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Applications of Jack's lemma for analytic functions involving α-convex functions

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Abstract

In this paper, we introduce two new subclasses of α-convex functions and study the applications of Jack's Lemma in the characterization of functions in these classes.

Keywords: Analytic functions, α-convex functions, Jack's Lemma.

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1. Introduction

Let $E := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and $\mathcal{H}(E)$ represent the class of all analytic functions f defined in E. For a positive integer n and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] := \left\{f \in \mathcal{H}(E) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots,\ z \in E\right\}.$$

Also, we denote

$$\mathcal{A}_n := \left\{ f \in \mathcal{H}[0,n] : \frac{f^{(n)}(0)}{n!} = 1 \right\}. \tag{1.1}$$

For $n = 1$, we have

$$\mathcal{A} = \mathcal{A}_1 := \left\{f \in \mathcal{H}[0,1] : f'(0) = 1\right\}.$$

Let $\mathcal{S}_n$ denote the subclass of $\mathcal{A}_n$ consisting of univalent functions defined in E. Also for $\alpha \in \mathbb{R}$, $0 \leqslant \delta < 1$ and $n \in \mathbb{N}$, some well investigated subclasses of $\mathcal{S}_n$ include the classes $\mathcal{S}^*(n,\delta)$, $\mathcal{C}(n,\delta)$ and $\mathcal{M}(n,\alpha,\delta)$ of starlike, convex and α-convex functions of order δ respectively. For $n = 1$, the above mentioned classes reduce to the classes of starlike, convex and α-convex functions of order δ respectively; for detail, see [10] with references therein. We also note that $\mathcal{M}(n,0,0) = S^*(n,0)$, $\mathcal{M}(n,1,0) = \mathcal{C}(n,0)$; for details see

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[1–5, 7, 9–12]. Moreover, throughout our discussion, we will assume that $n \in \mathbb{N}, \alpha \in \mathbb{R}, 0 \leqslant \delta < 1$. Also, for a function $f \in \mathcal{M}(n, \alpha, \delta)$, we write

$$\operatorname{Re}\left[(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > \delta, \qquad (z \in E). \tag{1.2}$$

For $f \in \mathcal{A}_n$ given by (1.1), we set the functional p such that

$$p(z) = z\left[\frac{f(z)}{z^{1-\alpha}}\left(\frac{f'(z)}{f(z)}\right)^{\alpha}\right]^{\frac{1}{1-\delta}}, \qquad (z \in E), \tag{1.3}$$

where $p(0) = 0$ and if the exponents are not integers, we can select a suitable branch so that p is analytic in E. On differentiating (1.3) and taking real part of the resultant positive, we have

$$\operatorname{Re}\left[\frac{zp'(z)}{p(z)}\right] = \frac{1}{1-\delta}\operatorname{Re}\left\{(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)-\delta\right\} > 0.$$

This shows that the function p is starlike in E.

Definition 1.1. Let $\mathcal{MS}_\rho(n, \alpha, \delta)$ be the subclass of $\mathcal{A}_n$ consisting of all α-convex functions f of order δ given by (1.2), which satisfy the condition

$$\left|\frac{p(z)}{zp'(z)} - \frac{1}{\rho}\right| < \frac{1}{\rho}, \qquad (z \in E),$$

for some $\rho\,(0 < \rho < 1)$, where p is defined by (1.3).

We also define

$$f \in \mathcal{MC}_\rho(n, \alpha, \delta), \quad \text{if and only if,} \quad zf' \in \mathcal{MS}_\rho(n, \alpha, \delta).$$

The following lemma due to Jack [6] is of fundamental importance.

Lemma 1.2. *Let w be analytic in the open unit disk E with*

$$w(0) = 0 \ \ \textit{and} \ \ |w(z)| < 1, \qquad (z \in E).$$

If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_0, then we have

$$z_0 w'(z_0) = k w(z_0),$$

where $k \geqslant 1$ is a real number.

In the past years various applications of Jack Lemma have been explored in the literature of the subject, for details, we refer [1–5, 7–9, 11–13] and others with references therein.

2. Main results

Theorem 2.1. *Let $f \in \mathcal{A}_n$ is an α-convex function of order δ and let p be given by (1.3) satisfy*

$$\left|1+\frac{zp'(z)}{p(z)} - \frac{zp''(z)}{p'(z)}\right| < 1-\rho, \qquad (z \in E),$$

for some $\rho : \frac{1}{2} \leqslant \rho < 1$. Then

$$\left|\frac{p(z)}{zp'(z)} - 1\right| = \left|(1-\delta)\left\{(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)-\delta\right\}^{-1} - 1\right| < \frac{1}{\rho} - 1.$$

Therefore, $f \in \mathcal{MS}_\rho(n, \alpha, \delta)$.

