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Preservation properties for some classes of discrete distributions

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Abstract

In this paper, some new families of discrete life distributions are discussed. Definitions and basic results are introduced. Several properties of these classes are presented, including the preservation under convolution, closer under the formation of parallel systems and mixing.

Keywords: Classes of life distributions, NBUCA, NBUC, GHNWUE, convolution, mixing, formation of parallel systems.

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1. Introduction

Nonparametric aging classes of life distributions have been found to be useful in reliability analysis, engineering applications, maintenance policies, economics, biometry, queuing theory and many other fields. There are many situations where a continuous time is inappropriate for describing the lifetime of devices and other systems. For example, the life time of many devices in industry such as switches and mechanical tools, depends essentially on the number of times they are turned on and off or the number of shocks they receive. In such cases, the time to failure is often more appropriately represented by the number of times they are used before they fail, which is a discrete random variable.

Discrete lifetimes usually arise through grouping or finite-precision measurement of continuous time phenomena. Let X be a non-negative discrete random variable representing the lifetime of the unit. Without loss of generality, it is assumed that N is a support of X. The probability mass function (p.m.f) is given by $f(x) = \Pr\{X = x\}, x = 0, 1, 2, \ldots$ the cumulative distribution function F of X satisfies $f(x) = \Pr\{X \leqslant x\} = \sum_{i=0}^{x} = f(i)$ for all $x \in N$ where $N = \{0, 1, \ldots\}$. The distribution of counting random variable is called a discrete life distribution. In particular, if $f(0) = \Pr(X = 0) = 0$, or a counting random variable X has a support on $N_+ = \{1, 2, \ldots\}$, we say that the discrete distribution is zero-truncated. Moreover, $N_- = \{-1, 0, 1, \ldots\}$.

Similar to continuous distributions, discrete distribution can also be classified by the properties of failure rates, mean residual lifetimes, survival function. These classes of discrete distribution aging have

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been used extensively in different fields of statistics and probability such insurance, finance, reliability, survival analysis, and others. See, for example, [5, 8, 10, 11, 13, 14, 16–18]. Some commonly used classes of discrete distributions include the classes of discrete decreasing failure rate (D-DFR), discrete decreasing failure rate average (D-DFRA), discrete new worse than used (D-NWU), discrete increasing mean residual life (D-IMRL), discrete harmonic new worse than used in expectation (D-HNWUE), and their dual ones including the classes of discrete increasing failure rate(D-IFR), discrete increasing failure rate average (D-IFRA), discrete new better than used (D-NBU), discrete decreasing mean residual life (D-DMRL), and discrete harmonic new better than used in expectation (D-HNBUE).

[4] defined a continuous random variable X (or its distribution function F) to be generalized harmonic new better than used in expectation if

$$\int_0^\infty \int_t^\infty \bar{F}(x)\,dx \leqslant \mu^2 \text{ if } 0 < \mu < \infty.$$

Also, [15] defined a continuous random variable X (or its distribution function F) to be new better than used of second order (NBU (2)) if

$$\int_0^x \bar{F}(t+y)\,dy \leqslant \bar{F}(t) \int_0^x \int_x^\infty \bar{F}(y)dydx \quad \text{for all } x, t \geqslant 0.$$

[3] defined the class of new better than used in increasing convex average order (NBUCA). This class is requiring the distribution function F of a random variable X to satisfy.

$$\int_0^\infty \int_x^\infty \bar{F}(u+t)\,dudt \leqslant \bar{F}(t) \int_0^\infty \int_x^\infty \bar{F}(u)dudx \quad \text{for all } t \geqslant 0.$$

[1] defined the class of new better than used in convex ordering of second order (NBUC2). Preservation properties under convolution, random maxima, mixing and formation of coherent structures are established. Stochastic comparisons of the excess lifetime when the inter-arrival times belong to the NBUC(2) class are developed. Some applications of Poisson shock models and a test of exponentiality against NBUC (2) alternative are presented. This class is requiring the distribution function F of a random variable X to satisfy

$$\int_z^\infty \int_x^\infty \bar{F}(u+t)\,dudx \leqslant \bar{F}(t) \int_z^\infty \int_x^\infty \bar{F}(u)dudx \quad \text{for all } t, z \geqslant 0.$$

The authors have demonstrated their usefulness in reliability applications as well as in other fields.

