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Fixed point theorems for rational type (α-Θ)-contractions in controlled metric spaces

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Abstract

This paper aims to define rational type (α-Θ)-contraction in controlled metric space and obtain some advanced fixed point theorems. The outcomes generalize and extend various famous results in the literature. An example and certain consequences are presented to illustrate the significance of established results.

Keywords: Fixed point, rational type (α-Θ)-contraction, controlled metric spaces.

2010 MSC: 54H25, 47H10.

1. Introduction

In 1906 Frechet provided the conception of metric space which was an axiomatic development in functional analysis. Due to its simplicity, it has been generalized by several researchers[1–19] in the recent past.

Czerwik [9] defined the conception of b-metric space in this way.

Definition 1.1 ([9]). Let $\mathcal{W} \neq \emptyset$ and $b \geqslant 1$. A mapping $\kappa_b : \mathcal{W} \times \mathcal{W} \to [0,\infty)$ is called a b-metric if these assertions hold:

(b1) $\kappa_b(\iota, s) = 0 \Longleftrightarrow \iota = s$;
(b2) $\kappa_b(\iota, s) = \kappa_b(s, \iota)$ for all $\iota, s \in \mathcal{W}$;
(b3) $\kappa_b(\iota, z) \leqslant b[\kappa_b(\iota, s) + \kappa_b(s, z)]$ for all $\iota, s, z \in \mathcal{W}$.

Then $(\mathcal{W}, \kappa_b)$ is called a b-metric space (b-MS).

Kamran et al. [14] gave the notion of extended b-metric spaces in 2017.

Definition 1.2. Let $\mathcal{W} \neq \emptyset$, $p : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ and $\kappa_e : \mathcal{W} \times \mathcal{W} \to [0, \infty)$. Then κ_e is called an extended b-metric if

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(i) $\kappa_e(\iota, s) = 0$ if and only if $\iota = s$;
(ii) $\kappa_e(\iota, s) = \kappa_e(s, \iota)$;
(iii) $\kappa_e(\iota, s) \leqslant p(\iota, s)[\kappa_e(\iota, z) + \kappa_e(s, z)]$

for all $\iota, s, z \in \mathcal{W}$. Then $(\mathcal{W}, \kappa_e)$ is said to be an extended b-metric space (Eb-MS).

Currently, a new type of a space was given by Mlaiki et al. [16].

Definition 1.3. Let $\mathcal{W} \neq \emptyset$, $p : \mathcal{W} \times \mathcal{W} \to [1, \infty)$, and $\kappa_p : \mathcal{W} \times \mathcal{W} \to [0, \infty)$. Then κ_p is said to be a controlled metric if

(i) $\kappa_p(\iota, s) = 0$ if and only if $\iota = s$;
(ii) $\kappa_p(\iota, s) = \kappa_p(s, \iota)$;
(iii) $\kappa_p(\iota, s) \leqslant p(\iota, z)\kappa_p(\iota, z) + p(s, z)\kappa_p(s, z)$

for all $\iota, s, z \in \mathcal{W}$. The pair $(\mathcal{W}, (\kappa_p)$ is called a controlled metric space (CMS).

In 2015, Samet et al. [13] gave the notion of Θ-contraction in this way.

Definition 1.4. Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying:

(Θ_1) Θ is nondecreasing;
(Θ_2) for each sequence $\{\iota_n\} \subseteq \iota^+$, $\lim_{n\to\infty} \Theta(\iota_n) = 1 \Leftrightarrow \lim_{n\to\infty}(\iota_n) = 0$;
(Θ_3) $\exists\, 0 < h < 1$ and $\vartheta \in (0, \infty]$ such that $\lim_{\iota\to 0^+} \frac{\Theta(\iota)-1}{\iota^h} = \vartheta$.

A self mapping $\mathcal{O} : \mathcal{W} \to \mathcal{W}$ is called Θ-contraction if $\exists\, \Theta$ satisfying (Θ_1)-(Θ_3) and $k \in (0, 1)$ such that

$$\kappa(\mathcal{O}\iota, \mathcal{O}s) \neq 0 \Longrightarrow \Theta(\kappa(\mathcal{O}\iota, \mathcal{O}s)) \leqslant [\Theta(\kappa(\iota, s))]^k$$

for all $\iota, s \in \mathcal{W}$.