Proof. Consider the function ω such that

$$\omega(z) = \frac{1}{\rho - 1}\left(\rho \frac{p(z)}{zp'(z)} - 1\right) - 1, \qquad (z \in E).$$

Then clearly $\omega(0) = 0$ and ω is analytic in E. We prove that ω satisfies the condition $|\omega(z)| < 1$ in E. By the above definition for ω, we write

$$\frac{p(z)}{zp'(z)} = \frac{\rho - 1}{\rho}\omega(z) + 1 = \frac{(\rho-1)\,\omega(z) + \rho}{\rho}.$$

On logarithmic differentiation of above equation, we write

$$\left|\frac{zp'(z)}{p(z)} - \frac{zp''(z)}{p'(z)} - 1\right| = |\rho - 1|\left|\frac{z\omega'(z)}{(\rho-1)\,\omega(z) + \rho}\right| < 1 - \rho.$$

Suppose that there exists a point $z_0 \in E$ such that

$$|z| \leqslant |z_0|\, |\omega(z)| = |\omega(z_0)| = 1. \tag{2.1}$$

Applying Lemma 1.2, we can write

$$\omega(z_0) = e^{i\theta} \text{ and } \frac{z_0\omega'(z_0)}{\omega(z_0)} = k, \quad (k \geqslant 1).$$

This implies that

$$\left|\frac{z_0p'(z_0)}{p(z_0)} - \frac{z_0p''(z_0)}{p'(z_0)} - 1\right| = |\rho - 1|\left|\frac{k}{\rho(1 + e^{-i\theta}) - 1}\right| \geqslant |\rho - 1|\left|\frac{1}{(\rho-1) + \rho e^{-i\theta}}\right|,$$

which proves that

$$\left|\frac{z_0p'(z_0)}{p(z_0)} - \frac{z_0p''(z_0)}{p'(z_0)} - 1\right|^2 \geqslant \frac{(1-\rho)^2}{(\rho-1)^2 + \rho^2 + 2\rho(\rho-1)\cos\theta}. \tag{2.2}$$

Since the right-hand side of (2.2) takes its minimum at $\cos\theta = -1$, so we have

$$\left|\frac{z_0p'(z_0)}{p(z_0)} - \frac{z_0p''(z_0)}{p'(z_0)} - 1\right| \geqslant 1 - \rho. \tag{2.3}$$

Thus (2.3) contradicts our assumption. Thus, there is no point $z_0 \in E$ which satisfies (2.1). This shows that $|\omega(z)| < 1$ for $z \in E$, which implies that

$$\left|\frac{p(z)}{zp'(z)} - \frac{1}{\rho}\right| < \frac{1}{\rho}, \qquad \left(\frac{1}{2} \leqslant \rho < 1, z \in E\right).$$

Hence, $f \in \mathcal{MS}_\rho(n, \alpha, \delta)$. □

For $f \in \mathcal{MC}_\rho(n, \alpha, \delta)$, we have the following theorem.

Theorem 2.2. *Let* $f \in \mathcal{A}_n$ *be an analytic function defined in the open unit disk* E *and let* p *be such that, as given by* (1.2) *and satisfy*

$$\left|\frac{zp''(z)}{p'(z)} - \frac{z\{2p''(z) + zp'''(z)\}}{p'(z) + zp''(z)}\right| < 1 - \rho, \qquad \left(\frac{1}{2} \leqslant \rho < 1, z \in E\right).$$

Then

$$\left|\frac{p'(z)}{p'(z) + zp''(z)} - 1\right| < \frac{1}{\rho} - 1,$$

and hence, $f \in \mathcal{MC}_\rho(n, \alpha, \delta)$.

Replacing $p(z)$ by $zp'(z)$ in Theorem 2.1, we obtain the required proof.

Theorem 2.3. *If* $f \in \mathcal{MS}_\rho(n, \alpha, \delta)$ *for* $\frac{1}{2} \leqslant \rho < 1$, *then*

$$\left|\left(\frac{z}{p(z)}\right)^\eta - 1\right| = \left|\left[\frac{f(z)}{z^{1-\alpha}}\left(\frac{f'(z)}{f(z)}\right)^\alpha\right]^{\frac{-\eta}{1-\delta}} - 1\right| < 1-\rho, \qquad (z \in E),$$

where p *is given by* (1.3), $0 \leqslant \rho < 1$ *and* $0 < \eta \leqslant 1-\rho$.

