Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 12 (2019), 829–845
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

Some recurrence relations of poly-Cauchy numbers

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Abstract

Poly-Cauchy numbers $c_n^{(k)}$ $(n \geqslant 0, k \geqslant 1)$ have explicit expressions in terms of the Stirling numbers of the first kind. When the index is negative, there exists a different expression. However, the sequence $\{c_n^{(-k)}\}_{n\geqslant 0}$ seem quite irregular for a fixed integer $k \geqslant 2$. In this paper we establish a certain kind of recurrence relations among the sequence $\{c_n^{(-k)}\}_{n\geqslant 0}$, analyzing the structure of poly-Cauchy numbers. We also study those of poly-Cauchy numbers of the second kind, poly-Euler numbers, and poly-Euler numbers of the second kind. Some different proofs are given. As applications, some leaping relations are shown.

Keywords: Poly-Cauchy numbers, poly-Euler numbers, recurrence, leaping relations, Vandermonde's determinant.

2010 MSC: 11B75, 11B37, 11B68, 11B73, 05A19, 11C20, 15A15.

1. Introduction

The poly-Cauchy numbers (of the first kind) are defined by ([9])

$$c_n^{(k)} = n! \underbrace{\int_0^1 \cdots \int_0^1}_{k} \binom{x_1 \cdots x_k}{n} dx_1 \dots dx_k, \quad (n \geqslant 0, k \geqslant 1).$$

Their generating function is given by ([9, Theorem 2])

$$\mathrm{Lif}_k\big(\log(1+x)\big) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!},$$

where

$$\mathrm{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k}$$

is the polylogarithm factorial (or polyfactorial) function. When $k = 1$, $c_n = c_n^{(1)}$ are the original Cauchy

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doi: 10.22436/jnsa.012.12.05

Received: 2019-06-01 Revised: 2019-07-01 Accepted: 2019-07-23

numbers, whose generating function is given by ([3, 12])

$$\frac{x}{\log(1+x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}.$$

In 2018 Sasaki [14] showed several recurrence relations concerning the poly-Bernoulli numbers with negative indices. Poly-Bernoulli numbers $B_n^{(k)}$ ([7]) are defined by

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!},$$

where

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}$$

is the polylogarithm function. When $k=1$, $B_n = B_n^{(1)}$ are the classical Bernoulli numbers with $B_1^{(1)} = 1/2$, whose generating function is given by

$$\frac{x}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Poly-Bernoulli numbers $B_n^{(k)}$ have explicit forms in terms of the Stirling numbers of the second kind ([7]):

$$B_n^{(k)} = \sum_{m=0}^{n} \left\{ {n \atop m} \right\} \frac{(-1)^{n-m} m!}{(m+1)^k}.$$

If the index is negative, another form is given by ([2, Theorem 2])

$$B_n^{(-k)} = \sum_{j=0}^{\min(n,k)} (j!)^2 \left\{ {n+1 \atop j+1} \right\} \left\{ {k+1 \atop j+1} \right\}.$$

Many relations and identities for Bernoulli numbers have been known and several ones for poly-Bernoulli numbers have been discovered. Sasaki has found some new relations for poly-Bernoulli numbers together with Ohno. The main result is stated as

$$\sum_{0 \leqslant i \leqslant j \leqslant m} (-1)^i \left[{m+2 \atop i+1} \right] B_{n+j}^{(-k)} = 0. \tag{1.1}$$

By using the falling factorial, we determine the coefficients $s_i^{(n)}$ by

$$(x-2)_n = s_n^{(n)} x^n + s_{n-1}^{(n)} x^{n-1} + \cdots + s_1^{(n)} x + s_0^{(n)}.$$

Each coefficient $s_{n-l}^{(n)}$ $(j = 0, 1, \ldots, n)$ can be written in terms of the (unsigned) Stirling numbers of the first kind as

$$s_{n-l}^{(n)} = \sum_{j=0}^{l} (-1)^j \left[{n+2 \atop n+2-j} \right].$$

Then, (1.1) is equivalent to the following convenient form.

Lemma 1.1. *For* $n \geqslant k$ *and* $m \geqslant k$,

$$\sum_{j=0}^{m} s_j^{(m)} B_{n-m+j}^{(-k)} = 0.$$

For example, by $(x-2)_2 = x^2 - 5x + 6$, we have

$$B_n^{(-2)} = 5B_{n-1}^{(-2)} - 6B_{n-2}^{(-2)}.$$

By $(x-2)_3 = x^3 - 9x^2 + 26x - 24$, we have

$$B_n^{(-3)} = 9B_{n-1}^{(-3)} - 26B_{n-2}^{(-3)} + 24B_{n-3}^{(-3)}.$$

In fact, $B_n^{(-2)} = 2 \cdot 3^n - 2^n$ ([15, A027649]) and $B_n^{(-3)} = 2^n - 2 \cdot 3^{n+1} + 6 \cdot 4^n$ ([15, A027650]).

Poly-Cauchy numbers (of the first kind) have an explicit form in terms of the (unsigned) Stirling numbers of the first kind ([9, Theorem 1]):

$$c_n^{(k)} = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^{n-m}}{(m+1)^k}, \quad (n \geqslant 0,\ k \geqslant 1).$$

The definition may restrict the index k as nonnegative values, but the above expression is possible for negative k too. If the index is negative, another form is given by ([5, Theorem 8 Remark])

$$c_n^{(-k)} = \sum_{j=0}^{k} (-1)^{n+j} j! \left(\begin{bmatrix} n \\ j \end{bmatrix} - n \begin{bmatrix} n-1 \\ j \end{bmatrix} \right) \left\{ \begin{matrix} k+1 \\ j+1 \end{matrix} \right\}, \quad (n \geqslant 0,\ k \geqslant 0). \tag{1.2}$$

It is easy to see that

$$c_n^{(-1)} = (-1)^n (n-2)! \quad (n \geqslant 2).$$

In [6], some congruence relations of poly-Cauchy numbers $c_n^{(-k)}$ are investigated by k being fixed for all sufficiently large n. However, more precise regularity of the following sequences ([15, A222627, A222636, A222748, A223023]) has not been found:

$$\{c_n^{(-2)}\}_{n\geqslant 0} = 1, 4, 5, -3, 4, -8, 20, -52, 72, 936, -17568, 238752, -3113280, \dots,$$
$$\{c_n^{(-3)}\}_{n\geqslant 0} = 1, 8, 19, -1, -10, 48, -234, 1302, -8328, 60672, -497688, \dots,$$
$$\{c_n^{(-4)}\}_{n\geqslant 0} = 1, 16, 65, 45, -116, 340, -1240, 5480, -28464, 169248, \dots,$$
$$\{c_n^{(-5)}\}_{n\geqslant 0} = 1, 32, 211, 359, -538, 984, -1866, 1110, 32640, -449760, \dots.$$

In this paper we establish a certain kind of recurrence relations among the sequence $\{c_n^{(-k)}\}_{n\geqslant 0}$, analyzing the structure of poly-Cauchy numbers. We also study those of poly-Cauchy numbers of the second kind, poly-Euler numbers and poly-Euler numbers of the second kind. Some different proofs are given. As applications, some leaping relations are shown.

2. Poly-Cauchy numbers with negative indices

Now, we are ready to present our main theorem, which can explain how the sequences $\{c_n^{(-k)}\}_{n\geqslant 0}$ are generated, though they do not seem to have any clear regularity.

Theorem 2.1. *For* $n \geqslant k+2$,

$$c_n^{(-k)} + \sum_{l=1}^{k} \left(\sum_{l+1 \leqslant i_1 \leqslant \cdots \leqslant i_l \leqslant k+1} (n-i_1)\cdots(n-i_l) \right) c_{n-l}^{(-k)} = 0.$$

Remark 2.2. When $k = 1, 2, 3$, we have

$$c_n^{(-1)} + (n-2)c_{n-1}^{(-1)} = 0 \quad (n \geqslant 3),$$
$$c_n^{(-2)} + (n-2+n-3)c_{n-1}^{(-2)} + (n-3)^2 c_{n-2}^{(-2)} = 0 \quad (n \geqslant 4),$$

$$\begin{aligned}&c_n^{(-3)}+(n-2+n-3+n-4)c_{n-1}^{(-3)}\\&\quad+\left((n-3)^2+(n-3)(n-4)+(n-4)^2\right)c_{n-2}^{(-3)}+(n-4)^2c_{n-3}^{(-3)}=0\quad(n\geqslant 5),\end{aligned}$$

respectively.