Theorem 1.5 ([13])**.** *If $\mathcal{O} : \mathcal{W} \to \mathcal{W}$ be a Θ-contraction on complete metric space $(\mathcal{W}, \kappa)$, then $\exists \iota^* \in \mathcal{W}$ such that $\iota^* = \mathcal{O}\iota^*$.*

We represent by the Ω, the family of all above mapping Θ satisfying the above assertions (Θ_1)-(Θ_3) to be consistent with Samet et al. [13].

In this paper, we define rational type (α, Θ)-contraction in the setting of complete CMS to obtain some generalized results.

2. Main results

Definition 2.1. Let $(\mathcal{W}, \kappa_p)$ be a CMS. A function $\mathcal{O} : \mathcal{W} \to \mathcal{W}$ is said to be a rational type (α, Θ)-contraction if $\exists\, \alpha : \mathcal{W} \times \mathcal{W} \to \mathbb{R}^+, k \in (0, 1)$ and $\Theta \in \Omega$ such that

$$\alpha(\iota, s)\Theta(\kappa_p(\mathcal{O}\iota, \mathcal{O}s)) \leqslant \Theta(M(\iota, s))^k, \tag{2.1}$$

where

$$M(\iota, s) = \max\left\{\kappa_p(\iota, s), \kappa_p(\iota, \mathcal{O}\iota), \kappa_p(s, \mathcal{O}s), \frac{\kappa_p(\iota, \mathcal{O}\iota)\kappa_p(s, \mathcal{O}s)}{1 + \kappa_p(\iota, s)}\right\}, \tag{2.2}$$

$\forall\, \iota, s \in \mathcal{W}$ with $\kappa_p(\mathcal{O}\iota, \mathcal{O}s) > 0$.

From now onward, we consider $(\mathcal{W}, \kappa_p)$ as complete controlled metric space.

Theorem 2.2. *Let $(\mathcal{W}, \kappa_p)$ is a complete CMS and $\mathcal{O} : \mathcal{W} \to \mathcal{W}$ be a rational type (α, Θ)-contraction such that:*

(i) *$\mathcal{O}$ is α-admissible;*
(ii) *$\exists\, \iota_0 \in \mathcal{W}$ such that $\alpha(\iota_0, \mathcal{O}\iota_0) \geqslant 1$;*

(iii) $\mathcal{O}$ *is continuous;*

(iv) $\sup_{m\geqslant 1} \lim_{i\to\infty} \frac{p(\iota_{i+1},\iota_{i+2})p(\iota_{i+1},\iota_m)}{p(\iota_i,\iota_{i+1})} < 1$.

In addition, assume that, for every $\iota \in \mathcal{W}$, *we have* $\lim_{n\to\infty} p(\iota_n, \iota)$ *and* $\lim_{n\to\infty} p(\iota, \iota_n)$ *exist and are finite. Then,* $\exists \iota^* \in \mathcal{W}$ *such that* $\iota^* = \mathcal{O}\iota^*$.

Proof. Suppose $\iota_0 \in \mathcal{W}$ is such that $\alpha(\iota_0, \mathcal{O}\iota_0) \geqslant 1$. We generate $\{\iota_n\}$ in $\mathcal{W}$ by $\iota_{n+1} = \mathcal{O}\iota_n$, $\forall\, n \in \mathbb{N}$. Obviously, if $\exists\, n_0 \in \mathbb{N}$ for which $\iota_{n_0+1} = \iota_{n_0}$, then $\mathcal{O}\iota_{n_0} = \iota_{n_0}$ and the proof is finished. Thus, we assume that $\iota_{n+1} \neq \iota_n$, $\forall n \in \mathbb{N}$. By using (i) and (ii), it is obvious that

$$\alpha(\iota_n, \iota_{n+1}) \geqslant 1,$$

$\forall\, n \in \mathbb{N}$. Thus by (2.1), we obtain

$$1 < \Theta(\kappa_p(\iota_n, \iota_{n+1})) = \Theta(\kappa_p(\mathcal{O}\iota_{n-1}, \mathcal{O}\iota_n)) \leqslant \alpha(\iota_n, \iota_{n+1})\Theta(\kappa_p(\mathcal{O}\iota_{n-1}, \mathcal{O}\iota_n)).$$

