\widetilde{t}_0^2 = \min(\frac{3}{8}, \frac{1}{3}) = \frac{1}{3},$$

$$t_0^2(2) = \begin{bmatrix} [0.25\alpha - 0.625, -0.375] \\ [-18, -2.8 - 15.2\alpha] \\ [-0.0625] \\ [-12.25] \end{bmatrix}, \quad \mu_{t_0^2}(2)\widetilde{t}_0^2 = \min(\frac{3}{4}, \frac{3}{4}) = \frac{3}{4},$$

$$t_0^2(3) = \begin{bmatrix} [-0.4375] \\ [-14.2] \\ [0.875\alpha - 0.5, 0.425] \\ [-14.25, -2.5\alpha - 11.75] \end{bmatrix}, \quad \mu_{t_0^2}(3)\widetilde{t}_0^2 = \min(\frac{1}{2}, 0.2) = 0.2,$$

$$t_0^2(4) = \begin{bmatrix} [-0.4375] \\ [-14.2] \\ [0.5\alpha - 0.1375, 0.3125] \\ [-17, -9.5\alpha - 7.5] \end{bmatrix}, \quad \mu_{t_0^2}(4)\widetilde{t}_0^2 = \min(\frac{1}{2}, 0.25) = 0.25.$$

By using the center average defuzzifier given by equation (5.6) the initial value $\widehat{T}_0 = (\widetilde{t}_0^1, \widetilde{t}_0^2, \widetilde{t}_0^3, \widetilde{t}_0^4)$ is given by

$$\widetilde{t}_0^1 = \frac{[-0.175 \times \frac{1}{3} + (-0.4375) \times \frac{1}{3}]}{\frac{1}{3} + \frac{1}{3}} = -0.30625, \qquad \widetilde{t}_0^2 = \frac{[-15.15 \times (0.6) + (-14.2) \times 0.75]}{0.6 + 0.75} = -14.62222,$$

$$\widetilde{t}_0^3 = \frac{[-0.025 \times (0.2) + (-0.0625) \times (0.2)]}{0.2 + 0.2} = -0.01875, \quad \widetilde{t}_0^4 = \frac{[-13.75 \times (0.5) + (-12.25) \times (0.25)]}{0.5 + 0.25} = -13.25.$$

By considering more rule bases the accuracy of the initial state can be improved.

6. Conclusions

In this article sufficient conditions for the controllability and observability of the fuzzy matrix discrete dynamical system are established. These are achieved through fuzzy rule base and difference inclusions approach. Novelty being the construction of rule base for the initial value without actually solving the system. Established theories are supported by numerical examples.

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