Proof of Theorem 2.1. For a fixed integer $k\geqslant 1$, assume that the numbers $c_n^{(-k)}$ satisfy the recurrence relation

$$c_n^{(-k)}+a_{1,n}c_{n-1}^{(-k)}+a_{2,n}c_{n-2}^{(-k)}+\cdots+a_{k,n}c_{n-k}^{(-k)}=0. \tag{2.1}$$

By the recurrence relation

$$\begin{bmatrix}n+1\\k\end{bmatrix}=n\begin{bmatrix}n\\k\end{bmatrix}+\begin{bmatrix}n\\k-1\end{bmatrix}\quad\text{with}\quad\begin{bmatrix}n\\0\end{bmatrix}=0\quad(n\geqslant 1), \tag{2.2}$$

the form (1.2) can be also written as

$$c_n^{(-k)}=\sum_{j=0}^{k}(-1)^{n+j+1}j!\left(\begin{bmatrix}n-1\\j\end{bmatrix}-\begin{bmatrix}n-1\\j-1\end{bmatrix}\right)\begin{Bmatrix}k+1\\j+1\end{Bmatrix}\quad(n\geqslant 0,\ k\geqslant 0).$$

Here, one determines [1, Table 2.2] that

$$\begin{bmatrix}-n'\\k\end{bmatrix}=\begin{cases}1, & \text{if } n'\geqslant 0,\ k=-n',\\0, & \text{if } n'\geqslant 0,\ k\geqslant -n'+1.\end{cases}$$

Since the relation (2.1) does not depend on the values $\left\{{k+1\atop j+1}\right\}$, we have the system of equations

$$\begin{aligned}&-\begin{bmatrix}n-1\\1\end{bmatrix}+\begin{bmatrix}n-2\\1\end{bmatrix}a_{1,n}-\begin{bmatrix}n-3\\1\end{bmatrix}a_{2,n}+\cdots+(-1)^{k+1}\begin{bmatrix}n-k-1\\1\end{bmatrix}a_{k,n}=0,\\
&-\left(\begin{bmatrix}n-1\\2\end{bmatrix}-\begin{bmatrix}n-1\\1\end{bmatrix}\right)+\left(\begin{bmatrix}n-2\\2\end{bmatrix}-\begin{bmatrix}n-2\\1\end{bmatrix}\right)a_{1,n}\\
&\quad-\left(\begin{bmatrix}n-3\\2\end{bmatrix}-\begin{bmatrix}n-3\\1\end{bmatrix}\right)a_{2,n}+\cdots+(-1)^{k+1}\left(\begin{bmatrix}n-k-1\\2\end{bmatrix}-\begin{bmatrix}n-k-1\\1\end{bmatrix}\right)a_{k,n}=0,\\
&-\left(\begin{bmatrix}n-1\\3\end{bmatrix}-\begin{bmatrix}n-1\\2\end{bmatrix}\right)+\left(\begin{bmatrix}n-2\\3\end{bmatrix}-\begin{bmatrix}n-2\\2\end{bmatrix}\right)a_{1,n}\\
&\quad-\left(\begin{bmatrix}n-3\\3\end{bmatrix}-\begin{bmatrix}n-3\\2\end{bmatrix}\right)a_{2,n}+\cdots+(-1)^{k+1}\left(\begin{bmatrix}n-k-1\\3\end{bmatrix}-\begin{bmatrix}n-k-1\\2\end{bmatrix}\right)a_{k,n}=0,\\
&\quad\vdots\\
&-\left(\begin{bmatrix}n-1\\k\end{bmatrix}-\begin{bmatrix}n-1\\k-1\end{bmatrix}\right)+\left(\begin{bmatrix}n-2\\k\end{bmatrix}-\begin{bmatrix}n-2\\k-1\end{bmatrix}\right)a_{1,n}\\
&\quad-\left(\begin{bmatrix}n-3\\k\end{bmatrix}-\begin{bmatrix}n-3\\k-1\end{bmatrix}\right)a_{2,n}+\cdots+(-1)^{k+1}\left(\begin{bmatrix}n-k-1\\k\end{bmatrix}-\begin{bmatrix}n-k-1\\k-1\end{bmatrix}\right)a_{k,n}=0,\end{aligned}$$

which is equivalent to the system

$$\begin{aligned}&-\begin{bmatrix}n-1\\1\end{bmatrix}+\begin{bmatrix}n-2\\1\end{bmatrix}a_{1,n}-\begin{bmatrix}n-3\\1\end{bmatrix}a_{2,n}+\cdots+(-1)^{k+1}\begin{bmatrix}n-k-1\\1\end{bmatrix}a_{k,n}=0,\\
&-\begin{bmatrix}n-1\\2\end{bmatrix}+\begin{bmatrix}n-2\\2\end{bmatrix}a_{1,n}-\begin{bmatrix}n-3\\2\end{bmatrix}a_{2,n}+\cdots+(-1)^{k+1}\begin{bmatrix}n-k-1\\2\end{bmatrix}a_{k,n}=0,\\
&-\begin{bmatrix}n-1\\3\end{bmatrix}+\begin{bmatrix}n-2\\3\end{bmatrix}a_{1,n}-\begin{bmatrix}n-3\\3\end{bmatrix}a_{2,n}+\cdots+(-1)^{k+1}\begin{bmatrix}n-k-1\\3\end{bmatrix}a_{k,n}=0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\vdots\end{aligned}$$

$$-\begin{bmatrix}n-1\\k\end{bmatrix}+\begin{bmatrix}n-2\\k\end{bmatrix}a_{1,n}-\begin{bmatrix}n-3\\k\end{bmatrix}a_{2,n}+\cdots+(-1)^{k+1}\begin{bmatrix}n-k-1\\k\end{bmatrix}a_{k,n}=0.$$

By Cramer, we have

$$a_{1,n}=\frac{\begin{vmatrix}\begin{bmatrix}n-1\\1\end{bmatrix} & -\begin{bmatrix}n-3\\1\end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix}n-k-1\\1\end{bmatrix}\\ \begin{bmatrix}n-1\\2\end{bmatrix} & -\begin{bmatrix}n-3\\2\end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix}n-k-1\\2\end{bmatrix}\\ \vdots & \vdots & & \vdots\\ \begin{bmatrix}n-1\\k\end{bmatrix} & -\begin{bmatrix}n-3\\k\end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix}n-k-1\\k\end{bmatrix}\end{vmatrix}}{\begin{vmatrix}\begin{bmatrix}n-2\\1\end{bmatrix} & -\begin{bmatrix}n-3\\1\end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix}n-k-1\\1\end{bmatrix}\\ \begin{bmatrix}n-2\\2\end{bmatrix} & -\begin{bmatrix}n-3\\2\end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix}n-k-1\\2\end{bmatrix}\\ \vdots & \vdots & & \vdots\\ \begin{bmatrix}n-2\\k\end{bmatrix} & -\begin{bmatrix}n-3\\k\end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix}n-k-1\\k\end{bmatrix}\end{vmatrix}},$$

$$a_{2,n}=\frac{\begin{vmatrix}\begin{bmatrix}n-2\\1\end{bmatrix} & \begin{bmatrix}n-1\\1\end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix}n-k-1\\1\end{bmatrix}\\ \begin{bmatrix}n-2\\2\end{bmatrix} & \begin{bmatrix}n-1\\2\end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix}n-k-1\\2\end{bmatrix}\\ \vdots & \vdots & & \vdots\\ \begin{bmatrix}n-2\\k\end{bmatrix} & \begin{bmatrix}n-1\\k\end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix}n-k-1\\k\end{bmatrix}\end{vmatrix}}{\begin{vmatrix}\begin{bmatrix}n-2\\1\end{bmatrix} & -\begin{bmatrix}n-3\\1\end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix}n-k-1\\1\end{bmatrix}\\ \begin{bmatrix}n-2\\2\end{bmatrix} & -\begin{bmatrix}n-3\\2\end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix}n-k-1\\2\end{bmatrix}\\ \vdots & \vdots & & \vdots\\ \begin{bmatrix}n-2\\k\end{bmatrix} & -\begin{bmatrix}n-3\\k\end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix}n-k-1\\k\end{bmatrix}\end{vmatrix}},$$

$$\vdots$$

$$a_{k,n}=\frac{\begin{vmatrix}\begin{bmatrix}n-2\\1\end{bmatrix} & -\begin{bmatrix}n-3\\1\end{bmatrix} & \cdots & \begin{bmatrix}n-1\\1\end{bmatrix}\\ \begin{bmatrix}n-2\\2\end{bmatrix} & -\begin{bmatrix}n-3\\2\end{bmatrix} & \cdots & \begin{bmatrix}n-1\\2\end{bmatrix}\\ \vdots & \vdots & & \vdots\\ \begin{bmatrix}n-2\\k\end{bmatrix} & -\begin{bmatrix}n-3\\k\end{bmatrix} & \cdots & \begin{bmatrix}n-1\\k\end{bmatrix}\end{vmatrix}}{\begin{vmatrix}\begin{bmatrix}n-2\\1\end{bmatrix} & -\begin{bmatrix}n-3\\1\end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix}n-k-1\\1\end{bmatrix}\\ \begin{bmatrix}n-2\\2\end{bmatrix} & -\begin{bmatrix}n-3\\2\end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix}n-k-1\\2\end{bmatrix}\\ \vdots & \vdots & & \vdots\\ \begin{bmatrix}n-2\\k\end{bmatrix} & -\begin{bmatrix}n-3\\k\end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix}n-k-1\\k\end{bmatrix}\end{vmatrix}}.$$

First, we show that the denominator

$$\begin{vmatrix}\begin{bmatrix}n-2\\1\end{bmatrix} & -\begin{bmatrix}n-3\\1\end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix}n-k-1\\1\end{bmatrix}\\ \begin{bmatrix}n-2\\2\end{bmatrix} & -\begin{bmatrix}n-3\\2\end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix}n-k-1\\2\end{bmatrix}\\ \vdots & \vdots & & \vdots\\ \begin{bmatrix}n-2\\k\end{bmatrix} & -\begin{bmatrix}n-3\\k\end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix}n-k-1\\k\end{bmatrix}\end{vmatrix}=\bigl((n-k-2)!\bigr)^{k}. \tag{2.3}$$