Since $\mathcal{O}$ is a rational type (α, Θ)-contraction, so $\forall n \in \mathbb{N}$, we can write

$$\begin{aligned}
1 &< \Theta(\kappa_p(\iota_n, \iota_{n+1})) \leqslant \alpha(\iota_n, \iota_{n+1})\Theta(\kappa_p(\mathcal{O}\iota_{n-1}, \mathcal{O}\iota_n)) \\
&\leqslant \Theta(M(\iota_{n-1}, \iota_n))^k \\
&= \Theta\left(\max\left\{\kappa_p(\iota_{n-1}, \iota_n), \kappa_p(\iota_{n-1}, \mathcal{O}\iota_{n-1}), \kappa_p(\iota_n, \mathcal{O}\iota_n), \frac{\kappa_p(\iota_{n-1}, \mathcal{O}\iota_{n-1})\kappa_p(\iota_n, \mathcal{O}\iota_n)}{1+\kappa_p(\iota_{n-1}, \iota_n)}\right\}\right)^k \\
&= \Theta\left(\max\left\{\kappa_p(\iota_{n-1}, \iota_n), \kappa_p(\iota_{n-1}, \iota_n), \kappa_p(\iota_n, \iota_{n+1}), \frac{\kappa_p(\iota_{n-1}, \iota_n)\kappa_p(\iota_n, \iota_{n+1})}{1+\kappa_p(\iota_{n-1}, \iota_n)}\right\}\right)^k \\
&\leqslant \Theta\left(\max\{\kappa_p(\iota_{n-1}, \iota_n), \kappa_p(\iota_{n-1}, \iota_n), \kappa_p(\iota_n, \iota_{n+1}), \kappa_p(\iota_n, \iota_{n+1})\}\right)^k \\
&= \Theta\left(\max\{\kappa_p(\iota_{n-1}, \iota_n), \kappa_p(\iota_n, \iota_{n+1})\}\right)^k.
\end{aligned}$$

Thus

$$1 < \Theta(\kappa_p(\iota_n, \iota_{n+1})) \leqslant \Theta\left(\max\{\kappa_p(\iota_{n-1}, \iota_n), \kappa_p(\iota_n, \iota_{n+1})\}\right)^k. \tag{2.3}$$

If there exists $n \in \mathbb{N}$ such that $\Theta\left(\max\{\kappa_p(\iota_{n-1}, \iota_n), \kappa_p(\iota_n, \iota_{n+1})\}\right)^k = \kappa_p(\iota_n, \iota_{n+1})$, then (2.3) becomes

$$1 < \Theta(\kappa_p(\iota_n, \iota_{n+1})) \leqslant \Theta\left(\kappa_p(\iota_n, \iota_{n+1})\right)^k < \Theta(\kappa_p(\iota_n, \iota_{n+1})),$$

which is a contradiction. Therefore $\Theta\left(\max\{\kappa_p(\iota_{n-1}, \iota_n), \kappa_p(\iota_n, \iota_{n+1})\}\right)^k = \kappa_p(\iota_{n-1}, \iota_n)$, for all $n \in \mathbb{N}$. Thus from (2.3), we get

$$\begin{aligned}
1 < \Theta(\kappa_p(\iota_n, \iota_{n+1})) &\leqslant \Theta\left(\kappa_p(\iota_n, \iota_{n+1})\right)^k \\
&\leqslant \Theta\left(\kappa_p(\iota_{n-1}, \iota_n)\right)^{k^2} \\
&\leqslant \Theta\left(\kappa_p(\iota_{n-2}, \iota_{n-1})\right)^{k^3} \\
&\;\;\vdots \\
&\leqslant \Theta\left(\kappa_p(\iota_0, \iota_1)\right)^{k^n}.
\end{aligned}$$

