Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 13 (2020), 97–99
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

Modeling turbulence with the Navier-Stokes equations

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Abstract

The Navier-Stokes differential equations describe the motion of fluids which are incompressible. The three-dimensional Navier-Stokes equations misbehave very badly although they are relatively simple-looking. The solutions could wind up being extremely unstable even with nice, smooth, reasonably harmless initial conditions. A mathematical understanding of the outrageous behavior of these equations would dramatically alter the field of fluid mechanics. This paper describes why the three-dimensional Navier-Stokes equations are not solvable, i.e., the equations cannot be used to model turbulence, which is a three-dimensional phenomenon.

Keywords: Navier-Stokes equations, turbulence, forecast, geometries, solutions, experimentalist.

2010 MSC: 76F02, 76F55.

1. The Navier-Stokes equations

The general equations of motion for a viscous fluid were obtained by Sir George Stokes in 1845. The following is the fundamental equation (in vectorial form) governing the flow of a viscous fluid:

$$\frac{\partial v}{\partial t} + (v.\bigtriangledown)v = -\frac{1}{\partial p}\bigtriangledown Pe - \bigtriangledown \phi + \frac{\partial \eta}{\partial p}\bigtriangledown 2v,$$

where v is the velocity of the fluid (as a function of position), Pe is the pressure, ϕ the gravitational potential, p the density, and η the viscosity.

A fluid in motion could be characterized by its velocity field (velocity as a function of position). However, because of the complex nature of the forces affecting fluids (in general, forces of both compression and viscosity) the result of applying basic principles such as Newton's second law is a set of nonlinear equations. Computational methods therefore play a large part in fluid dynamics. Newton's second law states that the rate of change of momentum p of a body equals the total force F acting upon it, as is described by the following equation:

$$F = \partial p/\partial t.$$

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doi: 10.22436/jnsa.013.02.03

Received: 2019-06-26 Revised: 2019-08-06 Accepted: 2019-08-31

If, as is normally the case, the mass of the body is constant, F = ∂(mv)/∂t reduces to F = m∂v/∂t or F = ma, where a is the acceleration of the body. Note that the force and acceleration are vectors. The first law is the null case of the second law (if $F = 0$, then $a = 0$).

The Navier-Stokes equation is a miracle of brevity, relating a fluid's velocity, pressure, density and viscosity. In two dimensions, fluid flow governed by this partial differential equation is deterministic and predictable. But this equation fails when the fluid becomes turbulent as turbulence represents three-dimensional flow of the fluid, for which the Navier-Stokes equation does very poorly. Whereas fluid flow under normal conditions tends to be laminar, in turbulence it becomes irregular and develops eddies, ripples, and whorls. But yet there is some sort of order found within this disorder or turbulence which could be described as self-similar or fractal. What mathematical technique could be used to describe this state?

The Navier-Stokes equations are nonlinear and do not submit to any general method of solution. Each new problem has to be carefully formulated as to geometry and proper boundary conditions. Then some scheme of attack might be adopted with the hope of reaching a solution. In most cases all attempts to obtain an exact solution fail. Approximate solutions have to make do. In a few cases exact solutions could be obtained. The possibility that perhaps the flow of the fluid is unidirectional, i.e., v (x, y, t) = 0, is not an assumption. It is rather an intuitive guess which is pursued until we either find a solution or become convinced that it does not lead to a solution, in which case we mark it as an unsuccessful trial.

Substitution of viscosity in the Navier-Stokes equations with viscosity= 0 reduces them to a form called the Euler equations:

$$p\frac{Dq}{Dt} = pg - \bigtriangledown p \ \text{ (in vectorial form).}$$

The Euler equations had been formulated earlier than the Navier-Stokes equations and considered an approximation. The Euler equations are of the first order and cannot in general satisfy the boundary conditions. We could therefore conclude that the Euler equations do not form a good approximation near a rigid boundary. Far from a boundary and where viscosity = 0 is a fair estimate, they have an important role as approximations and are generally easier to solve than the full Navier-Stokes equations.

The Navier-Stokes equations do need for their solution initial conditions as well as boundary conditions. The following are proper boundary conditions for a velocity on a rigid boundary:

$$q_n = q_t = 0,$$

where q_n is the normal component of the velocity relative to the solid boundary, and q_t is the tangential component. These conditions are also termed the no-penetration ($q_n = 0$) and no-slip ($q_t = 0$) viscous boundary conditions. When the region occupied by the fluid is not closed, i.e., the fluid is not completely confined, additional conditions are still required on some surfaces which completely enclose the domain of the solution. These might represent some real physical surfaces or they might be chosen quite arbitrarily, provided the velocity on them is known. The pressure, which is also a dependent variable, also requires boundary conditions. The Navier-Stokes equations are then satisfied and we now know the resulting pressure field. This flow can exist only if the obtained pressure is possible. An acceptable boundary condition might be: $p = p_\infty = \text{const}$ at $r \to \infty$, which then implies: $p = p_\infty - \frac{pQ^2}{8\pi^2}.\frac{1}{r^2}$. We also note that in the solution for the pressure there is no trace of the viscosity. This pressure therefore also satisfies the Euler equations. As viscosity in a fluid enables it to smooth out or overcome the ripples, eddies and whorls of turbulence, a viscous fluid is in effect not so much affected by turbulence than a non-viscous fluid. Thus, the Navier-Stokes equations, as they relate to viscous fluids, present a better solution for incompressible fluids which are viscous and subject to turbulence than the Euler equations for non-viscous fluids [1–4].

2. Modeling of turbulence

The scientist normally makes a forecast of the outcome of a flow and uses the Navier-Stokes equations to model this forecast. However, in the instance of turbulence, making this forecast will be fraught with difficulty, if it can be carried out at all. Putting it another way, if turbulence could be forecasted, predicted and described by the Navier-Stokes equations it could not be turbulence, for turbulence implies puzzlement, lack of order or pattern and lack of predictability.

The Navier-Stokes equations are nonlinear due to the acceleration terms such as $u\partial u/\partial x$. As a result, the solution to these equations may not be unique. For instance, the flow between two rotating cylinders can be solved using the Navier-Stokes equations to treat a relatively simple flow with circular streamlines; it can also be a flow with streamlines which are like a spring wound around the cylinders as a torus; there are also more complex flows which are solutions to the Navier-Stokes equations, all satisfying the identical boundary conditions.

For simple geometries, the Navier-Stokes equations can be solved with relative ease. However, the equations cannot be solved for a turbulent flow even for the simplest of examples. A turbulent flow is highly unsteady, nonlinear and three-dimensional and therefore requires that the three velocity components be specified at all points in a region of interest at some initial time, say $t = 0$. But, even for the simplest geometry, such information will be almost impossible to obtain [4].

3. Conclusion

Therefore, the solutions for turbulent flows have to be left to the experimentalist and are not attempted to solving by Navier-Stokes equations [4].

References

[1] R. L. Devaney, *An introduction to chaotic dynamical systems*, The Benjamin/Cummings Publishing Co., Menlo Park, (1986). 1

[2] J. Gleick, *Chaos: making a new science*, Viking Press, New York, (1987).

[3] J. M. T. Thompson, H. B. Stewart, *Nonlinear dynamics and chaos*, John Wiley & Sons, Chichester, (1986).

[4] B. Wong, *The mathematical theory of turbulence or chaos*, Int. J. Nonlinear Sci., **10** (2010), 264–278. 1, 2, 3