By doing the operation: the 1st column+$(n-3)\times$the 2nd column, the 2nd column+$(n-4)\times$the 3rd column, …, the $(k-1)$th column$+(n-k-1)\times$the kth column, using the recurrence relation (2.2), the left-hand side of (2.3) is equal to

$$\begin{vmatrix} 0 & \cdots & 0 & (-1)^{k+1}\begin{bmatrix} n-k-1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} n-3 \\ 1 \end{bmatrix} & \cdots & (-1)^k\begin{bmatrix} n-k-1 \\ 1 \end{bmatrix} & (-1)^{k+1}\begin{bmatrix} n-k-1 \\ 2 \end{bmatrix} \\ \vdots & & \vdots & \vdots \\ \begin{bmatrix} n-3 \\ k-1 \end{bmatrix} & \cdots & (-1)^k\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} & (-1)^{k+1}\begin{bmatrix} n-k-1 \\ k \end{bmatrix} \end{vmatrix} = \begin{bmatrix} n-k-1 \\ 1 \end{bmatrix}\cdot\begin{vmatrix} \begin{bmatrix} n-3 \\ 1 \end{bmatrix} & -\begin{bmatrix} n-4 \\ 1 \end{bmatrix} & \cdots & (-1)^k\begin{bmatrix} n-k-1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} n-3 \\ 2 \end{bmatrix} & -\begin{bmatrix} n-4 \\ 2 \end{bmatrix} & \cdots & (-1)^k\begin{bmatrix} n-k-1 \\ 2 \end{bmatrix} \\ \vdots & \vdots & & \vdots \\ \begin{bmatrix} n-3 \\ k-1 \end{bmatrix} & -\begin{bmatrix} n-4 \\ k-1 \end{bmatrix} & \cdots & (-1)^k\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} \end{vmatrix}.$$

Repeating the similar steps, the left-hand side of (2.3) is equal to

$$\left(\begin{bmatrix} n-k-1 \\ 1 \end{bmatrix}\right)^2\cdot\begin{vmatrix} \begin{bmatrix} n-4 \\ 1 \end{bmatrix} & -\begin{bmatrix} n-5 \\ 1 \end{bmatrix} & \cdots & (-1)^{k-1}\begin{bmatrix} n-k-1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} n-4 \\ 2 \end{bmatrix} & -\begin{bmatrix} n-5 \\ 2 \end{bmatrix} & \cdots & (-1)^{k-1}\begin{bmatrix} n-k-1 \\ 2 \end{bmatrix} \\ \vdots & \vdots & & \vdots \\ \begin{bmatrix} n-4 \\ k-2 \end{bmatrix} & -\begin{bmatrix} n-5 \\ k-2 \end{bmatrix} & \cdots & (-1)^{k-1}\begin{bmatrix} n-k-1 \\ k-2 \end{bmatrix} \end{vmatrix}$$

$$= \cdots = \left(\begin{bmatrix} n-k-1 \\ 1 \end{bmatrix}\right)^{k-2}\cdot\begin{vmatrix} 0 & -\begin{bmatrix} n-k-1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} n-k-1 \\ 1 \end{bmatrix} & -\begin{bmatrix} n-k-1 \\ 2 \end{bmatrix} \end{vmatrix} = \left(\begin{bmatrix} n-k-1 \\ 1 \end{bmatrix}\right)^{k} = \big((n-k-2)!\big)^k.$$

Now, we can write the numerators of $a_{j,n}$ $(j=1,2,\dots,k)$ as

$$f_j(n,k) := \begin{vmatrix} \begin{bmatrix} n-1 \\ 1 \end{bmatrix} & -\begin{bmatrix} n-2 \\ 1 \end{bmatrix} & \cdots & (-1)^{j-1}\begin{bmatrix} n-j \\ 1 \end{bmatrix} & (-1)^j\begin{bmatrix} n-j-2 \\ 1 \end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix} n-k-1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} n-1 \\ 2 \end{bmatrix} & -\begin{bmatrix} n-2 \\ 2 \end{bmatrix} & \cdots & (-1)^{j-1}\begin{bmatrix} n-j \\ 2 \end{bmatrix} & (-1)^j\begin{bmatrix} n-j-2 \\ 2 \end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix} n-k-1 \\ 2 \end{bmatrix} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \begin{bmatrix} n-1 \\ k \end{bmatrix} & -\begin{bmatrix} n-2 \\ k \end{bmatrix} & \cdots & (-1)^{j-1}\begin{bmatrix} n-j \\ k \end{bmatrix} & (-1)^j\begin{bmatrix} n-j-2 \\ k \end{bmatrix} & \cdots & (-1)^{k+1}\begin{bmatrix} n-k-1 \\ k \end{bmatrix} \end{vmatrix},$$

where there exists a gap without $\begin{bmatrix} n-j-1 \\ \ell \end{bmatrix}$ $(1\leqslant \ell\leqslant k)$. In this sense, we can put the denominator (2.3) as $f_0(n,k)$. We shall show that $f_j(n,k)$ has the recursion formula

$$f_j(n,k) = (n-k-2)!\big((n-j-1)f_{j-1}(n-1,k-1)+f_j(n-1,k-1)\big) \quad (n\geqslant k+2), \tag{2.4}$$

with $f_0(n,k)=\big((n-k-2)!\big)^k$ and $f_k(n,k)=\big((n-k-1)!\big)^k$ $(k\geqslant 0)$. We do the following operations:

1. the 1st column$+(n-2)\times$the 2nd column, the 2nd column$+(n-3)\times$the 3rd column, …, the $(j-1)$th column$+(n-j)\times$the jth column;
2. the determinant is divided into two parts by applying the recurrence relation (2.2) on the $(j-1)$th column;
3. for the first determinant the jth column$+(n-j-2)\times$the $(j+1)$th column, the $(j+1)$th column$+(n-j-3)\times$ $(j+2)$th column, …, the $(k-1)$th column$+(n-k-1)\times$the kth column, for the second determinant the $(j+1)$the column$+(n-j-3)\times$ $(j+2)$th column, …, the $(k-1)$th column$+(n-k-1)\times$the kth column.

Therefore, we have

$$f_j(n,k) = \begin{vmatrix} \begin{bmatrix} n-2 \\ 0 \end{bmatrix} & \cdots & (-1)^j\begin{bmatrix} n-j \\ 0 \end{bmatrix} & (-1)^{j-1}\begin{bmatrix} n-j \\ 1 \end{bmatrix} & (-1)^j\begin{bmatrix} n-j-2 \\ 1 \end{bmatrix} & \cdots & (-1)^k\begin{bmatrix} n-k \\ 1 \end{bmatrix} & (-1)^{k+1}\begin{bmatrix} n-k-1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} n-2 \\ 1 \end{bmatrix} & \cdots & (-1)^j\begin{bmatrix} n-j \\ 1 \end{bmatrix} & (-1)^{j-1}\begin{bmatrix} n-j \\ 2 \end{bmatrix} & (-1)^j\begin{bmatrix} n-j-2 \\ 2 \end{bmatrix} & \cdots & (-1)^k\begin{bmatrix} n-k \\ 2 \end{bmatrix} & (-1)^{k+1}\begin{bmatrix} n-k-1 \\ 2 \end{bmatrix} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \begin{bmatrix} n-2 \\ k-1 \end{bmatrix} & \cdots & (-1)^j\begin{bmatrix} n-j \\ k-1 \end{bmatrix} & (-1)^{j-1}\begin{bmatrix} n-j \\ k \end{bmatrix} & (-1)^j\begin{bmatrix} n-j-2 \\ k \end{bmatrix} & \cdots & (-1)^k\begin{bmatrix} n-k \\ k \end{bmatrix} & (-1)^{k+1}\begin{bmatrix} n-k-1 \\ k \end{bmatrix} \end{vmatrix}$$

