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On the solution of Wave-Schrodinger equation

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Abstract

In this paper, we are finding a solution of the fractional Wave-Schrodinger equation by Laplace transform in the sense of Caputo fractional derivative. It was found that the fundamental solution of the equation is related to Wright function.

Keywords: Dirac delta distribution, Laplacian operator, Wright function.

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1. Introduction

The Laplacian operator $\triangle^k$ iterated k-times is defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k,$$

where n is the dimension of space $\mathbb{R}^n$ and k is a non-negative integer. Kananthai [1] has proved that the generalized function $(-1)^k S_{2k}(x)$ is an elementary solution of the operator $\triangle^k$, that is

$$\triangle^k (-1)^k S_{2k}(x) = \delta,$$

where δ is the Dirac-delta distribution and $S_{2k}(x)$ is defined by

$$S_{2k}(x) = \frac{\pi^{-\frac{n}{2}} 2^{-2k} \Gamma\left(\frac{n-2k}{2}\right) \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{\frac{2k-n}{2}}}{\Gamma(k)}. \tag{1.1}$$

In 2002, Kananthai et al. [2] have first introduced the operator $\triangle_i^k$ and is defined by

$$\triangle_i^k = \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right), i = \sqrt{-1}.$$

They have proved the function $(-1)^k(-i)^{\frac{q}{2}}S_{2k}(x)$ is an elementary solution of the operator $\triangle_i^k$ and $S_{2k}(x)$ is defined by (1.1). It is well known that the linear Schrodinger equation can be written as the following form

$$\frac{\partial}{\partial t}u(x,t) = i\frac{\partial^2}{\partial x^2}u(x,t), i = \sqrt{-1}$$

with the initial condition

$$u(x,0) = f(x).$$

The Schrodinger equation has been widely in application in science and engineering, there are several integral transforms such as Laplace transform, Fourier transform, Wavelet transform, etc., for solving the equation.

The purpose of this work is to introduce a new function where related the Wright function [3, 4] and studied Laplace transform of a new function. After that, we are solving the fundamental solution of the Wave-Schrodinger equation as follows:

$$\frac{\partial^\alpha}{\partial t^\alpha}\phi(x,t) + i\frac{\partial^2}{\partial x^2}\phi(x,t) = 0 \ , \ i = \sqrt{-1} \ , \ 1 < \alpha \leqslant 2,$$

with the initial condition

$$\phi(x,0) = 0 \ , \ \phi_t(x,0) = \delta(x),$$

where δ is the Dirac-delta distribution and $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo derivative. Before going that point, the following definitions and some important concepts are needed.

2. Preliminaries

Definition 2.1. Let $f(t)$ be a function an exponential order and piecewise continuous. The Laplace transform of the function f is given by

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st}f(t)dt. \tag{2.1}$$

Definition 2.2. Let $f(t)$ be a function of the Schwart space, the Fourier transform of $f(t)$ is given by

$$\widehat{f}(w) = \int_{\mathbb{R}} f(t)e^{iwt}dt.$$

Definition 2.3. For m to be the smallest integer that exceeds α, the Caputo fractional derivatives of order α is defined by

$$D^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t(t-\tau)^{m-n-1}\frac{\partial^m}{\partial t^m}u(x,t), \\ \frac{\partial^m}{\partial t^m}u(x,t), \quad n = m. \end{cases}$$

Definition 2.4. The Laplace transform of the Caputo fractional derivative is defined by

$$\mathcal{L}[D^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}f^{(k)}(0) \ , \ n-1 < \alpha < n.$$

Definition 2.5. The Wright function $W_{\alpha,\beta}$ is defined by

$$W_{\alpha,\beta} = \sum_{n=0}^\infty \frac{Z^n}{n!\Gamma(n\alpha+\beta)} \ , \ \alpha > -1 \ , \ \beta \in \mathbb{C}, \tag{2.2}$$

where $\Gamma(x)$ is the Euler Gamma function given by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-6}dt.$$