Thus by (2.3), we have

$$1 < \Theta(\kappa_p(\iota_n, \iota_{n+1})) \leqslant \Theta\left(\kappa_p(\iota_0, \iota_1)\right)^{k^n}. \tag{2.4}$$

Taking $n \to \infty$ in (2.4), we get

$$\lim_{n\to\infty} \Theta(\kappa_p(\iota_n, \iota_{n+1})) = 1.$$

By (Θ_2), we get

$$\lim_{n\to\infty} \kappa_p(\iota_n, \iota_{n+1}) = 0.$$

By (Θ_3), $\exists\, 0 < h < 1$ and $\vartheta \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\Theta(\kappa_p(\mathcal{O}\iota_n, \mathcal{O}\iota_{n+1})) - 1}{\kappa_p(\iota_n, \iota_{n+1})^h} = \vartheta.$$

Assume that $\vartheta < \infty$. In this case, let $\lambda = \frac{\vartheta}{2} > 0$. By definition, $\exists\, n_1 \in \mathbb{N}$ so that

$$\left|\frac{\Theta(\kappa_p(\mathcal{O}\iota_n, \mathcal{O}\iota_{n+1})) - 1}{\kappa_p(\iota_n, \iota_{n+1})^h} - \vartheta\right| \leqslant \lambda, \quad \forall n > n_1.$$

This implies that

$$\frac{\Theta(\kappa_p(\mathcal{O}\iota_n, \mathcal{O}\iota_{n+1})) - 1}{\kappa_p(\iota_n, \iota_{n+1})^h} \geqslant \vartheta - \lambda = \frac{\vartheta}{2} = \lambda, \quad \forall n > n_1.$$

Then

$$n\kappa_p(\iota_n, \iota_{n+1})^h \leqslant \mu n[\Theta(\kappa_p(\mathcal{O}\iota_n, \mathcal{O}\iota_{n+1})) - 1], \quad \forall n > n_1,$$

with $\mu = \frac{1}{\lambda}$. Now we assume that $\vartheta = \infty$ and $\lambda > 0$. By definition, $\exists\, n_1 \in \mathbb{N}$ so that

$$\lambda \leqslant \frac{\Theta(\kappa_p(\mathcal{O}\iota_n, \mathcal{O}\iota_{n+1})) - 1}{\kappa_p(\iota_n, \iota_{n+1})^h}, \quad \forall n > n_1.$$

This implies that

$$n\kappa_p(\iota_n, \iota_{n+1})^h \leqslant \mu n[\Theta(\kappa_p(\mathcal{O}\iota_n, \mathcal{O}\iota_{n+1})) - 1], \quad \forall n > n_1,$$

where $\mu = \frac{1}{\lambda}$. Hence, in all situation, $\exists\, \mu > 0$ and $n_1 \in \mathbb{N}$ so that

$$n\kappa_p(\iota_n, \iota_{n+1})^h \leqslant \mu n[\Theta(\kappa_p(\mathcal{O}\iota_n, \mathcal{O}\iota_{n+1})) - 1] \tag{2.5}$$

for all $n > n_1$. Thus by (2.4) and (2.5), we get

$$n\kappa_p(\iota_n, \iota_{n+1})^h \leqslant \mu n([(\Theta(\kappa_p(\iota_0, \iota_1))]^{k^n} - 1).$$

Taking $n \to \infty$, we obtain

$$\lim_{n\to\infty} n\kappa_p(\iota_n, \iota_{n+1})^h = 0.$$

Hence, $\exists\, n_2 \in \mathbb{N}$ so that

$$\kappa_p(\iota_n, \iota_{n+1}) \leqslant \frac{1}{n^{1/h}}, \quad \forall n > n_2.$$