$$
\begin{aligned}
&= (n-j-1)\\
&\quad\times\begin{vmatrix}
\genfrac{[}{]}{0pt}{}{n-2}{0} & \dots & (-1)^j\genfrac{[}{]}{0pt}{}{n-j}{0} & (-1)^{j-1}\genfrac{[}{]}{0pt}{}{n-j-1}{1} & (-1)^j\genfrac{[}{]}{0pt}{}{n-j-2}{1} & \dots & (-1)^k\genfrac{[}{]}{0pt}{}{n-k}{1} & (-1)^{k+1}\genfrac{[}{]}{0pt}{}{n-k-1}{1}\\
\genfrac{[}{]}{0pt}{}{n-2}{1} & \dots & (-1)^j\genfrac{[}{]}{0pt}{}{n-j}{1} & (-1)^{j-1}\genfrac{[}{]}{0pt}{}{n-j-1}{2} & (-1)^j\genfrac{[}{]}{0pt}{}{n-j-2}{2} & \dots & (-1)^k\genfrac{[}{]}{0pt}{}{n-k}{2} & (-1)^{k+1}\genfrac{[}{]}{0pt}{}{n-k-1}{2}\\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots\\
\genfrac{[}{]}{0pt}{}{n-2}{k-1} & \dots & (-1)^j\genfrac{[}{]}{0pt}{}{n-j}{k-1} & (-1)^{j-1}\genfrac{[}{]}{0pt}{}{n-j-1}{k} & (-1)^j\genfrac{[}{]}{0pt}{}{n-j-2}{k} & \dots & (-1)^k\genfrac{[}{]}{0pt}{}{n-k}{k} & (-1)^{k+1}\genfrac{[}{]}{0pt}{}{n-k-1}{k}
\end{vmatrix}\\
&\quad+\begin{vmatrix}
\genfrac{[}{]}{0pt}{}{n-2}{0} & \dots & (-1)^j\genfrac{[}{]}{0pt}{}{n-j}{0} & (-1)^{j-1}\genfrac{[}{]}{0pt}{}{n-j-1}{0} & (-1)^j\genfrac{[}{]}{0pt}{}{n-j-2}{1} & \dots & (-1)^k\genfrac{[}{]}{0pt}{}{n-k}{1} & (-1)^{k+1}\genfrac{[}{]}{0pt}{}{n-k-1}{1}\\
\genfrac{[}{]}{0pt}{}{n-2}{1} & \dots & (-1)^j\genfrac{[}{]}{0pt}{}{n-j}{1} & (-1)^{j-1}\genfrac{[}{]}{0pt}{}{n-j-1}{1} & (-1)^j\genfrac{[}{]}{0pt}{}{n-j-2}{2} & \dots & (-1)^k\genfrac{[}{]}{0pt}{}{n-k}{2} & (-1)^{k+1}\genfrac{[}{]}{0pt}{}{n-k-1}{2}\\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots\\
\genfrac{[}{]}{0pt}{}{n-2}{k-1} & \dots & (-1)^j\genfrac{[}{]}{0pt}{}{n-j}{k-1} & (-1)^{j-1}\genfrac{[}{]}{0pt}{}{n-j-1}{k-1} & (-1)^j\genfrac{[}{]}{0pt}{}{n-j-2}{k} & \dots & (-1)^k\genfrac{[}{]}{0pt}{}{n-k}{k} & (-1)^{k+1}\genfrac{[}{]}{0pt}{}{n-k-1}{k}
\end{vmatrix}\\
&= (n-j-1)\\
&\quad\times\begin{vmatrix}
\genfrac{[}{]}{0pt}{}{n-2}{0} & \dots & (-1)^j\genfrac{[}{]}{0pt}{}{n-j}{0} & (-1)^{j-1}\genfrac{[}{]}{0pt}{}{n-j-2}{0} & (-1)^j\genfrac{[}{]}{0pt}{}{n-j-3}{0} & \dots & (-1)^k\genfrac{[}{]}{0pt}{}{n-k-1}{0} & (-1)^{k+1}\genfrac{[}{]}{0pt}{}{n-k-1}{1}\\
\genfrac{[}{]}{0pt}{}{n-2}{1} & \dots & (-1)^j\genfrac{[}{]}{0pt}{}{n-j}{1} & (-1)^{j-1}\genfrac{[}{]}{0pt}{}{n-j-2}{1} & (-1)^j\genfrac{[}{]}{0pt}{}{n-j-3}{1} & \dots & (-1)^k\genfrac{[}{]}{0pt}{}{n-k-1}{1} & (-1)^{k+1}\genfrac{[}{]}{0pt}{}{n-k-1}{2}\\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots\\
\genfrac{[}{]}{0pt}{}{n-2}{k-1} & \dots & (-1)^j\genfrac{[}{]}{0pt}{}{n-j}{k-1} & (-1)^{j-1}\genfrac{[}{]}{0pt}{}{n-j-2}{k-1} & (-1)^j\genfrac{[}{]}{0pt}{}{n-j-3}{k-1} & \dots & (-1)^k\genfrac{[}{]}{0pt}{}{n-k-1}{k-1} & (-1)^{k+1}\genfrac{[}{]}{0pt}{}{n-k-1}{k}
\end{vmatrix}\\
&\quad+\begin{vmatrix}
\genfrac{[}{]}{0pt}{}{n-2}{0} & \dots & (-1)^j\genfrac{[}{]}{0pt}{}{n-j}{0} & (-1)^{j-1}\genfrac{[}{]}{0pt}{}{n-j-1}{0} & (-1)^j\genfrac{[}{]}{0pt}{}{n-j-3}{0} & \dots & (-1)^k\genfrac{[}{]}{0pt}{}{n-k-1}{0} & (-1)^{k+1}\genfrac{[}{]}{0pt}{}{n-k-1}{1}\\
\genfrac{[}{]}{0pt}{}{n-2}{1} & \dots & (-1)^j\genfrac{[}{]}{0pt}{}{n-j}{1} & (-1)^{j-1}\genfrac{[}{]}{0pt}{}{n-j-1}{1} & (-1)^j\genfrac{[}{]}{0pt}{}{n-j-3}{1} & \dots & (-1)^k\genfrac{[}{]}{0pt}{}{n-k-1}{1} & (-1)^{k+1}\genfrac{[}{]}{0pt}{}{n-k-1}{2}\\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots\\
\genfrac{[}{]}{0pt}{}{n-2}{k-1} & \dots & (-1)^j\genfrac{[}{]}{0pt}{}{n-j}{k-1} & (-1)^{j-1}\genfrac{[}{]}{0pt}{}{n-j-1}{k-1} & (-1)^j\genfrac{[}{]}{0pt}{}{n-j-3}{k-1} & \dots & (-1)^k\genfrac{[}{]}{0pt}{}{n-k-1}{k-1} & (-1)^{k+1}\genfrac{[}{]}{0pt}{}{n-k-1}{k}
\end{vmatrix}\\
&= (n-k-2)!(n-j-1)f_{j-1}(n-1,k-1)+(n-k-2)!f_j(n-1,k-1),
\end{aligned}
$$

which is (2.4).

Now, by induction, we can show that

$$
f_j(n,k)=\bigl((n-k-2)!\bigr)^k\sum_{j+1\leqslant i_1\leqslant\cdots\leqslant i_j\leqslant k+1}(n-i_1)\cdots(n-i_j). \tag{2.5}
$$

The result is valid for $j=0$, and is clear for $k=1,2$. Assume that the result is valid up to $j-1$. Then by (2.4)

$$
\begin{aligned}
f_j(n,k)&=(n-k-2)!(n-j-1)\bigl((n-k-2)!\bigr)^{k-1}\sum_{j\leqslant i_1\leqslant\cdots\leqslant i_{j-1}\leqslant k}(n-i_1-1)\cdots(n-i_{j-1}-1)\\
&\qquad+(n-k-2)!f_j(n-1,k-1).
\end{aligned}
$$

Repeating the similar steps, we have

$$
\begin{aligned}
f_j(n,k)&=\bigl((n-k-2)!\bigr)^k(n-j-1)\sum_{j\leqslant i_1\leqslant\cdots\leqslant i_{j-1}\leqslant k}(n-i_1-1)\cdots(n-i_{j-1}-1)\\
&\quad+\bigl((n-k-2)!\bigr)^k(n-j-2)\sum_{j\leqslant i_1\leqslant\cdots\leqslant i_{j-1}\leqslant k-1}(n-i_1-2)\cdots(n-i_{j-1}-2)\\
&\quad+\bigl((n-k-2)!\bigr)^2f_j(n-2,k-2)\\
&=\bigl((n-k-2)!\bigr)^k(n-j-1)\sum_{j\leqslant i_1\leqslant\cdots\leqslant i_{j-1}\leqslant k}(n-i_1-1)\cdots(n-i_{j-1}-1)\\
&\quad+\bigl((n-k-2)!\bigr)^k(n-j-2)\sum_{j\leqslant i_1\leqslant\cdots\leqslant i_{j-1}\leqslant k-1}(n-i_1-2)\cdots(n-i_{j-1}-2)\\
&\quad\ \vdots\\
&\quad+\bigl((n-k-2)!\bigr)^k(n-k)\sum_{j\leqslant i_1\leqslant\cdots\leqslant i_{j-1}\leqslant j+1}(n-i_1+j-k)\cdots(n-i_{j-1}+j-k)
\end{aligned}
$$

$$+\bigl((n-k-2)!\bigr)^{k-j} f_j(n-k+j,j),$$

where

$$\begin{aligned}
f_j(n-k+j,j) &= \begin{vmatrix} \left[{n-k+j-1 \atop 1}\right] & -\left[{n-k+j-2 \atop 1}\right] & \cdots & (-1)^j\left[{n-k+1 \atop 1}\right] & (-1)^{j+1}\left[{n-k \atop 1}\right] \\ \left[{n-k+j-1 \atop 2}\right] & -\left[{n-k+j-2 \atop 2}\right] & \cdots & (-1)^j\left[{n-k+1 \atop 2}\right] & (-1)^{j+1}\left[{n-k \atop 2}\right] \\ \vdots & \vdots & & \vdots & \vdots \\ \left[{n-k+j-1 \atop j}\right] & -\left[{n-k+j-2 \atop j}\right] & \cdots & (-1)^j\left[{n-k+1 \atop j}\right] & (-1)^{j+1}\left[{n-k \atop j}\right] \end{vmatrix} \\
&= \begin{vmatrix} \left[{n-k+j-2 \atop 0}\right] & -\left[{n-k+j-3 \atop 0}\right] & \cdots & (-1)^j\left[{n-k \atop 0}\right] & (-1)^{j+1}\left[{n-k \atop 1}\right] \\ \left[{n-k+j-2 \atop 1}\right] & -\left[{n-k+j-3 \atop 1}\right] & \cdots & (-1)^j\left[{n-k \atop 1}\right] & (-1)^{j+1}\left[{n-k \atop 2}\right] \\ \vdots & \vdots & & \vdots & \vdots \\ \left[{n-k+j-2 \atop j-1}\right] & -\left[{n-k+j-3 \atop j-1}\right] & \cdots & (-1)^j\left[{n-k \atop j-1}\right] & (-1)^{j+1}\left[{n-k \atop j}\right] \end{vmatrix} \\
&= \left[{n-k \atop 1}\right] \cdot \begin{vmatrix} \left[{n-k+j-2 \atop 1}\right] & \cdots & (-1)^j\left[{n-k \atop 1}\right] \\ \vdots & & \vdots \\ \left[{n-k+j-2 \atop j-1}\right] & \cdots & (-1)^j\left[{n-k \atop j-1}\right] \end{vmatrix} \\
&\ \ \vdots \\
&= \left(\left[{n-k \atop 1}\right]\right)^{j-2} \begin{vmatrix} \left[{n-k+1 \atop 1}\right] & -\left[{n-k \atop 1}\right] \\ \left[{n-k+1 \atop 2}\right] & -\left[{n-k \atop 2}\right] \end{vmatrix} \\
&= \left(\left[{n-k \atop 1}\right]\right)^{j-2} \begin{vmatrix} \left[{n-k \atop 0}\right] & -\left[{n-k \atop 1}\right] \\ \left[{n-k \atop 1}\right] & -\left[{n-k \atop 2}\right] \end{vmatrix} \\
&= \left(\left[{n-k \atop 1}\right]\right)^{j} = \bigl((n-k-1)!\bigr)^j.
\end{aligned}$$

