Numerical quenching of a heat equation with nonlinear boundary conditions

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Abstract

In this paper, we study the quenching behavior of semidiscretizations of the heat equation with nonlinear boundary conditions. We obtain some conditions under which the positive solution of the semidiscrete problem quenches in a finite time and estimate its semidiscrete quenching time. We also establish the convergence of the semidiscrete quenching time and obtain some results on numerical quenching rate. Finally we give some numerical results to illustrate our analysis.

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1. Introduction

In this paper, we study the behavior of a semidiscrete approximation of the following heat equation involving nonlinear boundary flux conditions:

$$u_t(x,t) = u_{xx}(x,t), \quad x \in (0,a), \ t \in (0,T), \tag{1.1}$$

$$u_x(0,t) = (1 - u(0,t))^{-p}, \ u_x(a,t) = (1 - u(a,t))^{-q}, \ t \in (0,T), \tag{1.2}$$

$$0 \leqslant u(x,0) = u_0(x) < a, \ x \in [0,a], \tag{1.3}$$

where $m, n \geqslant 0, p, q > 0$ and u_0 satisfies the compatibility conditions, i.e., $u_0'(0) = (1 - u_0(0))^{-p}, u_0'(a) = (1 - u_0(a))^{-q}$. The problem is said to be quench if there exists a finite time T such that,

$$\lim_{t \to T^-} \max_{0 \leqslant x \leqslant a} u(x,t) = 1^-.$$

The study of heat equation with nonlinear boundary conditions has deserved a great deal of interest in recent years and has been used to model, for example, heat transfer, polarization phenomena in ionic

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conductor, chemical catalyst kinetics, etc. (see [3, 4, 6–9, 14, 15, 17] and the references cited therein). In [4] the authors have given some conditions under which the solution u of (1.1)-(1.3) quenches in a finite time and the quenching occurs only at the boundary $x = a$. They proved that u_t blows up at quenching set and also gave the quenching rate near the quenching time T, which in other word says that there exist constants $C_1, C_2 > 0$ such that $1 - C_1(T - t)^{\frac{1}{q+1}} \leqslant u(a, t) \leqslant 1 - C_2(T - t)^{\frac{1}{q+1}}$.

From our knowledge, the numerical approximation of equations (1.1)-(1.3) has not been studied. Thus, in this paper, we are investigated in the numerical study using a semidiscrete form of (1.1)-(1.3). For that, we consider the case where $a = 1$. Let $I \in \mathbb{N}$. We consider a uniform mesh on the interval $[0, 1]$ as

$$x_i = (i - 1)h, \ i = 1, \ldots, I, \ h = 1/(I - 1),$$

$U_h = U_h(t) = (U_1(t), \ldots, U_I(t))^\mathsf{T}$ are the values of the numerical approximation at the nodes x_i at time t of the solution of (1.1)-(1.3). By the finite difference method we obtain the following system of ODEs whose solution is U_h:

$$U_i'(t) = \delta^2 U_i(t), \ i = 2, \ldots, I - 1, \ t \in (0, T_h), \tag{1.4}$$

$$U_1'(t) = \delta^2 U_1(t) - \frac{2(1 - U_1(t))^{-p}}{h}, \ t \in (0, T_h), \tag{1.5}$$

$$U_I'(t) = \delta^2 U_I(t) + \frac{2(1 - U_I(t))^{-q}}{h}, \ t \in (0, T_h), \tag{1.6}$$

$$U_i(0) = \varphi_i \geqslant 0, \ i = 1, \ldots, I, \tag{1.7}$$

where $0 \leqslant \varphi_i < 1, \ 1 \leqslant i \leqslant I$ and

$$\delta^2 U_i(t) = \frac{U_{i-1}(t) - 2U_i(t) + U_{i+1}(t)}{h^2}, \ 2 \leqslant i \leqslant I - 1, \ t > 0,$$

$$\delta^2 U_1(t) = \frac{2U_2(t) - 2U_1(t)}{h^2}, \ t > 0, \quad \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2}, \ t > 0.$$

We say that the solution of (1.4)-(1.7) quenches in a finite time T_h if

$$\lim_{t \to T_h} \|U_h(t)\|_\infty = 1.$$

T_h is called the semidiscrete quenching time of (1.4)-(1.7).