Consider the triangle inequality for $q \geqslant 1$, we have

$$\begin{aligned}
\kappa_p(\iota_n, \iota_{n+q}) &\leqslant p(\iota_n, \iota_{n+1})\kappa_p(\iota_n, \iota_{n+1}) + p(\iota_{n+1}, \iota_{n+q})\kappa_p(\iota_{n+1}, \iota_{n+q}) \\
&\leqslant p(\iota_n, \iota_{n+1})\kappa_p(\iota_n, \iota_{n+1}) + p(\iota_{n+1}, \iota_{n+q})p(\iota_{n+1}, \iota_{n+2})\kappa_p(\iota_{n+1}, \iota_{n+2}) \\
&\quad + p(\iota_{n+1}, \iota_{n+q})p(\iota_{n+2}, \iota_{n+q})\kappa_p(\iota_{n+2}, \iota_{n+q}) \\
&\leqslant p(\iota_n, \iota_{n+1})\kappa_p(\iota_n, \iota_{n+1}) + p(\iota_{n+1}, \iota_{n+q})p(\iota_{n+1}, \iota_{n+2})\kappa_p(\iota_{n+1}, \iota_{n+2}) \\
&\quad + p(\iota_{n+1}, \iota_{n+q})p(\iota_{n+2}, \iota_{n+q})p(\iota_{n+2}, \iota_{n+3})\kappa_p(\iota_{n+2}, \iota_{n+3}) \\
&\quad + p(\iota_{n+1}, \iota_{n+q})p(\iota_{n+2}, \iota_{n+q})p(\iota_{n+3}, \iota_{n+q})\kappa_p(\iota_{n+3}, \iota_{n+q}) \\
&\;\;\vdots \\
&\leqslant p(\iota_n, \iota_{n+1})\kappa_p(\iota_n, \iota_{n+1}) + \sum_{i=n+1}^{n+q-2} \left(\prod_{j=n+1}^{i} p(\iota_j, \iota_{n+q}) \right) p(\iota_i, \iota_{i+1})\kappa_p(\iota_i, \iota_{i+1})
\end{aligned}$$

$$+\prod_{i=n+1}^{n+q-1} p(\iota_i,\iota_{n+q})\kappa_p(\iota_{n+q-1},\iota_{n+q}).$$

which further implies that

$$\begin{aligned}\kappa_p(\iota_n,\iota_{n+q}) &\leqslant p(\iota_n,\iota_{n+1})\kappa_p(\iota_n,\iota_{n+1})+\sum_{i=n+1}^{n+q-2}\left(\prod_{j=n+1}^{i} p(\iota_j,\iota_{n+q})\right)p(\iota_i,\iota_{i+1})\kappa_p(\iota_i,\iota_{i+1})\\ &\quad+\left(\prod_{i=n+1}^{n+q-1} p(\iota_i,\iota_{n+q})\right)p(\iota_{n+q-1},\iota_{n+q})\kappa_p(\iota_{n+q-1},\iota_{n+q})\\ &= p(\iota_n,\iota_{n+1})\kappa_p(\iota_n,\iota_{n+1})+\sum_{i=n+1}^{n+q-1}\left(\prod_{j=n+1}^{i} p(\iota_j,\iota_{n+q})\right)p(\iota_i,\iota_{i+1})\kappa_p(\iota_i,\iota_{i+1})\\ &\leqslant p(\iota_n,\iota_{n+1})\kappa_p(\iota_n,\iota_{n+1})+\sum_{i=n+1}^{n+q-1}\left(\prod_{j=0}^{i} p(\iota_j,\iota_{n+q})\right)p(\iota_i,\iota_{i+1})\kappa_p(\iota_i,\iota_{i+1})\\ &\leqslant p(\iota_n,\iota_{n+1})\kappa_p(\iota_n,\iota_{n+1})+\sum_{i=n+1}^{n+q-1}\left(\prod_{j=0}^{i} p(\iota_j,\iota_{n+q})\right)p(\iota_i,\iota_{i+1})\frac{1}{i^{\frac{1}{k}}}.\end{aligned}$$