Thus, we obtain the formula (2.5). □

3. Poly-Cauchy numbers of the second kind

The poly-Cauchy numbers of the second kind are defined by ([9])

$$\widehat{c}_n^{(k)} = n!\underbrace{\int_0^1\cdots\int_0^1}_{k}\binom{-x_1\cdots x_k}{n}dx_1\dots dx_k.$$

Their generating function is given by ([9, Theorem 5])

$$\mathrm{Lif}_k\bigl(-\log(1+x)\bigr) = \sum_{n=0}^\infty \widehat{c}_n^{(k)}\frac{x^n}{n!}.$$

When $k=1$, $\widehat{c}_n = \widehat{c}_n^{(1)}$ are the original Cauchy numbers of the second kind, whose generating function is given by

$$\frac{x}{(1+x)\log(1+x)} = \sum_{n=0}^\infty \widehat{c}_n\frac{x^n}{n!}.$$

Poly-Cauchy numbers of the second kind have an explicit form in terms of the (unsigned) Stirling numbers of the first kind ([9, Theorem 4]):

$$\widehat{c}_n^{(k)} = (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{1}{(m+1)^k}.$$

If the index is negative, poly-Cauchy numbers of the second kind have a different expression ([5, Theorem 8 Remark]):

$$\widehat{c}_n^{(-k)} = \sum_{j=0}^{k} (-1)^n j! \begin{bmatrix} n+1 \\ j+1 \end{bmatrix} \begin{Bmatrix} k+1 \\ j+1 \end{Bmatrix}.$$

Similarly to Theorem 2.1, poly-Cauchy numbers of the second kind with negative indices have the following recurrence relations.

Theorem 3.1. *For* $n \geqslant k+1$,

$$\widehat{c}_n^{(-k)} + \sum_{l=1}^{k+1} \left(\sum_{l-1 \leqslant i_1 \leqslant \cdots \leqslant i_l \leqslant k} (n-i_1)\cdots(n-i_l) \right) \widehat{c}_{n-l}^{(-k)} = 0.$$

Remark 3.2. When $k = 0, 1, 2$, we have

$$\begin{aligned}
&\widehat{c}_n^{(0)} + n\widehat{c}_{n-1}^{(0)} = 0 \quad (n \geqslant 1),\\
&\widehat{c}_n^{(-1)} + (n+n-1)\widehat{c}_{n-1}^{(-1)} + (n-1)^2\widehat{c}_{n-2}^{(-1)} = 0 \quad (n \geqslant 2),\\
&\widehat{c}_n^{(-2)} + (n+n-1+n-2)\widehat{c}_{n-1}^{(-2)}\\
&\quad + \left((n-1)^2 + (n-1)(n-2) + (n-2)^2\right)\widehat{c}_{n-2}^{(-2)} + (n-2)^3\widehat{c}_{n-2}^{(-2)} = 0 \quad (n \geqslant 3),
\end{aligned}$$

respectively. Since

$$\widehat{c}_n^{(0)} = (-1)^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = (-1)^n n!,$$

the first relation is clear.

4. Poly-Euler numbers

For an integer k, poly-Euler numbers $E_n^{(k)}$ $(n \geqslant 0)$ are defined by ([13])

$$\frac{\mathrm{Li}_k(1-e^{-4x})}{4x\cosh x} = \sum_{n=0}^{\infty} E_n^{(k)} \frac{x^n}{n!}.$$

When $k = 1$, $E_n = E_n^{(1)}$ are the classical Euler numbers, defined by

$$\frac{1}{\cosh x} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.$$

Poly-Euler numbers can be expressed in terms of poly-Bernoulli numbers ([13, Theorem 2.1]):

$$E_n^{(k)} = \frac{1}{2(n+1)} \sum_{m=0}^{n+1} \binom{n+1}{m} 4^m \left((-1)^{n-m+1} - (-3)^{n-m+1}\right) B_m^{(k)}.$$

In the case of negative index, poly-Euler numbers have an explicit expression ([13, Theorem 6.1]):

$$(n+1)E_n^{(-k)}=\frac{(-1)^k}{2}\sum_{l=0}^{k}(-1)^l l!\left\{ {k \atop l} \right\}\left((4l+3)^{n+1}-(4l+1)^{n+1}\right)\quad (k\geqslant 0).$$

Notice that $E_n^{(-k)}$ are not necessarily integers but $(n+1)E_n^{(-k)}$ are all integers. For convenience, put $\widetilde{E}_n^{(-k)}=(n+1)E_n^{(-k)}$. By using the generalized falling factorial, determine the coefficients $\sigma_i^{(n)}$ by

$$(x-5|2)_n=(x-5)(x-7)\cdots(x-2n-3)=\sigma_n^{(n)}x^n+\sigma_{n-1}^{(n)}x^{n-1}+\cdots+\sigma_1^{(n)}x+\sigma_0^{(n)}.$$

Then Euler numbers with negative indices satisfy the following recurrence formula.

Theorem 4.1. *For* $k=0$,

$$\widetilde{E}_n^{(0)}-4\widetilde{E}_{n-1}^{(0)}+3\widetilde{E}_{n-2}^{(0)}=0\quad (n\geqslant 2).$$

For $k\geqslant 1$,

$$\sum_{j=0}^{2k}\sigma_j^{(2k)}\widetilde{E}_{n-2k+j}^{(-k)}=0\quad (n\geqslant 2k).$$

Remark 4.2. For $k=1,2,3$, we have

$$\{\widetilde{E}_n^{(-1)}\}_{n\geqslant 0}=1,12,109,888,6841,51012,372709,2687088,19200241,136354812,\dots,$$
$$\{\widetilde{E}_n^{(-2)}\}_{n\geqslant 0}=1,28,493,7192,95161,1189108,14331493,168625072,1951326961,\dots,$$
$$\{\widetilde{E}_n^{(-3)}\}_{n\geqslant 0}=1,60,1837,42840,865081,16022100,280592677,4730230320,77624198641,\dots,$$

and

$$\widetilde{E}_n^{(-1)}-12\widetilde{E}_{n-1}^{(-1)}+35\widetilde{E}_{n-2}^{(-1)}=0,$$
$$\widetilde{E}_n^{(-2)}-32\widetilde{E}_{n-1}^{(-2)}+374\widetilde{E}_{n-2}^{(-2)}-1888\widetilde{E}_{n-3}^{(-2)}+3465\widetilde{E}_{n-4}^{(-2)}=0,$$
$$\widetilde{E}_n^{(-3)}-60\widetilde{E}_{n-1}^{(-3)}+1465\widetilde{E}_{n-2}^{(-3)}-18600\widetilde{E}_{n-3}^{(-3)}+129259\widetilde{E}_{n-4}^{(-3)}-465180\widetilde{E}_{n-5}^{(-3)}+675675\widetilde{E}_{n-6}^{(-3)}=0.$$

In fact, $\widetilde{E}_n^{(0)}=(3^{n+1}-1)/2$ (*Cf.* [15, A003462]). The sequence $\{\widetilde{E}_n^{(-1)}\}_{n\geqslant 0}$ is the 6th binomial transform of $0,1,0,1,0,1,\dots$ ([15, A081200]).

The proof of Theorem 4.1 depends on the following Lemma.