For study on numerical approximations of heat equation with non-linear boundary conditions we refer to [1, 5, 10–13, 16]. Nabongo et al. in [12] were interested in the numerical study using a semidiscrete form of

$$u_t(x, t) = u_{xx}(x, t), \quad x \in (0, 1), \ t \in (0, T),$$

$$u_x(0, t) = 0, \ u_x(1, t) = -u(1, t)^{-\beta}, \ t \in (0, T), \quad u(x, 0) = u_0(x), \ x \in [0, 1],$$

where $\beta > 0$. They showed that some conditions under which the positive solution of the numerical approximation for this heat equation quenches in a finite time, they also established the convergence of the semidiscrete quenching time and obtained some results on numerical quenching rate and set.

Inspired by [4, 5, 12], we give some conditions under which any positive solution of semidiscrete scheme of (1.1)-(1.3) quenches in a finite time. We also show that the semidiscrete quenching time converges to the theoretical one when the mesh size goes to zero. The rest of the paper is organized as follows : in the next section, we give some properties concerning our semidiscrete scheme. In Section 3, under some conditions, we prove that the solution of the semidiscrete scheme of (1.1)-(1.3) quenches in a finite time, we give a result on numerical quenching rate. In Section 4, we show that the quenching time converges to the theoretical one when the mesh size goes to zero. Finally, in the last section, we give some numerical results to illustrate our analysis.

2. Properties of the semidiscrete scheme

In this section, we give some auxiliary results for the problem (1.4)-(1.7).

Definition 2.1. A function $V_h \in C^1((0, T_h), \mathbf{R}^I)$ is a lower solution of (1.4)-(1.7) if

$$V_i'(t) - \delta^2 V_i(t) \leqslant 0, \quad i = 2, \ldots, I-1, \ t \in (0, T_h),$$

$$V_1'(t) - \delta^2 V_1(t) + \frac{2(1 - V_1(t))^{-p}}{h} \leqslant 0, \ t \in (0, T_h),$$

$$V_I'(t) - \delta^2 V_I(t) - \frac{2(1 - V_I(t))^{-q}}{h} \leqslant 0, \ t \in (0, T_h),$$

$$V_i(0) \leqslant U_i(0), \quad i = 1, \ldots, I,$$

where $U_h = (U_1, \ldots, U_I)^\top$ is solution of (1.4)-(1.7). On the other hand, we say that $V_h \in C^1((0, T_h), \mathbf{R}^I)$ is an upper solution of (1.4)-(1.7) if these inequalities are reversed.

Lemma 2.2. *Let $b_h \in C^0([0, T_h], \mathbf{R}^I)$ and let $V_h \in C^1([0, T_h], \mathbf{R}^I)$ such that*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + b_i(t)V_i(t) \geqslant 0, \ 1 \leqslant i \leqslant I, \ t \in (0, T_h), \quad V_i(0) \geqslant 0, \ 1 \leqslant i \leqslant I.$$

Then we have $V_i(t) \geqslant 0, \ 1 \leqslant i \leqslant I, \ t \in (0, T_h)$.

Proof. Let us define the vector $Z_h(t) = V_h(t)e^{\lambda t}$ with λ a real. We have

$$Z_i'(t) - \delta^2 Z_i(t) + (b_i(t) - \lambda)Z_i(t) \geqslant 0, \quad i = 1, \ldots, I, \ t \in (0, T_h), \tag{2.1}$$

$$Z_i(0) \geqslant 0, \quad i = 1, \ldots, I. \tag{2.2}$$

Denote $m = \min_{1 \leqslant i \leqslant I, \ t \in [0, T_0]} Z_i(t)$, where $T_0 \in (0, T_h)$. Since for $i \in \{1, \ldots, I\}$, $Z_i(t)$ is a continuous function on the compact $[0, T_0]$, there exists $i_0 \in \{1, \ldots, I\}$ such that $m = Z_{i_0}(t_{i_0})$. Assume $m < 0$. Taking λ such that $b_{i_0}(t_{i_0}) - \lambda > 0$. If $t_{i_0} = 0$, then $Z_{i_0}(0) < 0$, which contradicts (2.2), hence $t_{i_0} > 0$.