2. Basic definitions

Most of the nonparametric discrete classes of distributions that are commonly found in the reliability literature are based on some notion of aging. In this section we present the definitions of some classes of discrete distributions, which are used in the sequel.

Definition 2.1. Let X and Y be two non-negative random variables with distribution functions $F(x)$ and $G(y)$, and survival functions $\bar{F}(x)$ and $\bar{G}(x)$, respectively. X is said to be smaller than Y in the

1. stochastic ordering, denoted by $X \leqslant_{st} Y$ if

$$\bar{F}(x) \leqslant \bar{G}(y) \quad \text{for all } x \geqslant 0;$$

2. discrete increasing convex order, denoted by$X \leqslant_{dicx} Y$ if

$$\sum_{i=x}^{\infty} \bar{F}(i) \leqslant \sum_{i=y}^{\infty} \bar{G}(i) \quad \text{for all } i \in N;$$

3. discrete increasing convex average order, denoted by $X \leqslant_{dicxa} Y$ if

$$\sum_{x=0}^{\infty}\sum_{i=x}^{\infty}\bar{F}(i) \leqslant \sum_{y=0}^{\infty}\sum_{i=y}^{\infty}\bar{G}(i) \quad \text{for all} \quad i \in N;$$

4. discrete increasing concave order, denoted by $X \leqslant_{dicv} Y$ if

$$\sum_{i=0}^{x}\bar{F}(i) \leqslant \sum_{i=0}^{y}\bar{G}(i) \quad \text{for all} \quad i \in N.$$

Definition 2.2. A discrete distribution F is called discrete new better than used in convex ordering (D-NBUC) (or discrete new worse than used in convex ordering (D-NWUC)) if

$$\sum_{i=x+y}^{x}\bar{F}(i) \leqslant \bar{F}(x)\sum_{i=y}^{\infty}\bar{F}(j) \quad \text{for all} \quad x, y \in N.$$

Definition 2.3. An integer valued random variable X (or its cdf F) is said to be discrete new better than used in convex average order (D-NBUCA) if

$$\sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\bar{F}(i) \leqslant \bar{F}(j)\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\bar{F}(i) \quad \text{for all} \quad j \in N.$$

Or equivalently

$$\sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\bar{F}(x+y+i) \leqslant \sum_{y=0}^{\infty}\bar{F}(y)\sum_{i=0}^{\infty}\bar{F}(x+i).$$

Definition 2.4. An integer valued random variable X (or its cdf F) F is said to be discrete new better than used in second order (D-NBU (2)) if

$$\sum_{j=0}^{k}\bar{F}(i+j) \leqslant \bar{F}(i)\sum_{j=0}^{k}\bar{F}(j) \quad \text{for all} \quad i, k \in N.$$

Definition 2.5. A discrete distribution F with finite mean μ is called discrete new better than used in expectation (D-NBUE) if for all $x \in N$

$$\sum_{i=x}^{\infty}\bar{F}(i) \leqslant \bar{F}(x)\sum_{j=0}^{\infty}\bar{F}(j) \quad \text{for all} \quad x \in N.$$

Or equivalently

$$\sum_{i=x}^{\infty}\bar{F}(i) \leqslant \mu\bar{F}(x).$$

Definition 2.6. A discrete distribution F with expectation$\sum_{i=0}^{\infty}\bar{F}(i) = \mu$ is called discrete generalized harmonic new better than used in expectation (D-GHNBUE) if for all $x \in N$.

$$\sum_{k=0}^{\infty}\sum_{j=k}^{\infty}\bar{F}_j \leqslant \mu^2.$$

Theorem 2.7. *If* $X \in D-NBU(2)$, *then* $X \in D-NBUE$.