Lemma 4.3. *Put*

$$V_n:=\begin{vmatrix}1 & x_1 & \cdots & x_1^{n-1}\\ 1 & x_2 & \cdots & x_2^{n-1}\\ \vdots & \vdots & & \vdots\\ 1 & x_n & \cdots & x_n^{n-1}\end{vmatrix},$$

and let

$$V_{n,j}:=\begin{vmatrix}1 & x_1 & \cdots & x_1^{j-2} & x_1^n & x_1^j & \cdots & x_1^{n-1}\\ 1 & x_2 & \cdots & x_2^{j-2} & x_1^n & x_2^j & \cdots & x_2^{n-1}\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ 1 & x_n & \cdots & x_n^{j-2} & x_n^n & x_n^j & \cdots & x_n^{n-1}\end{vmatrix}\quad (j=1,2,\dots,n),$$

where the j*th column* $x_1^{j-1},x_2^{j-1},\dots,x_n^{j-1}$ *of* V_n *is replaced by* $x_1^n,x_2^n,\dots,x_n^n$. *Then for* $i=1,2,\dots,n$

$$V_{n,n-i+1}=(-1)^{i-1}\left(\sum_{1\leqslant \ell_1<\cdots<\ell_i\leqslant n}x_{\ell_1}\cdots x_{\ell_i}\right)V_n.$$

Proof. As is well-known, Vandermonde's determinant is given by $V_n = \prod_{1 \leqslant i<j \leqslant n}(x_j - x_i)$. After the jth column is moved to the nth column, we get $V_{n,j} = (-1)^{n-j} W_{n,j}$, where

$$W_{n,j} = \begin{vmatrix} 1 & x_1 & \cdots & x_1^{j-2} & x_1^j & \cdots & x_1^{n-1} & x_1^n \\ 1 & x_2 & \cdots & x_2^{j-2} & x_2^j & \cdots & x_2^{n-1} & x_2^n \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{j-2} & x_n^j & \cdots & x_n^{n-1} & x_n^n \end{vmatrix}.$$

Then, it is sufficient to prove that

$$W_{n,j} = \left(\sum_{1 \leqslant \ell_1 < \cdots < \ell_{n-j+1} \leqslant n} x_{\ell_1} \cdots x_{\ell_{n-j+1}} \right) V_n. \tag{4.1}$$

When $j = 1$, (4.1) is valid because

$$W_{n,1} = \begin{vmatrix} x_1 & x_1^2 & \cdots & x_1^n \\ x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & & \vdots \\ x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} = x_1 x_2 \cdots x_n V_n.$$

Let $2 \leqslant j \leqslant n-1$. By the column operations from the jth to the second, we have

$$\begin{aligned} W_{n,j} &= \begin{vmatrix} 1 & x_1 - x_n & \cdots & x_1^{j-3}(x_1 - x_n) & x_1^{j-2}(x_1^2 - x_n^2) & x_1^j(x_1 - x_n) & \cdots & x_1^{n-1}(x_1 - x_n) \\ 1 & x_2 - x_n & \cdots & x_2^{j-3}(x_2 - x_n) & x_2^{j-2}(x_2^2 - x_n^2) & x_2^j(x_2 - x_n) & \cdots & x_2^{n-1}(x_2 - x_n) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n-1} - x_n & \cdots & x_{n-1}^{j-3}(x_1 - x_n) & x_{n-1}^{j-2}(x_{n-1}^2 - x_n^2) & x_{n-1}^j(x_{n-1} - x_n) & \cdots & x_1^{n-1}(x_{n-1} - x_n) \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{vmatrix} \\ &= (-1)^{n+1} \prod_{\ell=1}^{n-1} (x_\ell - x_n) \cdot \begin{vmatrix} 1 & \cdots & x_1^{j-3} & x_1^{j-2}(x_1 + x_n) & x_1^j & \cdots & x_1^{n-1} \\ 1 & \cdots & x_2^{j-3} & x_2^{j-2}(x_2 + x_n) & x_2^j & \cdots & x_2^{n-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & \cdots & x_{n-1}^{j-3} & x_{n-1}^{j-2}(x_{n-1} + x_n) & x_{n-1}^j & \cdots & x_{n-1}^{n-1} \end{vmatrix} \\ &= \prod_{\ell=1}^{n-1} (x_n - x_\ell) \cdot (W_{n-1,j-1} + x_n W_{n-1,j}). \end{aligned}$$

By induction on j about the relation (4.2), using (4.1) up to $j-1$, we have

$$\begin{aligned} W_{n,j} &= \prod_{\ell=1}^{n-1} (x_n - x_\ell) \left(\sum_{1 \leqslant \ell_1 < \cdots < \ell_{n-j+1} \leqslant n-1} x_{\ell_1} \cdots x_{\ell_{n-j+1}} \right) V_{n-1} + x_n \prod_{\ell=1}^{n-1} (x_n - x_\ell) \cdot W_{n-1,j} \\ &= \left(\sum_{1 \leqslant \ell_1 < \cdots < \ell_{n-j+1} \leqslant n-1} x_{\ell_1} \cdots x_{\ell_{n-j+1}} \right) V_n + x_n \prod_{\ell=1}^{n-1} (x_n - x_\ell) \cdot W_{n-1,j}. \end{aligned} \tag{4.2}$$

On the other hand,

$$W_{n,n} = \begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-2} & x_1^n \\ 1 & x_2 & \cdots & x_2^{n-2} & x_2^n \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} & x_n^n \end{vmatrix} = x_n V_n + \prod_{\ell=1}^{n-1} (x_n - x_\ell) \cdot W_{n-1,n-1}$$

$$
\begin{aligned}
&= (x_{n-1}+x_n)V_n + \prod_{\ell_1=1}^{n-1}(x_n - x_{\ell_1})\prod_{\ell_2=1}^{n-2}(x_{n-1}-x_{\ell_2})\cdot W_{n-2,n-2}\\
&= (x_n + x_{n-1}+\cdots+x_3)V_n\\
&\quad + \prod_{\ell_1=1}^{n-1}(x_n - x_{\ell_1})\prod_{\ell_2=1}^{n-2}(x_{n-1}-x_{\ell_2})\cdots\prod_{\ell_{j-2}=1}^{2}(x_3 - x_{\ell_{j-2}})\cdot W_{2,2}\\
&= (x_1+x_2+x_3+\cdots+x_{n-1}+x_n)V_n.
\end{aligned}
$$

Thus, (4.1) is also valid when $j = n$. Now, for $2 \leqslant j \leqslant n-1$, from (4.2) we obtain

$$
\begin{aligned}
W_{n,j} &= \left(\sum_{1\leqslant \ell_1<\cdots<\ell_{n-j+1}\leqslant n-1} x_{\ell_1}\cdots x_{\ell_{n-j+1}}\right)V_n + x_n\left(\sum_{1\leqslant \ell_1<\cdots<\ell_{n-j}\leqslant n-2} x_{\ell_1}\cdots x_{\ell_{n-j}}\right)V_n\\
&\quad + x_n x_{n-1}\prod_{\ell_1=1}^{n-1}(x_n - x_{\ell_1})\prod_{\ell_2=1}^{n-2}(x_{n-1}-x_{\ell_2})\cdot W_{n-2,j}\\
&= \left(\sum_{1\leqslant \ell_1<\cdots<\ell_{n-j+1}\leqslant n-1} x_{\ell_1}\cdots x_{\ell_{n-j+1}}\right)V_n + x_n\left(\sum_{1\leqslant \ell_1<\cdots<\ell_{n-j}\leqslant n-2} x_{\ell_1}\cdots x_{\ell_{n-j}}\right)V_n\\
&\quad + x_n x_{n-1}\left(\sum_{1\leqslant \ell_1<\cdots<\ell_{n-j-1}\leqslant n-3} x_{\ell_1}\cdots x_{\ell_{n-j-1}}\right)V_n + \cdots + x_n x_{n-1}\cdots x_{j+2}\left(\sum_{1\leqslant \ell_1<\ell_2\leqslant j} x_{\ell_1}x_{\ell_2}\right)V_n\\
&\quad + x_n x_{n-1}\cdots x_{j+1}\prod_{\ell_1=1}^{n-1}(x_n - x_{\ell_1})\prod_{\ell_2=1}^{n-2}(x_{n-1}-x_{\ell_2})\cdots\prod_{\ell_{n-j}=1}^{j}(x_{j+1}-x_{\ell_{n-j}})\cdot W_{j,j}.
\end{aligned}
$$

Since the last term is equal to $x_n x_{n-1}\cdots x_{j+1}(x_1+x_2+\cdots+x_j)$, we get

$$
W_{n,j} = \left(\sum_{1\leqslant \ell_1<\cdots<\ell_{n-j+1}\leqslant n} x_{\ell_1}\cdots x_{\ell_{n-j+1}}\right)V_n,
$$

as desired. □

Proof of Theorem 4.1. When $k = 0$, by $\widetilde{E}_n^{(0)} = (3^{n+1}-1)/2$ $(n \geqslant 0)$, the recurrence relation is clear. Let $k \geqslant 1$. Assume that for a fixed k, the numbers $\widetilde{E}_n^{(-k)}$'s satisfy the recurrence relation

$$
\widetilde{E}_n^{(-k)} + \tau_{k,1}\widetilde{E}_{n-1}^{(-k)} + \tau_{k,2}\widetilde{E}_{n-2}^{(-k)} + \cdots + \tau_{k,2k}\widetilde{E}_{n-2k}^{(-k)} = 0. \tag{4.3}
$$

Since k is fixed, this relation depends only on the part $\left((4l+3)^{n+1}-(4l+1)^{n+1}\right)$ in the explicit expression. As $0 \leqslant l \leqslant k$ and $\left\{ {k \atop 0} \right\} = 0$ $(k \geqslant 1)$, the relation (4.3) is equivalent to the system

$$
(2\ell+3)^{n+1} + (2\ell+3)^n\tau_{k,1} + (2\ell+3)^{n-1}\tau_{k,2} + \cdots + (2\ell+3)^{n-2k+1}\tau_{k,2k} = 0 \quad (1 \leqslant \ell \leqslant 2k)
$$

or the system

$$
\begin{aligned}
5^{2k} + 5^{2k-1}\tau_{k,1} + 5^{2k-2}\tau_{k,2} + \cdots + 5\tau_{k,2k-1} + \tau_{k,2k} &= 0,\\
7^{2k} + 7^{2k-1}\tau_{k,1} + 7^{2k-2}\tau_{k,2} + \cdots + 7\tau_{k,2k-1} + \tau_{k,2k} &= 0,\\
&\vdots\\
(4k+3)^{2k} + (4k+3)^{2k-1}\tau_{k,1} + (4k+3)^{2k-2}\tau_{k,2} + \cdots + (4k+3)\tau_{k,2k-1} + \tau_{k,2k} &= 0.
\end{aligned}
$$