It is not hard to see that

$$Z_{i_0}'(t_{i_0}) = \lim_{k \to 0} \frac{Z_{i_0}(t_{i_0}) - Z_{i_0}(t_{i_0} - k)}{k} \leqslant 0,$$

moreover by a straightforward computation we get

$$Z_{i_0}'(t_{i_0}) - \delta^2 Z_{i_0}(t_{i_0}) + (b_{i_0}(t_{i_0}) - \lambda)Z_{i_0}(t_{i_0}) < 0,$$

but these inequalities contradict (2.1) and the proof is complete. $\qquad\square$

Another form of the maximum principle for semidiscrete equations is the following comparison lemma.

Lemma 2.3. *Let $W_h, V_h \in C^1((0, T_h), \mathbf{R}^I)$ such that for $t \in (0, T_h)$, we have*

$$W_i'(t) - \delta^2 W_i(t) \leqslant V_i'(t) - \delta^2 V_i(t), \quad i = 2, \ldots, I-1,$$

$$W_1'(t) - \delta^2 W_1(t) + g(W_1(t)) \leqslant V_1'(t) - \delta^2 V_1(t) + g(V_1(t)),$$

$$W_I'(t) - \delta^2 W_I(t) + f(W_I(t)) \leqslant V_I'(t) - \delta^2 V_I(t) + f(W_I(t)),$$

$$0 \leqslant W_i(0) \leqslant V_i(0), \quad i = 1, \ldots, I.$$

Then we have

$$W_h(t) \leqslant V_h(t), \ \forall t \in (0, T_h).$$

Proof. Let us define the vector $Z_h(t) = V_h(t) - W_h(t)$ with λ a real. We have

$$Z'_i(t) - \delta^2 Z_i(t) \geqslant 0, \quad i = 2, \ldots, I - 1, \ t \in (0, T_h), \tag{2.3}$$

$$Z'_1(t) - \delta^2 Z_1(t) + g'(\zeta(t)) Z_1(t) \geqslant 0, \ t \in (0, T_h), \tag{2.4}$$

$$Z'_{I-1}(t) - \delta^2 Z_{I-1}(t) - f'(\xi(t)) Z_I(t) \geqslant 0, \ t \in (0, T_h), \tag{2.5}$$

$$Z_i(0) \geqslant 0, \quad i = 1, \ldots, I, \tag{2.6}$$

where $\zeta(t)$ and $\xi(t)$ lie, respectively, between $V_1(t)$ and $W_1(t)$, and between $V_I(t)$ and $W_I(t)$. We can rewrite (2.3)-(2.6) as

$$Z'_i(t) - \delta^2 Z_i(t) + b_i(t) Z_i(t) \geqslant 0, \quad i = 1, \ldots, I, \ t \in (0, T_h),$$

where $b_1(t) = g'(\zeta(t))$, $b_I(t) = -f'(\xi(t))$ and $b_i(t) = 0$, $i = 2, \ldots, I - 1$, $\forall t \in (0, T_h)$. According to lemma 2.2, $Z_h(t) \geqslant 0$, $\forall t \in (0, T_h)$ and the proof is complete. $\qquad\square$

The results of the next lemma are analogue to those of continuous problem.

Lemma 2.4. *Let $U_h \in C^1((0, T_h), \mathbf{R}^I)$ be solution of (1.4)-(1.7) with a nonnegative initial data φ_h such that φ_h is a lower solution and $\varphi_{i+1} > \varphi_i$, for $i = 1, \ldots, I - 1$. Then we have*

(i) $U_i(t) \geqslant 0$ *and* $U_i(t) \geqslant \varphi_i$, $i = 1, \ldots, I$, $t \in (0, T_h)$;

(ii) $U_{i+1}(t) > U_i(t)$, $i = 1, \ldots, I - 1$, $t \in (0, T_h)$;

(iii) $\dfrac{dU_i(t)}{dt} > 0$, $i = 1, \ldots, I$, $t \in (0, T_h)$.

Proof.

(i) Since φ_h is a lower solution of (1.4)-(1.7), we have

$$U_i(t) \geqslant \varphi_i \geqslant 0, \quad \text{for} \quad i = 1, \ldots, I, \ t \in (0, T_h),$$

by Lemma 2.3.