Proof. Let us define the function

$$\omega(z) = \frac{1}{1-\rho}\left\{\left(\frac{z}{p(z)}\right)^\eta - 1\right\} = \frac{1}{1-\rho}\left[\left\{\frac{f(z)}{z^{1-\alpha}}\left(\frac{f'(z)}{f(z)}\right)^\alpha\right\}^{-\frac{\eta}{1-\delta}} - 1\right], \qquad (z \in E). \tag{2.4}$$

Then clearly $\omega(0) = 0$ and ω is analytic in E. We want to prove that $|\omega(z)| < 1$ in E. Since

$$\left(\frac{z}{p(z)}\right)^\eta = (1-\rho)\,\omega(z) + 1, \qquad (z \in E), \tag{2.5}$$

so on differentiating (2.5) and in view of (2.4), we write

$$\begin{aligned}\eta\left(1 - \frac{zp'(z)}{p(z)}\right) &= \frac{(1-\rho)\,z\omega'(z)}{(1-\rho)\,\omega(z)+1} \\ &= \eta\left(1 - \frac{1}{1-\delta}\left\{(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - \delta\right\}\right).\end{aligned}$$

This implies that

$$\begin{aligned}\frac{zp'(z)}{p(z)} &= \frac{1}{1-\delta}\left\{(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - \delta\right\} \\ &= \frac{\eta(1-\rho)\,\omega(z) + \eta - (1-\rho)\,z\omega'(z)}{\eta(1-\rho)\,\omega(z)+\eta}, \qquad (z \in E),\end{aligned}$$

which can be written as

$$\begin{aligned}\left|\rho\frac{p(z)}{zp'(z)} - 1\right| &= \left|\rho(1-\delta)\left\{(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - \delta\right\}^{-1} - 1\right| \\ &= \left|\frac{(\rho-1)\left[(\eta(1-\rho)\,\omega(z)+\eta)\right] + (1-\rho)\,z\omega'(z)}{\eta(1-\rho)\,\omega(z) + \eta - (1-\rho)\,z\omega'(z)}\right|.\end{aligned}$$

To proceed further, we assume that there exists a point $z_0 \in E$ such that

$$|z| \leqslant |z_0|\,|\omega(z)| = |\omega(z_0)| = 1, \qquad (z \in E). \tag{2.6}$$

Applying Lemma 1.2, we have

$$\omega(z_0) = e^{i\theta} \quad \text{and} \quad \frac{z_0\omega'(z_0)}{\omega(z_0)} = k, \qquad (k \geqslant 1, z \in E).$$

This gives that

$$\left|\rho\frac{p(z_0)}{z_0p'(z_0)} - 1\right| = \left|\rho(1-\delta)\left\{(1-\alpha)\left(\frac{z_0f'(z_0)}{f(z_0)}\right) + \alpha\left(1 + \frac{zf''(z_0)}{f'(z_0)}\right) - \delta\right\}^{-1} - 1\right|$$

$$= \left| \frac{(\rho-1)\left[(\eta(1-\rho)\omega(z_0)+\eta)\right]+(1-\rho)z_0\omega'(z_0)}{\eta(1-\rho)\omega(z_0)+\eta-(1-\rho)z_0\omega'(z_0)} \right|,$$

or we can write

$$\left|\rho\frac{p(z_0)}{z_0p'(z_0)}-1\right|^2 = \left| \frac{(\rho-1)\left[(\eta(1-\rho)\omega(z_0)+\eta)\right]+(1-\rho)z_0\omega'(z_0)}{\eta(1-\rho)\omega(z_0)+\eta-(1-\rho)z_0\omega'(z_0)} \right|^2$$
$$= \left| \frac{(1-\rho)^2(\rho\eta+k-\eta)^2+\eta^2(\rho-1)^2+2\eta(1-\rho)(\rho-1)(\rho\eta+k-\eta)\cos\theta}{(1-\rho)^2(\eta-k)^2+\eta^2+2\eta(1-\rho)(\eta-k)\cos\theta} \right|.$$