Hence, the coefficients $\tau_{k,1}, \tau_{k,2}, \ldots, \tau_{k,2k}$ are the solution of the matrix equation

$$\begin{pmatrix} 5^{2k-1} & 5^{2k-2} & \cdots & 5 & 1 \\ 7^{2k-1} & 7^{2k-2} & \cdots & 7 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ (4k+3)^{2k-1} & (4k+3)^{2k-2} & \cdots & 4k+3 & 1 \end{pmatrix} \begin{pmatrix} \tau_{k,1} \\ \tau_{k,2} \\ \vdots \\ \tau_{k,2k-1} \\ \tau_{k,2k} \end{pmatrix} = \begin{pmatrix} -5^{2k} \\ -7^{2k} \\ \vdots \\ -(4k+1)^{2k} \\ -(4k+3)^{2k} \end{pmatrix}.$$

The determinant of the $2k$ dimensional Vandermonde matrix on the left-hand side is equal to

$$V := \prod_{1 \leqslant \ell_1 < \ell_2 \leqslant 2k} \big((2\ell_1+3)-(2\ell_2+3)\big) = (-2)^{k(2k-1)} V_{2k},$$

which is the denominator part of $\tau_{k,j}$ by Cramer. By applying Lemma 4.3 with $n = 2m$ and $x_i = 2i+3$ ($i = 1, 2, \ldots, 2k$), the numerator part of $\tau_{k,j}$ is given by

$$(-1)^j \left(\sum_{1 \leqslant \ell_1 < \cdots < \ell_j \leqslant 2k} (2\ell_1+3)\cdots(2\ell_j+3) \right) \cdot V.$$

Hence, we have

$$\tau_{k,j} = (-1)^j \sum_{1 \leqslant \ell_1 < \cdots < \ell_j \leqslant 2k} (2\ell_1+3)\cdots(2\ell_j+3) = \sigma_{2k-j}^{(2k)} \quad (j = 1, 2, \ldots, 2k).$$

□

For an integer k, poly-Euler numbers of the second kind $\widehat{E}_n^{(k)}$ ($n \geqslant 0$) are defined by ([10, 11])

$$\frac{\mathrm{Li}_k(1-e^{-4x})}{4\sinh x} = \sum_{n=0}^{\infty} \widehat{E}_n^{(k)} \frac{x^n}{n!}.$$

When $k = 1$, $\widehat{E}_n = \widehat{E}_n^{(1)}$ are the original Euler numbers of the second kind (or complimentary Euler numbers), defined by

$$\frac{x}{\sinh x} = \sum_{n=0}^{\infty} \widehat{E}_n \frac{x^n}{n!}.$$

Poly-Euler numbers of the second kind can be expressed in terms of poly-Bernoulli numbers ([10, Theorem 3.1], [11, Lemma 3.1]):

$$\widehat{E}_n^{(k)} = \frac{1}{2} \sum_{m=0}^{n} \binom{n}{m} 4^m \big((-1)^{n-m} - (-3)^{n-m}\big) B_m^{(k)}.$$

In the case of negative index, poly-Euler numbers of the second kind have an explicit expression ([10, Theorem 3.3], [11, Lemma 3.2]):

$$\widehat{E}_n^{(-k)} = \frac{(-1)^k}{2} \sum_{l=0}^{k} (-1)^l l! \left\{ {k \atop l} \right\} \big((4l+3)^n + (4l+1)^n\big) \quad (k \geqslant 0).$$

Notice that $E_n^{(-k)}$ are all integers. Euler numbers of the second kind with negative indices satisfy the same recurrence formula as those in Theorem 4.1.

Theorem 4.4. *For* $k=0$,
$$\widehat{E}_n^{(0)}-4\widehat{E}_{n-1}^{(-k)}+3\widehat{E}_{n-2}^{(-k)}=0\quad(n\geqslant 2).$$
For $k\geqslant 1$,
$$\sum_{j=0}^{2k}\sigma_j^{(2k)}\widehat{E}_{n-2k+j}^{(-k)}=0\quad(n\geqslant 2k).$$

Remark 4.5. For $k=0,1,2,3$, we have
$$\{\widehat{E}_n^{(0)}\}_{n\geqslant 0}=1,2,5,14,41,122,365,1094,3281,9842,29525,88574,\dots,$$
$$\{\widehat{E}_n^{(-1)}\}_{n\geqslant 0}=1,6,37,234,1513,9966,66637,450834,3077713,21153366,\dots,$$
$$\{\widehat{E}_n^{(-2)}\}_{n\geqslant 0}=1,14,165,1826,19689,210134,2236365,23819306,254327889,\dots,$$
$$\{\widehat{E}_n^{(-3)}\}_{n\geqslant 0}=1,30,613,10770,175465,2741670,41809933,628464090,9366724945,\dots.$$

In fact, $\widehat{E}_n^{(0)}=(3^n+1)/2$ ([15, A007051]). The sequence $\{\widehat{E}_n^{(-1)}\}_{n\geqslant 0}$ is the 6th binomial transform of $1,0,1,0,1,\dots$ ([15, A081188]).

Proof of Theorem 4.4. The proof can be done as the same as that of Theorem 4.1. However, we shall prove the similar result by a different method, which is also applicable to that of Theorem 4.1. Since
$$x(x-1)\cdots(x-n+1)=\sum_{\ell=0}^{n}(-1)^{n-\ell}\begin{bmatrix}n\\ \ell\end{bmatrix}x^\ell,$$
we get
$$(x-5|2)_n=\sum_{\ell=0}^{n}(-2)^{n-\ell}\begin{bmatrix}n\\ \ell\end{bmatrix}(x-5)^\ell=\sum_{\ell=0}^{n}(-2)^{n-\ell}\begin{bmatrix}n\\ \ell\end{bmatrix}\sum_{j=0}^{\ell}\binom{\ell}{j}(-5)^{\ell-j}x^j$$
$$=\sum_{j=0}^{n}\sum_{\ell=j}^{n}(-2)^{n-\ell}(-5)^{\ell-j}\binom{\ell}{j}\begin{bmatrix}n\\ \ell\end{bmatrix}x^j.$$
Hence, for $j=0,1,\dots,n$, we have
$$\sigma_j^{(n)}=\sum_{\ell=j}^{n}(-2)^{n-\ell}(-5)^{\ell-j}\binom{\ell}{j}\begin{bmatrix}n\\ \ell\end{bmatrix}.$$
Therefore, for $k\geqslant 1$ and $n\geqslant 2k$,
$$\sum_{j=0}^{2k}\sigma_j^{(2k)}\widehat{E}_{n-2k+j}^{(-k)}$$
$$=\sum_{j=0}^{2k}\sum_{\ell=j}^{2k}(-2)^{2k-\ell}(-5)^{\ell-j}\binom{\ell}{j}\begin{bmatrix}2k\\ \ell\end{bmatrix}\frac{(-1)^k}{2}\sum_{r=0}^{k}(-1)^r r!\begin{Bmatrix}k\\ r\end{Bmatrix}\big((4r+3)^{n-2k+j}+(4r+1)^{n-2k+j}\big).$$
Now
$$\sum_{j=0}^{2k}\sum_{\ell=j}^{2k}(-2)^{2k-\ell}(-5)^{\ell-j}\binom{\ell}{j}\begin{bmatrix}2k\\ \ell\end{bmatrix}\frac{(-1)^k}{2}\sum_{r=0}^{k}(-1)^r r!\begin{Bmatrix}k\\ r\end{Bmatrix}(4r+3)^{n-2k+j}$$

$$
\begin{aligned}
&= 2^{2k}\sum_{r=0}^{k}(-1)^r r!\left\{ k \atop r \right\}(4r+3)^{n-2k}\sum_{\ell=0}^{2k}\left(\frac{5}{2}\right)^{\ell}\left[2k \atop \ell \right]\sum_{j=0}^{\ell}\binom{\ell}{j}\left(-\frac{4r+3}{5}\right)^{j}\\
&= 2^{2k}\sum_{r=0}^{k}(-1)^r r!\left\{ k \atop r \right\}(4r+3)^{n-2k}\sum_{\ell=0}^{2k}\left(\frac{5}{2}\right)^{\ell}\left[2k \atop \ell \right]\left(1-\frac{4r+3}{5}\right)^{\ell}\\
&= 2^{2k}\sum_{r=0}^{k}(-1)^r r!\left\{ k \atop r \right\}(4r+3)^{n-2k}\sum_{\ell=0}^{2k}\left[2k \atop \ell \right](1-2r)^{\ell}\\
&= 2^{2k}\sum_{r=0}^{k}(-1)^r r!\left\{ k \atop r \right\}(4r+3)^{n-2k}(2r-1)(2r-2)\cdots(2r-2k),
\end{aligned}
$$

which is equal to 0 because $(2r-1)(2r-2)\cdots(2r-2k)=0$ for $r=1,2,\dots,k$ and $\left\{ k \atop 0 \right\}=0$ for $k\geqslant 1$. Similarly,

$$
\begin{aligned}
&\sum_{j=0}^{2k}\sum_{\ell=j}^{2k}(-2)^{2k-\ell}(-5)^{\ell-j}\binom{\ell}{j}\left[2k \atop \ell \right]\frac{(-1)^k}{2}\sum_{r=0}^{k}(-1)^r r!\left\{ k \atop r \right\}(4r+1)^{n-2k+j}\\
&\quad = 2^{2k}\sum_{r=0}^{k}(-1)^r r!\left\{ k \atop r \right\}(4r+1)^{n-2k}(2r-2)(2r-3)\cdots(2r-2k-1),
\end{aligned}
$$

which is equal to 0 because $(2r-2)(2r-3)\cdots(2r-2k-1)=0$ for $r=1,2,\dots,k$ and $\left\{ k \atop 0 \right\}=0$ for $k\geqslant 1$. □