(ii) Let t_0 be the first $t > 0$, such that $K_i(t) = U_{i+1}(t) - U_i(t) > 0$, for $1 \leqslant i \leqslant I - 1$, but $K_{i_0}(t_0) = U_{i_0+1}(t_0) - U_{i_0}(t_0) = 0$ for a certain $i_0 \in \{1, \ldots, I - 1\}$. Without lost of generality, we can suppose that i_0 is the smallest integer which satisfies the above equality. We have

$$\frac{dK_{i_0}(t_0)}{dt} = \lim_{\epsilon \to 0} \frac{K_{i_0}(t_0) - K_{i_0}(t_0 - \epsilon)}{\epsilon} \leqslant 0,$$

$$\frac{K_{i_0+1}(t_0) - 2K_{i_0}(t_0) + K_{i_0-1}(t_0)}{h^2} > 0, \quad \text{if } 2 \leqslant i_0 \leqslant I - 2,$$

$$K_{i_0+1}(t_0) - 3K_{i_0}(t_0)h^2 > 0, \quad \text{if } i_0 = 1,$$

$$\frac{-3K_{i_0}(t_0) + K_{i_0-1}(t_0)}{h^2} > 0, \quad \text{if } i_0 = I - 1,$$

which imply that

$$\frac{dK_{i_0}(t_0)}{dt} - \frac{K_{i_0+1}(t_0) - 2K_{i_0}(t_0) + K_{i_0-1}(t_0)}{h^2} < 0, \quad \text{if } 2 \leqslant i_0 \leqslant I - 2,$$

$$\frac{dK_{i_0}(t_0)}{dt} - \frac{K_{i_0+1}(t_0) - 3K_{i_0}(t_0)}{h^2} - \frac{2(1 - U_{i_0}(t))^{-p}}{h} < 0, \quad \text{if } i_0 = 1,$$

$$\frac{dK_{i_0}(t_0)}{dt} - \frac{-3K_{i_0}(t_0) + K_{i_0-1}(t_0)}{h^2} - \frac{2(1 - U_{i_0+1}(t))^{-q}}{h} < 0, \quad \text{if } i_0 = I - 1,$$

but these inequalities contradict (1.4)-(1.6) and we obtain the desired result.

(iii) Denote $Z_i(t) = U_i(t + \varepsilon) - U_i(t)$, $i = 1, \ldots, I$, using (i) we obtain $Z_i(0) \geqslant 0$.
A straightforward calculation yields

$$\frac{dZ_i(t)}{dt} = \frac{Z_{i-1}(t) - 2Z_i(t) + Z_{i+1}(t)}{h^2}, \quad 2 \leqslant i \leqslant I - 1, \ t \in (0, T_h),$$

$$\frac{dZ_1(t)}{dt} = \frac{2Z_2(t) - 2Z_1(t)}{h^2} - \frac{2p(1 - \eta(t))^{-p-1}Z_1(t)}{h}, \quad t \in (0, T_h),$$

$$\frac{dZ_I(t)}{dt} = \frac{2Z_{I-1}(t) - 2Z_I(t)}{h^2} + \frac{2q(1 - \xi(t))^{-q-1}Z_I(t)}{h}, \quad t \in (0, T_h),$$

$$Z_i(0) \geqslant 0, \quad i = 1, \ldots, I,$$

where $\eta(t)$ and $\xi(t)$ lie, respectively, between $U_1(t + \varepsilon)$ and $U_1(t)$ and between $U_I(t + \varepsilon)$ and $U_I(t)$. We obtain the below inequality by the same manner as Lemma 2.3

$$Z_i(t) \geqslant 0, \quad i = 1, \ldots, I, \forall t \in (0, T_h).$$

This fact implies the desired result. $\qquad\square$

3. Quenching and quenching rate

In this section, under some assumptions, we show that the solution U_h of (1.4)-(1.7) quenches in a finite time and estimate its semidiscrete quenching time. Moreover we determine the numerical quenching rate and quenching set of the solution.