Proof. Since X is D-NBU(2)

$$\bar{F}(i)\sum_{j=0}^{k}\bar{F}(j) \geqslant \sum_{j=0}^{k}\bar{F}(i+j) \quad \text{for all} \quad k \in N. \tag{2.1}$$

Letting k tending to infinity in (2.1), one gets

$$\bar{F}(i)\sum_{j=0}^{\infty}\bar{F}(j) \geqslant \sum_{j=0}^{\infty}\bar{F}(i+j),$$

or equivalently

$$\mu\bar{F}(i) \geqslant \sum_{j=0}^{\infty}\bar{F}(i+j),$$

which is D-NBUE. □

3. Preservation properties

In this section, commonly used properties of aging classes, such as convolution, closure under formation of parallel systems, and mixing (see, e.g., [1, 3, 5, 9, 13]) are derived for D-NBU(2), D-NBUCA, and D-GHNWUE classes.

3.1. Convolutions

In the next theorems, we establish the closure property of D-NBU(2), D-NBUCA, and D-GHNWUE classes under the convolution.

Theorem 3.1. *Suppose thatF_1 and F_2 are two independent D-NBU(2) life distributions. Then their convolution is also D-NBU(2).*

Proof.

$$\begin{aligned}
\sum_{s=0}^{x}\bar{F}(i+s) &= \sum_{k=0}^{x}\sum_{s=0}^{\infty}\bar{F}(t+s-k)dF_2(k),\\
&= \left\{\sum_{s=0}^{x}\sum_{k=0}^{t}\bar{F}_1(t+s-k)dF_2(k)\sum_{s=0}^{x}\sum_{k=t}^{\infty}\bar{F}_1(t+s-k)dF_2(k)\right\} \qquad (3.1)\\
&= \left\{\sum_{k=0}^{t}\left[\sum_{s=0}^{x}\bar{F}_1(t+s-k)\right]dF_2(k)+\sum_{s=0}^{x}\left[\sum_{\nu=0}^{\infty}\bar{F}_1(s-\nu)dF_2(\nu+t)\right]\right\} = A_1+A_2,
\end{aligned}$$

where

$$A_1 = \sum_{k=0}^{t}\bar{F}_1(t-k)\sum_{s=0}^{x}dF_2(k) = [F(t)-F_2(t)]\sum_{k=0}^{x}\bar{F}_1(k), \tag{3.2}$$

and

$$\begin{aligned}
A_2 &\leqslant \sum_{s=0}^{x}\left[\bar{F}_1(t)\bar{F}_2(t)+\sum_{s=0}^{\infty}\sum_{\nu=0}^{\infty}\bar{F}_2(\nu+t)d_\nu\bar{F}_1(s-\nu)\right]\\
&= \bar{F}_2(t)\sum_{k=0}^{x}\bar{F}_1(k)+\sum_{s=0}^{x}\left[\sum_{\nu=0}^{s}\bar{F}_2(\nu+t)d_\nu\bar{F}_1(s-\nu)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \psi + \sum_{s=0}^{x}\left[\sum_{k=0}^{s} \bar{F}_2(t+s-k)dF_1(k)\right] \\
&= \psi + \sum_{k=0}^{x}\left[\sum_{s=k}^{x} \bar{F}_2(t+s-k)dF_1(k)\right] \\
&= \psi + \sum_{k=0}^{x}\left[\sum_{s=0}^{x-k} \bar{F}_2(t+s)\right]dF_1(k) \\
&\leqslant \psi + \bar{F}_2(t)\sum_{k=0}^{x}\sum_{s=0}^{x-k} \bar{F}_2(s)dF_1(k) \\
&= \psi + \bar{F}_2(t)\sum_{k=0}^{x}\sum_{k=0}^{s} \bar{F}_2(s-k)dF_1(k),
\end{aligned} \tag{3.3}
$$