5. Leaping relations

Since the numbers $c_n^{(-2)}$ satisfy the three-term recurrence relation $c_n^{(-2)}+(2n-5)c_{n-1}^{(-2)}+(n-3)^2c_{n-2}^{(-2)}$ for $n\geqslant 4$, they also can satisfy the leaping recurrence relations of the form

$$A(n)c_n^{(-2)}+B(n)c_{n-r}^{(-2)}+C(n)c_{n-2r}^{(-2)}=0\quad (n\geqslant 2r+2)$$

for a positive integer r.

Given a three-term recurrence formula $Z_n=T(n)Z_{n-1}+U(n)Z_{n-2}$ $(n\geqslant 2)$ with arbitrary initial values Z_0 and Z_1, and two sequence of integers are eventually quasi-periodic as

$$
\begin{aligned}
\{T(n)\}_{n\geqslant 0}&=a_0,a_1\dots,a_\rho,\overline{T_1(k),T_2(k),\dots,T_w(k)}_{k=1}^{\infty},\\
\{U(n)\}_{n\geqslant 0}&=b_0,b_1\dots,b_\rho,\overline{U_1(k),U_2(k),\dots,U_w(k)}_{k=1}^{\infty}.
\end{aligned}
$$

For integers a and l with $l\geqslant 1$, we define the determinant

$$
K_l(a)=\begin{vmatrix}
T(a) & 1 & 0 & & & \\
-U(a+1) & T(a+1) & 1 & & & \\
0 & -U(a+2) & T(a+2) & & & \\
 & & & \ddots & & 0\\
 & & & & T(a+l-2) & 1\\
 & & & 0 & -U(a+l-1) & T(a+l-1)
\end{vmatrix}
$$

with $K_0(a)=1$. Let $\Omega(M)=U(M-r)U(M-r+1)\cdots U(M-1)$ with $M=(n-1)r+i+2$. Then we have the following ([4, Theorem 2]).

Lemma 5.1. *For any integers* r *and* i *with* $r \geqslant 2$, $0 \leqslant \rho \leqslant i < \rho + r$, *and* $n \geqslant 2$,

$$K_{r-1}(M-r)\cdot Z_{rn+i} - \big(K_{r-1}(M)K_r(M-r) + U(M)K_{r-1}(M-r)K_{r-2}(M+1)\big)\cdot Z_{rn-r+i} + (-1)^r\Omega(M)K_{r-1}(M)\cdot Z_{rn-2r+i} = 0.$$

We apply Lemma 5.1 as $T(n) = -(2n-5)$ and $U(n) = -(n-3)^2$. When $r = 2$ and $i = 0$, we get $M = 2n$, $\Omega(M) = (2n-4)^2(2n-5)^2$, $K_1(M) = -(4n-5)$, $K_1(M-r) = -(4n-9)$, and $K_2(M-r) = 12n^2 - 48n + 47$. Thus, we have

$$-(4n-9)c_n^{(-2)} + 2(4n-7)(4n^2-14n+11)c_{n-2}^{(-2)} - (2n-4)^2(2n-5)^2(4n-5)c_{n-4}^{(-2)}.$$

When $r = 2$ and $i = 1$, we get $M = 2n+1$, $\Omega(M) = (2n-3)^2(2n-4)^2$, $K_1(M) = -(4n-3)$, $K_1(M-r) = -(4n-7)$, and $K_2(M-r) = 12n^2 - 36n + 26$. Thus, we have

$$-(4n-7)c_{2n+1}^{(-2)} + 2(4n-5)(4n^2-10n+5)c_{2n-1}^{(-2)} - (2n-3)^2(2n-4)^2(4n-3)c_{2n-3}^{(-2)}.$$

Proposition 5.2. *For* $n \geqslant 3$,

$$(4n-9)c_{2n}^{(-2)} - 2(4n-7)(4n^2-14n+11)c_{2n-2}^{(-2)} + (2n-3)^2(2n-4)^2(4n-5)c_{2n-4}^{(-2)} = 0,$$
$$(4n-7)c_{2n+1}^{(-2)} - 2(4n-5)(4n^2-10n+5)c_{2n-1}^{(-2)} + (2n-3)^2(2n-4)^2(4n-3)c_{2n-3}^{(-2)} = 0.$$

For example,

$$\begin{aligned} 3c_6^{(-2)} - 50c_4^{(-2)} + 28c_2^{(-2)} &= 0,\\ 5c_7^{(-2)} - 154c_5^{(-2)} + 324c_3^{(-2)} &= 0,\\ 7c_8^{(-2)} - 342c_6^{(-2)} + 1584c_4^{(-2)} &= 0,\\ 9c_9^{(-2)} - 638c_7^{(-2)} + 5200c_5^{(-2)} &= 0. \end{aligned}$$

It is similarly shown for the cases when $r \geqslant 3$, but the expressions become more complicated.

For the three-term recurrence relation of Bernoulli numbers $B_n^{(-2)} - 5B_{n-1}^{(-2)} + 6B_{n-2}^{(-2)}$, leaping recurrence relations are much simpler. When $r = 2$, we have $5B_n^{(-2)} - 65B_{n-2}^{(-2)} + 180B_{n-4}^{(-2)} = 0$ or

$$B_n^{(-2)} - 13B_{n-2}^{(-2)} + 36B_{n-4}^{(-2)} = 0.$$

In fact, we can get a more general case by using the following result ([8, Theorem 1]).

Lemma 5.3. *If the sequence* $\{Z_n\}_n$ *satisfies the three-term relation* $Z_n = a_1 Z_{n-1} + a_2 Z_{n-2}$, *then for any positive integer* r *we have*

$$Z_n = r\sum_{i=0}^{\lfloor r/2\rfloor}\frac{(r-i-1)!}{i!(r-2i)!}a_1^{r-2i}a_2^i\cdot Z_{n-r} - (-1)^r a_2^r\cdot Z_{n-2r} \quad (n \geqslant 2r).$$

By applying Lemma 5.3 as $a_1 = 5$ and $a_2 = -6$, we have the following.

Proposition 5.4.

$$B_n^{(-2)} = r\sum_{i=0}^{\lfloor r/2\rfloor}\frac{(r-i-1)!}{i!(r-2i)!}5^{r-2i}(-6)^i\cdot B_{n-r}^{(-2)} - 6^r\cdot B_{n-2r}^{(-2)} \quad (n \geqslant 2r).$$

Concerning the sequence $\{c_n^{(-3)}\}$, a four-term recurrence relation

$$c_n^{(-3)}+3(n-3)c_{n-1}^{(-3)}+(3n^2-21n+37)c_{n-2}^{(-3)}+(n-4)^3c_{n-3}^{(-3)}\quad(n\geqslant 5)$$

holds. However, no general formula for leaping relation has been found. As for Bernoulli numbers $B_n^{(-3)}$, a four-term relation

$$B_n^{(-3)}-9B_{n-1}^{(-3)}+26B_{n-2}^{(-3)}-24B_{n-3}^{(-3)}=0$$

has only constant coefficients, by applying [8, Theorem 2], we have a leaping relation

$$\begin{aligned}B_n^{(-3)}&=r\sum_{j=0}^{\lfloor r/3\rfloor}\sum_{i=0}^{(r-3j)/2}\frac{(r-i-2j-1)!}{i!j!(r-2i-3j)!}9^{r-2i-3j}(-26)^i\cdot 24^jB_{n-r}^{(-3)}\\&\quad-r\sum_{j=0}^{\lfloor r/3\rfloor}\sum_{i=0}^{\lfloor(r-3j)/2\rfloor}(-1)^{r+i+j}\frac{(r-i-2j-1)!}{i!j!(r-2i-3j)!}9^i(-26)^{r-2i-3j}\cdot 24^{i+2j}\cdot B_{n-2r}^{(-3)}+24^rB_{n-3r}^{(-3)}\quad(n>3r).\end{aligned}$$

6. Comments

Sasaki [14] announced more recurrence formulas of poly-Bernoulli numbers from the duality theorem $B_n^{(-k)}=B_k^{(-n)}$, and combinatorial interpretations about some relations. However, poly-Cauchy numbers and poly-Euler numbers do not satisfy the duality theorem. Any combinatorial interpretation of poly-Cauchy numbers has not been known yet, though a few are discovered about poly-Euler numbers [13].

Acknowledgment

The author thanks the anonymous referee for the detailed comments and suggestions, which improved the quality of the paper.

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