We now define a function g for $\cos\theta = t$ such that

$$g(t) = \frac{(1-\rho)^2(\rho\eta+k-\eta)^2+\eta^2(\rho-1)^2+2\eta(1-\rho)(\rho-1)(\rho\eta+k-\eta)t}{(1-\rho)^2(\eta-k)^2+\eta^2+2\eta(1-\rho)(\eta-k)t}. \tag{2.7}$$

Taking derivative of (2.7) with respect to t, we see that $g'(t) > 0$, for

$$2\rho\eta k(1-\rho)\{\eta^2(\rho-1)-(1-\rho)^2(\eta-k)(\rho\eta+k-\eta)\}$$
$$= 2\rho\eta k(1-\rho)\left\{\eta^2(\rho-1)-(1-\rho)^2(\eta-k)\eta(\rho-1)-(1-\rho)^2(\eta-k)k\right\} > 0,$$

because $\eta-k<0$ for $0\leqslant\rho<1, 0\leqslant\eta<1-\rho$, $k\geqslant 1$. Therefore, g is monotonically increasing for t, when $\frac{1}{2}\leqslant\rho<1$ and

$$g(t) \geqslant g(-1) = \frac{(1-\rho)(\rho+k-\eta)-\eta(\rho-1)}{\eta-(1-\rho)(\eta-k)} = 1+\frac{\rho(1-\eta-\rho)}{(1-\rho)(k-\eta)+\eta} \geqslant 1.$$

This contradicts the condition that $f \in \mathcal{MS}_\rho(n,\alpha,\delta)$. Thus, there does not exist a point $z_0 \in E$ which satisfies (2.6). This shows that

$$\left|\left(\frac{z}{p(z)}\right)^\eta - 1\right| < 1-\rho, \qquad (z\in E),$$

which completes the desired proof. □

For $f \in \mathcal{MC}_\rho(n,\alpha,\delta)$, we have the following theorem.

Theorem 2.4. *If $f \in \mathcal{MC}_\rho(n,\alpha,\delta)$ for $\frac{1}{2}\leqslant\delta<1$, then*

$$\left|\left(\frac{1}{p'(z)}\right)^\eta - 1\right| < 1-\rho, \qquad (z\in E),$$

where p is given by (1.3), $0\leqslant\rho<1$ and $0<\eta\leqslant 1-\rho$.

Replacing $p(z)$ by $zp'(z)$ in Theorem 2.3, we obtain the required proof.

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References

[1] H. Al-Amiri, P. T. Mocanu, *Some simple criteria of starlikeness and convexity for meromorphic functions*, Mathematica, **37** (1995), 11–21. 1, 1

[2] M. Arif, I. Ahmad, M. Raza, K. Khan, *Sufficient condition of a subclass of analytic functions defined by Hadamard product*, Life Sci. J., **9** (2012), 2487–2489.

[3] J. Dziok, *Applications of the Jack lemma*, Acta Math. Hungar., **105** (2004), 93–102.

[4] B. A. Frasin, *Some sufficient conditions for certain integral operators*, J. Math. Inequal., **2** (2008), 527–535.

[5] A. W. Goodman, *Univalent Functions*, Vol: I & II, Mariner Publishing Co., Tampa, (1983). 1, 1

[6] I. S. Jack, *Functions starlike and convex of order* α, J. London Math. Soc. (2), **3** (1971), 469–474. 1

[7] S. S. Miller, P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., **65** (1978), 289–305. 1, 1

[8] P. T. Mocanu, *Some starlikeness conditions for analytic functions*, Rev. Roumaine Math. Pures Appl., **33** (1988), 117–124.

[9] M. Nunokawa, S. Owa, Y. Polatoglu, M. Çaglar, E. Yavuz Duman, *Some sufficient conditions for starlikeness and convexity*, Turkish J. Math., **34** (2010), 333–337. 1, 1

[10] K. Ochiai, S. Owa, M. Acu, *Applications of Jack's lemma for certain subclasses of analytic functions*, Gen. Math., **13** (2005), 73–82. 1

[11] J. A. Pfaltzgraff, M. O. Reade, T. Umezawa, *Sufficient conditions for univalence*, Ann. Fac. Sci. Univ. Nat. Zaïre (Kinshasa) Sect. Math.-Phys., **2** (1976), 211–218. 1

[12] H. Shiraishi, S. Owa, *Starlikeness and convexity for analytic functions concerned with Jack's Lemma*, Int. J. Open Probl. Comput. Sci. Math., **2** (2009), 37–47. 1

[13] R. Singh, S. Singh, *Some sufficient conditions for univalence and starlikeness*, Colloq. Math., **47** (1982), 309–314. 1