where the equality follows from the D-NBU(2) property of F_2 is given by

$$\psi = \bar{F}_2(t)\sum_{k=0}^{x} \bar{F}_1(k).$$

On the other hand

$$
\begin{aligned}
F(t)\sum_{s=0}^{x} \bar{F}_1(s) &= \bar{F}_(t)\sum_{k=0}^{x}\sum_{s=0}^{\infty} \bar{F}_2(s-k)dF_1(k) \\
&= \bar{F}(t)\left[\sum_{k=0}^{x} \bar{F}_1(k) + \sum_{s=0}^{x}\sum_{k=0}^{s} \bar{F}_2(s-k)dF_1(k)\right] \geqslant A_1 + A_2 = \sum_{s=0}^{x} \bar{F}(t+s).
\end{aligned}
$$

From (3.1), (3.2), and (3.3) and the fact that $\bar{F}(t) \geqslant \bar{F}_2(t)$ for all $t \geqslant 0$, this proves that F is D-NBU(2). □

Theorem 3.2. *Suppose that*F_1 *and* F_2 *are two independent D-NBUCA life distributions. Then their convolution is also D-NBUCA.*

Proof. The survival functions of convolution of two life distribution F_1 and F_2 is

$$\bar{F}(u) = \sum_{t=0}^{u} \bar{F}_1(u-t)f_2(t) \quad \text{for all} \quad u \in N.$$

Let $x, y \in N$,

$$
\begin{aligned}
\sum_{y=0}^{\infty}\sum_{i=0}^{\infty} \bar{F}(x+y+i) &= \sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \bar{F}_1(x+y+i-j)f_2(j) \\
&= \sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{x-1} \bar{F}_1(x+y+i-j)f_2(j) + \sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=x}^{\infty} \bar{F}_1(x+y+i-j)f_2(j) \\
&= \sum_{y=0}^{\infty}\sum_{j=0}^{x-1} f_2(j)\sum_{i=0}^{\infty} \bar{F}_1(x+y+i-j) + \sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \bar{F}_1(y+i-j)f_2(x+j) = I_1 + I_2.
\end{aligned}
$$

We observe that

$$I_1 \leqslant \sum_{y=0}^{\infty} \bar{F}_1(y)\sum_{j=0}^{x-1} f_2(j)\sum_{i=0}^{\infty} \bar{F}_1(x+i-j). \tag{3.4}$$

Also,

$$\begin{aligned}
I_2 &= \sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\bar{F}_2(x-1)\sum_{i=y}^{\infty}\bar{F}_1(i)+\sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{F}_2(x+j)f_1(y+i-j)\\
&= \sum_{y=0}^{\infty}\sum_{i=y}^{\infty}\bar{F}_1(i)+\sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{y+i}\bar{F}_2(x+y+i-j)f_1(j)\\
&= \bar{F}_2(x-1)\sum_{y=0}^{\infty}\sum_{i=y}^{\infty}\bar{F}_1(i)+\sum_{y=0}^{\infty}\sum_{j=y+1}^{\infty}\sum_{i=j-y}^{\infty}\bar{F}_2(x+y+i-j)f_1(j)\\
&\quad+\sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{F}_2(x+y+i-j)f_1(j)=A_1+A_2+A_3,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
A_1 &= \sum_{y=0}^{\infty}\bar{F}_2(x-1)\mu_1\bar{F}_1(y)\leqslant \mu_1\mu\bar{F}_2(x-1),\\
A_2 &= \sum_{y=0}^{\infty}\sum_{j=y+1}^{\infty}f_1(j)\sum_{i=0}^{\infty}\bar{F}_2(x+i)=\sum_{y=0}^{\infty}\bar{F}_1(y)\sum_{i=0}^{\infty}\bar{F}_2(x+i),\\
A_3 &\leqslant \sum_{y=0}^{\infty}\sum_{j=0}^{\infty}f_1(j)\left[\bar{F}_2(y-j)\sum_{i=0}^{\infty}\bar{F}_2(x+i)\right]\\
&= \sum_{y=0}^{\infty}\left[\sum_{j=0}^{y}\bar{F}_2(y-j)f_1(j)\right]\sum_{i=0}^{\infty}\bar{F}_2(x+i)\\
&= \sum_{y=0}^{\infty}\left[\bar{F}(y)-\bar{F}_1(y)\right]\sum_{i=0}^{\infty}\bar{F}_2(x+i)=(\mu-\mu_1)\sum_{i=0}^{\infty}\bar{F}_2(x+i).
\end{aligned}$$