Thus

$$\kappa_p(\iota_n,\iota_{n+q}) \leqslant p(\iota_n,\iota_{n+1})\kappa_p(\iota_n,\iota_{n+1})+\sum_{i=n+1}^{n+q-1}\left(\prod_{j=0}^{i} p(\iota_j,\iota_{n+q})\right)p(\iota_i,\iota_{i+1})\frac{1}{i^{\frac{1}{k}}}. \tag{2.6}$$

Now, consider

$$\begin{aligned}\sum_{i=n+1}^{n+q-1}\left(\prod_{j=0}^{i} p(\iota_j,\iota_{n+q})\right)p(\iota_i,\iota_{i+1})\frac{1}{n^{\frac{1}{k}}} &= \sum_{i=n+1}^{n+q-1}\frac{1}{i^{\frac{1}{k}}}\left(\prod_{j=0}^{i} p(\iota_j,\iota_{n+q})\right)p(\iota_i,\iota_{i+1})\\ &\leqslant \sum_{i=n+1}^{\infty}\frac{1}{i^{\frac{1}{k}}}\left(\prod_{j=0}^{i} p(\iota_j,\iota_{n+q})\right)p(\iota_i,\iota_{i+1}) = \sum_{i=n+1}^{\infty} U_iV_i,\end{aligned}$$

where

$$U_i = \frac{1}{i^{\frac{1}{k}}}$$

and

$$V_i = \left(\prod_{j=0}^{i} p(\iota_j,\iota_{n+q})\right)p(\iota_i,\iota_{i+1}).$$

Since $\frac{1}{k} > 0$, $\sum\limits_{i=n+1}^{\infty}\frac{1}{i^{\frac{1}{k}}}$ converges and also $V_i = \left(\prod\limits_{j=0}^{i} p(\iota_j,\iota_{n+q})\right)p(\iota_i,\iota_{i+1})$ is increasing and bounded above. Thus $\lim_{i\to\infty}\{V_i\} = \sup(V_i)$, exists and is non zero. Hence, the product $\left(\prod\limits_{j=0}^{i} p(\iota_j,\iota_{n+q})\right)p(\iota_i,\iota_{i+1})$ converges. Thus $\sum\limits_{i=n+1}^{\infty} U_iV_i$ converges. Let us consider the partial sum

$$S_q = \sum_{i=0}^{q}\left(\prod_{j=0}^{i} p(\iota_j,\iota_{n+q})\right)p(\iota_i,\iota_{i+1})\frac{1}{i^{\frac{1}{k}}}.$$

Now from (2.6), we have

$$\kappa_p(\iota_n, \iota_{n+q}) \leqslant p(\iota_n, \iota_{n+1})\kappa_p(\iota_n, \iota_{n+1}) + (S_{n+q-1} - S_n). \tag{2.7}$$

By ratio test and using the condition (2.2), we get guaranty of existence of $\lim_{n\to\infty} S_n$. Hence $\{S_n\}$ is Cauchy. Now taking $n \to +\infty$ in (2.7), we have

$$\lim_{n\to\infty} \kappa_p(\iota_n, \iota_{n+q}) = 0,$$

that is, $\{\iota_n\}$ is a Cauchy sequence in $(\mathcal{W}, \kappa_p)$, so $\{\iota_n\}$ converges to some $u \in \mathcal{W}$. Now we prove that $u = \mathcal{O}u$. Since $\iota_n \to u$ as $n \to \infty$ and the mapping $\mathcal{O}$ is continuous, so we have $\mathcal{O}\iota_n \to \mathcal{O}u$ as $n \to \infty$. Thus we have

$$\kappa_p(u, \mathcal{O}u) = \lim_{n\to\infty} \kappa_p(\iota_{n+1}, \mathcal{O}u) \lim_{n\to\infty} \kappa_p(\mathcal{O}\iota_n, \mathcal{O}u) = 0,$$

and hence $u = \mathcal{O}u$. □

Theorem 2.3. *Let $(\mathcal{W}, \kappa_e)$ be a complete Eb-MS and $\mathcal{O} : \mathcal{W} \to \mathcal{W}$. Assume that $\exists k \in (0, 1)$ and $\Theta \in \Omega$ such that*