Theorem 3.1. *Let $U_h(t)$ be the solution of (1.4)-(1.7) such that the initial data φ_h is a lower solution, assumption $q \geqslant p > 0$. Then there exists a positive constant γ such that U_h quenches in a finite time T_h and we have the following estimate*

$$T_h \leqslant \frac{(1 - \|\varphi_h\|_\infty)^{q+1}}{\gamma(q + 1)}.$$

Proof. Let $(0, T_h)$ be the maximal time interval on which $0 \leqslant U_h(t) < 1$. Our aim is to show that T_h is finite and satisfies the above inequality. Introduce the function $J_h(t)$ such that

$$J_i(t) = \frac{dU_i(t)}{dt} - \gamma(1 - U_i(t))^{-q}, \ i = 1 \ldots, I.$$

A straightforward calculation gives

$$\frac{dJ_1(t)}{dt} - \delta^2 J_1(t) + \frac{2p}{h}(1 - U_1(t))^{-p-1}J_1(t) = \frac{2\gamma}{h}(q - p)(1 - U_1(t))^{-p-q-1} + \gamma(\delta^+ U_1(t))^2 g''(\mu_1(t)),$$

$$\frac{dJ_I(t)}{dt} - \delta^2 J_I(t) - \frac{2q}{h}(1 - U_I(t))^{-q-1}J_I(t) = \gamma(\delta^- U_I(t))^2 g''(\theta_I(t)),$$

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) = \gamma(\delta^+ U_i(t))^2 g''(\mu_i(t)) + \gamma(\delta^- U_i(t))^2 g''(\theta_i(t)) \ i = 2, \ldots, I - 1,$$

with $g''(x) = q(q + 1)(1 - x)^{-q-2}$, $\delta^+ U_i = \frac{U_{i+1} - U_i}{h}$, $\delta^- U_i = \frac{U_i - U_{i-1}}{h}$, where θ_i and μ_i lie, respectively, between U_i and U_{i+1}, and between U_i and U_{i-1}. It is not hard to see that

$$\frac{dJ_1(t)}{dt} - \delta^2 J_1(t) + \frac{2p}{h}(1 - U_1(t))^{-p-1}J_1(t) \geqslant 0,$$

$$\frac{dJ_I(t)}{dt} - \delta^2 J_I(t) - \frac{2q}{h}(1 - U_I(t))^{-q-1}J_I(t) \geqslant 0,$$

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) \geqslant 0, \ i = 2, \ldots, I - 1.$$

Moreover, using the fact that φ_h is a lower solution, we obtain $J_h(0) \geqslant 0$. We deduce from Lemma 2.2 that $J_i(t) \geqslant 0$, $i = 1, \ldots, I$. As a consequence we get

$$(1 - U_i(t))^q dU_i(t) \geqslant \gamma dt. \tag{3.1}$$

Integrating (3.1) from 0 to t we obtain

$$(1 - U_i(t))^{q+1} \leqslant (1 - \varphi_i)^{q+1} - \gamma(q+1)t, \quad i = 1, \ldots, I.$$

Denote $\|\varphi_h\|_\infty = \varphi_{i_0}$ and $\|U_h(t)\|_\infty = U_{i_0}(t)$ for a certain $i_0 \in \{1, \ldots, I\}$. Which means that $\|U_h(T_0)\|_\infty = 1$ for some $T_0 = \dfrac{(1 - \|\varphi_h\|_\infty)^{q+1}}{\gamma(q+1)}$. Thus U_h quenches in finite time $T_h \leqslant T_0$. $\qquad\square$

Theorem 3.2. *Under the assumptions of Theorem 3.1 and Lemma 2.4, the solution U_h of (1.4)-(1.7) quenches in a finite time T_h and*

$$1 - \|U_h(t)\|_\infty \sim (T_h - t)^{\frac{1}{q+1}} \ \ for \ \ t \in (0, T_h).$$

Proof. From Theorem 3.1 U_h quenches in a finite time T_h. Integrating the above inequalities (3.1) over (t, T_h) we obtain

$$T_h - t \leqslant \frac{(1 - U_i(t))^{q+1}}{\gamma(q+1)}, \quad i = 1, \ldots, I, \tag{3.2}$$

which implies that

$$\|U_h(t)\|_\infty \leqslant 1 - C_2(T_h - t)^{1/(q+1)},$$

where $C_2 = (\gamma(q+1))^{1/(q+1)}$. From Lemma 2.4, $U_i < U_{i+1}$, for $1 \leqslant i \leqslant I - 1$, hence $\|U_h(t)\|_\infty = U_I(t)$, for $t \in (0, T_h)$. It follows from (1.6) that