Hence

$$I_2\leqslant \mu_1\mu\bar{F}_2(x-1)+\mu\sum_{i=0}^{\infty}\bar{F}_2(x+i). \tag{3.6}$$

Combining (3.4), (3.5), and (3.6), then

$$\sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\bar{F}_(x+y+i)\leqslant \mu\left\{\sum_{j=0}^{\infty}\sum_{i=0}^{x-1}\bar{F}_1(x+i-j)f_2(j)\right\}\left\{\mu_1\bar{F}_2(x-1)+\bar{F}_2(x+i)\right\}.$$

Finally, implementing (3.5) with y=0, we get

$$\begin{aligned}
\sum_{y=0}^{\infty}\bar{F}(y)\sum_{i=0}^{\infty}\bar{F}(x+i) &= \sum_{y=0}^{\infty}\bar{F}(y)\left\{\sum_{j=0}^{\infty}\sum_{i=0}^{x-1}\bar{F}_1(x+i-j)f_2(j)+\sum_{j=0}^{\infty}\sum_{i=j}^{\infty}\bar{F}_2(x+i-j)f_1(j)\right\}\\
&= \sum_{y=0}^{\infty}\bar{F}(y)\left\{\sum_{j=0}^{\infty}\sum_{i=0}^{x-1}\bar{F}_1(x+i-j)f_2(j)+\mu_1\bar{F}_2(x-1)+\sum_{j=0}^{\infty}f_1(j)\sum_{i=0}^{\infty}\bar{F}_2(x+i)\right\}\\
&\geqslant \sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\bar{F}_(x+y+i),
\end{aligned}$$

which means that F is D-NBUCA. □

Theorem 3.3. *Suppose that F_1 and F_2 are two independent D-GHNBUE life distributions. Then their convolution is also D-GHNBUE.*

Proof. With F^*G denoting the convolution of the life distributions F and G, the result follows from the following fact. If $\sigma_F \leqslant \mu_F$, $\sigma_G \leqslant \mu_G$ and $\sigma_{F^*G} \leqslant \mu_{F^*G}$, then either μ_F or μ_G. □

3.2. Closure of the D-NBU(2) under formation of parallel systems

In this subsection we show that the discrete new better than used in second order is closed under formation of parallel systems.

Theorem 3.4. *Let $X_1, X_2, \dots, X_n$ be independent and identically distributed integer valued random variables with distribution function F and F is D-NBU(2). Then the random variable $Y = \max_{1\leqslant i\leqslant n} X_i$ which has the distribution F_n is also D-NBU(2).*

Proof. Since F is D-NBU(2), then

$$\sum_{j=0}^{k} \bar{F}_(i+j) \leqslant \bar{F}_(i) \sum_{j=0}^{k} \bar{F}_(j),$$

which implies

$$\sum_{j=i}^{i+k} \bar{F}_(i+j) \leqslant \bar{F}_(i) \sum_{j=0}^{i+k} \bar{F}_(j),$$

or

$$\sum_{j=i}^{i+k} [1-F(j)] \leqslant \bar{F}_(i) \left[\sum_{j=0}^{i} \bar{F}_(j) + \sum_{j=i+1}^{i+k} \bar{F}_(j)\right],$$

or equivalently

$$\sum_{j=i}^{i+k} \left[\frac{1-F(j)}{1-F(i)} - 1 - F(j)\right] \leqslant \sum_{j=0}^{k} [1-F(j)].$$