$$\Theta(\kappa_p(\mathcal{O}\iota, \mathcal{O}s)) \leqslant \Theta(M(\iota, s))^k,$$

where

$$M(\iota, s) = \max\left\{\kappa_p(\iota, s), \kappa_p(\iota, \mathcal{O}\iota), \kappa_p(s, \mathcal{O}s), \frac{\kappa_p(\iota, \mathcal{O}\iota)\kappa_p(s, \mathcal{O}s)}{1 + \kappa_p(\iota, s)}\right\}, \quad \forall \iota, s \in \mathcal{W}$$

with $\kappa_p(\mathcal{O}\iota, \mathcal{O}s) > 0$. Suppose that these assertions also hold

(i) *$\mathcal{O}$ is continuous;*
(ii) $\sup_{m\geqslant 1} \lim_{i\to\infty} \frac{p(\iota_{i+1},\iota_{i+2})p(\iota_{i+1},\iota_m)}{p(\iota_i,\iota_{i+1})} < 1$.

In addition, assume that, for every $\iota \in \mathcal{W}$, we have $\lim_{n\to\infty} p(\iota_n, \iota)$ and $\lim_{n\to\infty} p(\iota, \iota_n)$ exist and are finite. Then, $\exists \iota^* \in \mathcal{W}$ such that $\iota^* = \mathcal{O}\iota^*$.

Corollary 2.4. *Let $(\mathcal{W}, \kappa_e)$ be a complete Eb-MS and $\mathcal{O} : \mathcal{W} \to \mathcal{W}$ be a rational type (α, Θ)-contraction such that:*

(i) *$\mathcal{O}$ is α-admissible;*
(ii) *$\exists\, \iota_0 \in \mathcal{W}$ such that $\alpha(\iota_0, \mathcal{O}\iota_0) \geqslant 1$;*
(iii) *$\mathcal{O}$ is continuous;*
(iv) $\sup_{m\geqslant 1} \lim_{i\to\infty} \frac{p(\iota_{i+1},\iota_{i+2})p(\iota_{i+1},\iota_m)}{p(\iota_i,\iota_{i+1})} < 1$.

In addition, assume that, for every $\iota \in \mathcal{W}$, we have $\lim_{n\to\infty} p(\iota_n, \iota)$ and $\lim_{n\to\infty} p(\iota, \iota_n)$ exist and are finite. Then, $\exists \iota^* \in \mathcal{W}$ such that $\iota^* = \mathcal{O}\iota^*$.

Proof. If we take $p(\iota, z) = p(z, s)$ in above Theorem 2.2, we get the conclusion. □

Corollary 2.5. *Let $(\mathcal{W}, \kappa_b)$ be a complete b- MS and $\mathcal{O} : \mathcal{W} \to \mathcal{W}$ be a rational type (α, Θ)-contraction such that:*

(i) *$\mathcal{O}$ is α-admissible;*
(ii) *$\exists\, \iota_0 \in \mathcal{W}$ such that $\alpha(\iota_0, \mathcal{O}\iota_0) \geqslant 1$;*
(iii) *$\mathcal{O}$ is continuous.*

Then, $\exists u \in \mathcal{W}$ such that $\mathcal{O}u = u$.

Proof. If we take $p(\iota, z) = p(z, s) = b \geqslant 1$ in above Theorem 2.2. □

Corollary 2.6. *Let $(\mathcal{W}, \kappa)$ be a complete metric space and $\mathcal{O} : \mathcal{W} \to \mathcal{W}$ be a rational type (α, Θ)-contraction such that:*

(i) *$\mathcal{O}$ is α-admissible;*
(ii) *$\exists\ \iota_0 \in \mathcal{W}$ such that $\alpha(\iota_0, \mathcal{O}\iota_0) \geqslant 1$;*
(iii) *$\mathcal{O}$ is continuous.*

Then, $\exists u \in \mathcal{W}$ such that $\mathcal{O}u = u$.