$$\frac{dU_I(t)}{dt} \leqslant \frac{2}{h}(1 - U_I(t))^{-q},$$

which implies that $(1 - U_I(t))^q dU_I(t) \leqslant \dfrac{2}{h} dt$. Thus we have

$$1 - C_1(T_h - t)^{1/(q+1)} \leqslant \|U_h(t)\|_\infty, \ \ \text{where} \ \ C_1 = \left(\frac{2(q+1)}{h}\right)^{1/(q+1)},$$

and we have the desired result. $\qquad\square$

Theorem 3.3. *If $\lim_{t \to T_h} U_I(t) = 1$, then $U_h'(t)$ blows up.*

Proof. Suppose $U_h'(t)$ is bounded. Then, there exists a positive constant M such that $U_h'(t) < M$. We have

$$\sum_{i=2}^{I} \sum_{j=i}^{I} h^2 U_j'(t) \leqslant \sum_{i=2}^{I} \sum_{j=i}^{I} h^2 M.$$

From (1.4)-(1.6) and lemma 2.4 we get

$$\frac{1}{(1 - U_I(t))^q} < \frac{M}{2} + \frac{hM}{2} + U_I(t) - U_1(t).$$

As $t \to T_h^-$, the left-hand side tends to infinity while the right-side is finite. This contradiction shows that $U_h'(t)$ blows up. $\qquad\square$

4. Convergence of the semidiscrete quenching time

In this section, we study the convergence of the semidiscrete quenching time. Now we will show that for each fixed time interval $[0, T]$, where u is defined, the solution U_h of (1.4)-(1.7) approximates u when the mesh parameter h goes to zero. We denote

$$u_h(t) = (u(x_1, t), \dots, u(x_I, t))^T \text{ and } \|U_h(t)\|_\infty = \max_{1 \leqslant i \leqslant I} |U_i(t)|.$$

Theorem 4.1. *Assume that the problem* (1.1)-(1.3) *has a solution* $u \in C^{4,1}([0,1] \times [0, T^*])$ *and the initial condition* φ_h *at* (1.7) *verifies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1), \quad h \to 0. \tag{4.1}$$

Then, for h *small enough, the semidiscrete problem* (1.4)-(1.7) *has a unique solution* $U_h \in C^1([0, T^*], \mathbf{R}^I)$ *such that*

$$\max_{t \in [0, T^*]} \|U_h(t) - u_h(t)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + h^2), \quad \text{as } h \to 0.$$

Proof. Let $\sigma > 0$ be such that

$$\|u\|_\infty < \sigma, \ t \in [0, T^*].$$

Then the problem (1.4)-(1.7), for each h, has a unique solution $U_h \in C^1([0, T^*], \mathbf{R}^I)$. Let $t(h) \leqslant T^*$ be the greatest value of $t > 0$ such that

$$\|U_h(t) - u_h(t)\|_\infty < 1. \tag{4.2}$$

The relation (4.1) implies $t(h) > 0$ for h small enough. Using the triangle inequality, we obtain

$$\|U_h(t)\|_\infty \leqslant \|U_h(t) - u_h(t)\|_\infty + \|u(\cdot, t)\|_\infty, \text{ for } t \in (0, t(h)),$$

which implies that

$$\|U_h(t)\|_\infty \leqslant 1 + \sigma, \text{ for } t \in (0, t(h)).$$

Let $e_h(t) = U_h(t) - u_h(t)$, $\forall t \in [0, T^*]$ be the discretization error and let $W \in C^{4,1}([0,1], [0, T^*])$ be such that $W(x, t) = (\|\varphi_h - u_h(0)\|_\infty + Mh^2) e^{(M+1)t + \sqrt{M+1}x}$, $\forall (x, t) \in [0,1] \times [0, T^*]$, with M a positive constant. We can prove by Lemma 2.3 that

$$|e_i(t)| < W(x_i, t), \ 1 \leqslant i \leqslant I, \text{ for } t \in (0, t(h)).$$

We deduce that

$$\|U_h(t) - u_h(t)\|_\infty \leqslant (\|\varphi_h - u_h(0)\|_\infty + Mh^2) e^{(M+1)t + \sqrt{M+1}}, \text{ for } t \in (0, t(h)).$$