Since F is a distribution, we have the following

$$\sum_{j=0}^{k} [1-F(j)] \leqslant \sum_{j=0}^{k} F^n(j). \tag{3.7}$$

Hence

$$\sum_{j=i}^{i+k} \left[\frac{(1-F(j))F(i)}{1-F(j)}\right] \geqslant \sum_{j=i}^{i+k} \left[\frac{(1-F^n(j))F^n(i)}{1-F^n(j)}\right]. \tag{3.8}$$

But

$$\sum_{j=i}^{i+k} \left[\frac{(1-F(j))F(i)}{1-F(j)} - \frac{(1-F^n(j))F^n(i)}{1-F^n(j)}\right] \geqslant 0,$$

which implies that

$$\begin{aligned}
&\sum_{j=i}^{i+k} \frac{(1-F(j))F(i)}{1-F(i)} \left\{1 - F^{n-1}(j) \left[\frac{1-F^n(j)}{1-F^n(j)} \times \frac{1-F(i)}{1-F^n(i)}\right]\right\} \\
&= \sum_{j=i}^{i+k} \frac{(1-F(j))F(i)}{1-F(i)} \left\{1 - F^{n-1}(j) \left[\frac{1+F(j)+\cdots+F^{n-1}(j)}{1+F(i)+\cdots+F^{n-1}(i)}\right]\right\}
\end{aligned}$$

$$\geqslant \sum_{j=i}^{i+k} \frac{(1-F(j))F(i)}{1-F(i)} \left\{1 - F^{n-1}(j)\left[\frac{1+F^{-1}(j)+\cdots+F^{-(n-1)}(j)}{1+F(i)+\cdots+F^{n-1}(i)}\right]\right\} \geqslant 0.$$

Since

$$F(j) \leqslant F^{-1}(j) \leqslant F^{-1}(i) \text{ for all } j \leqslant i,$$

from (3.7) and (3.8) we get

$$\sum_{j=i}^{i+k} \frac{F_n(i)\bar{F}_n(j)}{\bar{F}_n(i)} \leqslant \sum_{j=0}^{k} \bar{F}_n(j).$$

The above inequality may be written as follows

$$\sum_{j=i}^{k} \frac{\bar{F}_n(j)}{\bar{F}_n(i)} \leqslant \sum_{j=0}^{i} \bar{F}_n(j) + \sum_{j=i}^{k} \bar{F}_n(j),$$

whence

$$\sum_{j=0}^{k} \frac{\bar{F}_n(i+j)}{\bar{F}_n(i)} \leqslant \sum_{j=0}^{k} \bar{F}_n(j),$$

or

$$\sum_{j=0}^{k} \bar{F}_n(i+j) \leqslant \bar{F}_n(i) \sum_{j=0}^{k} \bar{F}_n(j),$$

thus, F_n is also D-NBU(2). □

3.3. *Mixtures*

In this subsection preservation of the D-NWU(2), D-NWUCA, and D-GHNWUE under mixing are discussed.

Theorem 3.5. *The D-NWU(2) Class is preserved under mixing.*

Proof. Let g be the mixing p.m.f. Applying Chebyshev's inequality, we obtain

$$\begin{aligned}
\bar{F}_{(}i) \sum_{j=0}^{k} \bar{F}_{(}j) &= \sum_{l=0}^{\infty} \bar{F}_l(i) g(l) \left[\sum_{j=0}^{k} \sum_{l=0}^{\infty} \bar{F}_l(j) g(l)\right] \\
&\leqslant \sum_{l=0}^{\infty} \left[\bar{F}_l(i) \sum_{j=0}^{k} \bar{F}_l(j)\right] g(l) \\
&\leqslant \sum_{l=0}^{\infty} \sum_{j=0}^{k} \bar{F}_l(j)\bar{F}_l(i) g(l) \leqslant \sum_{l=0}^{\infty} \sum_{j=0}^{k} \bar{F}_l(i+j) g(l) = \sum_{j=0}^{k} \sum_{l=0}^{\infty} \bar{F}_l(i+j) g(l) = \sum_{j=0}^{k} \bar{F}(i+j),
\end{aligned}$$