Proof. If we take $p(\iota, z) = p(z, s) = 1$ in above Theorem 2.2. □

Example 2.7. Let $\mathcal{W} = \{0, 1, 2\}$. Define $p : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ and $\kappa_p : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ as $p(\iota, s) = 1 + \iota s$ and

$$\begin{aligned}
&\kappa_p(2,2) = \kappa_p(0,0) = \kappa_p(1,1) = 0,\\
&\kappa_p(2,0) = \kappa_p(0,2) = 5, \kappa_p(1,0) = \kappa_p(0,1) = 10,\\
&\kappa_p(1,2) = \kappa_p(2,1) = 30.
\end{aligned}$$

Now, define

$$\mathcal{O} : \mathcal{W} \to \mathcal{W}$$

by

$$\mathcal{O}\iota = \begin{cases} 0, & \text{if } \iota \in \{0, 2\},\\ 2, & \text{if } \iota = 1, \end{cases}$$

and choose $k = \frac{3}{4}$. Define $\Theta(\beta) = e^{\sqrt{\beta}}$. Now we discuss various cases to prove the assumptions of our main result.

Case 01: If $\iota = 0, s = 1$, we have

$$\begin{aligned}
\Theta(\kappa_p(\mathcal{O}0, \mathcal{O}1)) = \Theta(\kappa_p(0,2)) &= e^{\sqrt{5}} < (e^{\sqrt{10}})^{\frac{3}{4}}\\
&= \left[\Theta\left(\max\left\{\kappa_p(0,1), \kappa_p(0,\mathcal{O}0), \kappa_p(1,\mathcal{O}1), \frac{\kappa_p(0,\mathcal{O}0)\kappa_p(1,\mathcal{O}1)}{1+\kappa_p(0,1)}\right\}\right)\right]^{\frac{3}{4}}.
\end{aligned}$$

Case 02: If $\iota = 0, s = 2$, we have

$$\begin{aligned}
\Theta(\kappa_p(\mathcal{O}0, \mathcal{O}2)) = \Theta(\kappa_p(0,0)) &= e^{0}\\
&< \left[\Theta\left(\max\left\{\kappa_p(0,2), \kappa_p(0,\mathcal{O}0), \kappa_p(2,\mathcal{O}2), \frac{\kappa_p(0,\mathcal{O}0)\kappa_p(2,\mathcal{O}2)}{1+\kappa_p(0,2)}\right\}\right)\right]^{\frac{3}{4}}.
\end{aligned}$$

Case 03: If $\iota = 1, s = 2$, we have

$$\begin{aligned}
&\Theta(\kappa_p(\mathcal{O}1, \mathcal{O}2)) = \Theta(\kappa_p(2,0))\\
&\quad = e^{\sqrt{5}} < (e^{\sqrt{30}})^{\frac{3}{4}} < \left[\Theta\left(\max\left\{\kappa_p(1,2), \kappa_p(1,\mathcal{O}1), \kappa_p(2,\mathcal{O}2), \frac{\kappa_p(1,\mathcal{O}1)\kappa_p(2,\mathcal{O}2)}{1+\kappa_p(1,2)}\right\}\right)\right]^{\frac{3}{4}}.
\end{aligned}$$

Case 04: If $\iota = s = 0, \iota = s = 1, \iota = s = 2$, we have

$$\begin{aligned}
&\Theta(\kappa_p(\mathcal{O}\iota, \mathcal{O}s)) = \Theta(\kappa_p(0,0))\\
&\quad = e^{0} < \left[\Theta\left(\max\left\{\kappa_p(\iota,s), \kappa_p(\iota,\mathcal{O}\iota), \kappa_p(s,\mathcal{O}s), \frac{\kappa_p(\iota,\mathcal{O}\iota)\kappa_p(s,\mathcal{O}s)}{1+\kappa_p(\iota,s)}\right\}\right)\right]^{\frac{3}{4}}, \quad \forall s, \iota \in \mathcal{W}.
\end{aligned}$$

Hence, all the conditions of above theorem are satisfied and $\mathcal{O}$ has a unique fixed point, which is, $\iota = 0$.

We can establish variety of results as special cases of our main Theorem 2.2.

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