Suppose that $T^* > t(h)$ from (4.2), we obtain

$$1 = \|U_h(t(h)) - u_h(t(h))\|_\infty \leqslant (\|\varphi_h - u_h(0)\|_\infty + Mh^2) e^{(M+1)t + \sqrt{M+1}}.$$

Since the term on the right hand side of the above inequality goes to zero as h tends to zero, we deduce that, $1 \leqslant 0$, which is impossible. Hence we have $t(h) = T^*$, and the proof is complete. $\square$

Theorem 4.2. *Suppose that the solution* u *of* (1.1)-(1.3) *quenches in a finite time* T *such that* $u \in C^{4,1}([0,1] \times [0, T))$ *and the initial condition at* (1.7) *satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1), \quad h \to 0.$$

Then the solution U_h *of* (1.4)-(1.7) *quenches in a finite time* T_h *and we have*

$$\lim_{h \to 0} T_h = T.$$

Proof. Set $\varepsilon > 0$. There exists $\eta > 0$ such that

$$\frac{(1-y)^{q+1}}{\gamma(q+1)} \leqslant \frac{\varepsilon}{2}, \quad 0 \leqslant y \leqslant \eta. \tag{4.3}$$

Since $u(x,t)$ quenches in a finite time T, there exists a time $T_0 < T$ such that $|T_0 - T| < \frac{\varepsilon}{2}$ and $0 \leqslant \|u(x,t)\|_\infty \leqslant \frac{\eta}{2}$ for $t \in [T_0, T)$. Setting $T_1 = \frac{T_0+T}{2}$, it is not hard to see that $\|u(x,t)\|_\infty < 1$ for $t \in [0, T_1]$. From Theorem 4.1, we have $\|U_h(T_1) - u_h(T_1)\|_\infty \leqslant \frac{\eta}{2}$. Applying the triangle inequality, we get

$$\|U_h(T_1)\|_\infty \leqslant \|U_h(T_1) - u_h(T_1)\|_\infty + \|u_h(T_1)\|_\infty \leqslant \eta.$$

From Theorem 3.1, U_h quenches in a finite time T_h. We deduce from (3.2) and (4.3) that

$$|T_h - T| \leqslant |T_h - T_1| + |T_1 - T| \leqslant \frac{(1 - \|U_h(T_1)\|_\infty)^{q+1}}{\gamma(q+1)} + \frac{\varepsilon}{2} \leqslant \varepsilon. \qquad \square$$

5. Numerical experiments

In this section, we present some numerical approximations to the quenching time of (1.4)-(1.7), the initial data φ_h such

$$\alpha\varphi_2 \geqslant (1 - \varphi_1)^{-p}, \tag{5.1}$$

where $\alpha > 0$. We use the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_h^{ne}} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2}, \quad 2 \leqslant i \leqslant I-1,$$

$$\frac{U_1^{(n+1)} - U_1^{(n)}}{\Delta t_h^{ne}} = \frac{2U_2^{(n)} - 2U_1^{(n)}}{h^2} - \frac{2}{h}\left(1 - U_1^{(n)}\right)^{-p},$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_h^{ne}} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2} + \frac{2}{h}\left(1 - U_I^{(n)}\right)^{-q},$$

$$U_i^{(0)} = \varphi_i, \quad 1 \leqslant i \leqslant I,$$

where $n \geqslant 0$, $\Delta t_h^{ne} = \min\left\{\frac{h^2}{2}, h^2\|U_h^{(n)}\|_\infty^{q+1}\right\}$. We also consider the implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_h^n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2}, \quad 2 \leqslant i \leqslant I-1,$$

$$\frac{U_1^{(n+1)} - U_1^{(n)}}{\Delta t_h^n} = \frac{2U_2^{(n+1)} - 2U_1^{(n+1)}}{h^2} - \frac{2}{h}\left(1 - U_1^{(n)}\right)^{-p},$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_h^n} = \frac{2U_{I-1}^{(n+1)} - 2U_I^{(n+1)}}{h^2} + \frac{2}{h}\left(1 - U_I^{(n)}\right)^{-q},$$

$$U_i^{(0)} = \varphi_i, \quad 1 \leqslant i \leqslant I,$$

where $n \geqslant 0$, $\Delta t_h^n = h^2\|U_h^{(n)}\|_\infty^{q+1}$.