as required. □

Theorem 3.6. *The D-NWUCA class is preserved under mixing.*

Proof. Let g be the mixing p.m.f. Applying Chebyshev's inequality, we obtain

$$\bar{F}(j) \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \bar{F}(i) = \left[\sum_{l=0}^{\infty} \bar{F}_l(j) g(l)\right] \left[\sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \bar{F}_l(j) g(l)\right]$$

$$
\begin{aligned}
&= \left[\sum_{l=0}^{\infty} \bar{F}_l(j)g(l)\right]\left\{\sum_{l=0}^{\infty}\left[\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\bar{F}_l(j)\right]g(l)\right\}\\
&\leqslant \sum_{m=0}^{\infty}\left[\bar{F}_m(j)\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\bar{F}_m(i)\right]g(m)\\
&\leqslant \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\bar{F}_m(i)g(m) = \sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\sum_{m=0}^{\infty}\bar{F}_m(i)g(m) = \sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\bar{F}(i),
\end{aligned}
$$

as required. □

Theorem 3.7. *The D-GHNWUE class is preserved under mixing.*

Proof. Let$F_i(j), i \in N$, be D-GHNWUE and $F(j) = \sum_{i=0}^{\infty} F_i(j)g(i)$ where $g(i), i \in N$is a probability function then $\mu_j = \sum_{i=0}^{\infty} \bar{F}_j(i) > 1, i \in N$ and $\sum_{j=0}^{\infty} \bar{F}_(j) > 1$. Utilizing Taylors expansion for two-dimensional functions, it can be show that $\varphi(s,t) = S^{1-j}(t-1)^j, s > 0, t > 1, j \in N$, is a convex real function. Then the two-dimensional Jensens inequality leads to

$$\varphi(E(s), E(t)) \leqslant E\varphi(s,t),$$

where S and T are identically distributed random variables with joint distribution defined by

$$F_r(S = \mu_i, T = \mu_l) = \begin{cases} g(i), & i = l,\\ 0, & \text{otherwise,}\end{cases}$$

and with marginal distributions defined by $F_r(S = \mu_i) = g(i), i \in N$. Therefore

$$[E(S)]^{1-j}\,[E(T)-1]^j = [E(S)]^{1-j}\,[E(T-1)]^j \leqslant E\left[S^{1-j}(T-1)\right],$$

which is equivalent to

$$\left[\sum_{i=0}^{\infty}\mu_i g(i)\right]^{1-j}\left[\sum_{l=0}^{\infty}(\mu_l-1)g(l)\right]^j \leqslant \sum_{i=0}^{\infty}\mu_i^{1-j}(\mu_l-1)^j g(i),$$

or

$$\left[\sum_{i=0}^{\infty}\mu_i g(i)\right]^{1-j}\left[1-\frac{1}{\sum_{l=0}^{\infty}\mu_l g(l)}\right] \leqslant \sum_{i=0}^{\infty}\mu_i^{1-j}(1-\frac{1}{\mu_i})^j g(i).$$

The desired result is now established since

$$\sum_{k=0}^{\infty}\sum_{j=k}^{\infty}\bar{F}(j) = \sum_{i=0}^{\infty}\left[\sum_{k=0}^{\infty}\sum_{j=k}^{\infty}\bar{F}_i(j)\right]g(i) \geqslant \sum_{k=0}^{\infty}\sum_{j=k}^{\infty}\mu_i\left(1-\frac{1}{\mu_i}\right)^k g(i) \geqslant \sum_{i=0}^{\infty}\mu_i^2 g(i) = \mu^2.$$

□

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