Let us notice that (5.1) and the restriction on the time step guarantee the positivity of the discrete solution. We need the following definition.

Definition 5.1. We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme quenches in a finite time if $\lim_{n\to\infty}\|U_h^{(n)}\|_\infty = 1$ and the series $\sum_{n=0}^{+\infty}\Delta t_h^n$ converges. The quantity $t_h^n = \sum_{j=0}^{n-1}\Delta t_h^j$ is called the numerical quenching time of the solution $U_h^{(n)}$ and $T_h = \sum_{n=0}^{+\infty}\Delta t_h^n$ is called the numerical quenching time of the solution U_h.

In Tables 1, 2, 3, and 4, in rows, we present the numerical quenching times, the numbers of iterations and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256, 512, 1024. We take for the numerical quenching time $T_h = \sum_{n=0}^{+\infty} \Delta t_h^n$ which is computed at the first time when $\Delta t_h^n = |t_h^{n+1} - t_h^n| \leqslant 10^{-16}$. The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}, \quad \text{where } h = 1/(I-1).$$

For the numerical values, we take $\varphi_i = \sqrt{1.5} - \cos(\frac{\Pi(i-1)h}{4})$ for $i = 1, \ldots, I$.

Table 1: Numerical quenching times obtained with the explicit Euler method for $q = 0.5$, $p = 0.5$.

I	T_h	n	s
16	0.04139596	236	-
32	0.03955588	679	-
64	0.03900581	2164	1.74
128	0.03884205	7556	1.75
256	0.03879396	28153	1.77
512	0.03878005	108750	1.79
1024	0.03877610	427914	1.82

Table 2: Numerical quenching times obtained with the implicit Euler method for $q = 0.5$, $p = 0.5$.

I	T_h	n	s
16	0.04221929	237	-
32	0.03975759	680	-
64	0.03905612	2163	1.81
128	0.03885465	7558	1.80
256	0.03879711	28155	1.81
512	0.03878084	108753	1.82
1024	0.03877629	427917	1.84

Table 3: Numerical quenching times, obtained with the explicit Euler method for $q = 1$, $p = 0.5$.

I	T_h	n	s
16	0.00972448	129	-
32	0.00813955	310	-
64	0.00763971	816	1.66
128	0.00748449	2434	1.69
256	0.00743743	8155	1.72
512	0.00742349	29675	1.76
1024	0.00741945	113290	1.79

Table 4: Numerical quenching times obtained with the implicit Euler method for $q = 1$, $p = 0.5$.

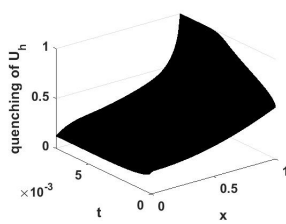| I	T_h	n	s
16	0.01036564	129	-
32	0.00830903	311	-
64	0.00768367	818	1.72
128	0.00749576	2436	1.73
256	0.00744028	8158	1.76
512	0.00742421	29678	1.79
1024	0.00741963	113293	1.81

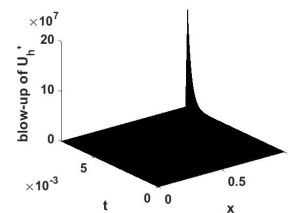Figure 1: Evolution of the numerical solution U_h for $I = 64$, $q = 1$ and $p = 0.5$.

Figure 2: Evolution of U_h' for $I = 64$, $q = 1$ and $p = 0.5$.

Remark 5.2. The various tables of our numerical results show that there is a relationship between the quenching time and the flows on the boundaries. If we consider the problem (1.4)-(1.7) in the case where the initial data $\varphi(x) = \sqrt{1.5} - \cos(\frac{\Pi x}{4})$ and $q = 0.5$, we observe from Tables 1 and 2 that the numerical quenching time is approximately equal to 0.038. When $q = 1$, we observe from Tables 3 and 4 that the numerical blow-up time is approximately equal to 0.0074. Thus we can said that when rise q we have an acceleration of quench of the solution. Also, from the tables we observe the convergence of blow-up time T_h of the solution of (1.4)-(1.7), since the rate of convergence is near 2. This result does not surprise us because of the result established in the previous section.

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