Lemma 2.6. *The function $\gamma(a, t)$ is defined by the following expressions*

$$\gamma(a, t) = t^{-\alpha+1} W_{-\alpha, 2-\alpha}(at^{-\alpha})$$

and Laplace transform of $\gamma(a, t)$ is given by

$$\mathcal{L}[\gamma(a, t)] = s^{\alpha-2} e^{as^\alpha}.$$

Proof. By (2.1), we have

$$
\begin{aligned}
\mathcal{L}[\gamma(a, t)] &= \int_0^\infty e^{-st} t^{-\alpha+1} W_{-\alpha, 2-\alpha}(at^{-\alpha}) dt \\
&= \int_0^\infty e^{-st} t^{-\alpha+1} \sum_{k=0}^\infty \frac{(at^{-\alpha})^k}{k! \Gamma(-\alpha k + 2 - \alpha)} dt \\
&= \sum_{k=0}^\infty \frac{a^k}{k! \Gamma(-\alpha k + 2 - \alpha)} \int_0^\infty e^{-st} t^{-\alpha-\alpha k+1} dt \\
&= \sum_{k=0}^\infty \frac{a^k}{k! \Gamma(-\alpha k + 2 - \alpha)} \mathcal{L}[t^{-\alpha-\alpha k+1}] \\
&= \sum_{k=0}^\infty \frac{a^k}{k! \Gamma(-\alpha k + 2 - \alpha)} \frac{\Gamma(-\alpha - \alpha k + 2)}{s^{-\alpha-\alpha k+2}} \\
&= \sum_{k=0}^\infty \frac{a^k}{k! s^{-\alpha-\alpha k+2}} = s^{\alpha-2} \sum_{k=0}^\infty \frac{(as^\alpha)^k}{k!} = s^{\alpha-2} e^{as^\alpha}.
\end{aligned}
$$

That completes the proof. □

3. Main results

Theorem 3.1. *Consider the Fractional Wave-Schrodinger equation*

$$\frac{\partial^\alpha}{\partial t^\alpha} \phi(x, t) + i \frac{\partial^2}{\partial x^2} \phi(x, t) = 0, \quad i = \sqrt{-1}, \quad 1 < \alpha \leqslant 2 \tag{3.1}$$

with the initial condition

$$\phi(x, 0) = 0, \quad \phi_t(x, 0) = \delta(x),$$

where $\delta(x)$ is the dirac delta distribution. By the Laplace and Fourier transform we obtain that the fundamental solution of the equation (3.1) is given by

$$\phi(x, t) = \frac{1}{2} \sqrt{i} t^{-\frac{\alpha}{2}+1} W_{-\frac{\alpha}{2}, 2-\frac{\alpha}{2}}(-\sqrt{i}|x|t^{-\frac{\alpha}{2}}),$$

where $W_{\alpha,\beta}$ is the Wright function is defined by (2.2). If we put $\alpha = 2$ in (3.1) the fractional Wave-Schrodinger equation reduces to

$$\frac{\partial^2}{\partial t^2} \phi(x, t) + i \frac{\partial^2}{\partial x^2} \phi(x, t) = 0 \tag{3.2}$$

and the solution of (3.2) is given by

$$\phi(x, t) = \frac{1}{2} \sqrt{i} W_{-1,1}(-\sqrt{i}|x|t^{-1}).$$

Proof. By (3.1), we have

$$\frac{\partial^\alpha}{\partial t^\alpha}\phi(x,t) + i\frac{\partial^2}{\partial x^2}\phi(x,t) = 0. \tag{3.3}$$

Taking Laplace transform on both sides of (3.3) and and using definition 2.1 we get

$$\mathcal{L}\left[\frac{\partial^\alpha}{\partial t^\alpha}\phi(x,t)\right] + i\mathcal{L}\left[\frac{\partial^2}{\partial x^2}\phi(x,t)\right] = 0,$$
$$s^\alpha\phi(x,s) - s^{\alpha-2}\delta(x) = -i\frac{\partial^2}{\partial x^2}\phi(x,s). \tag{3.4}$$

Applying Fourier transform respect to variable x on both sides of (3.4), we obtain

$$s^\alpha\mathcal{F}\phi(x,s) - s^{\alpha-2}\mathcal{F}[\delta(x)] = -i\mathcal{F}\frac{\partial^2}{\partial x^2}\phi(x,s),$$
$$s^\alpha\phi(\omega,s) - s^{\alpha-2} = i\omega^2\phi(\omega,s),$$
$$\phi(\omega,s) = \frac{s^{\alpha-2}}{s^\alpha + (-i)\omega^2} = \frac{is^{\alpha-2}}{is^\alpha + \omega^2}. \tag{3.5}$$

Applying inverse Fourier transform on both sides of (3.5), we obtain

$$\phi(x,s) = \mathcal{F}^{-1}\left[\frac{is^{\alpha-2}}{is^\alpha + \omega^2}\right] = \frac{\sqrt{i}s^{\alpha-2}e^{-|x|\sqrt{i}s^{\frac{\alpha}{2}}}}{2s^{\frac{\alpha}{2}}} = \frac{1}{2}\sqrt{i}s^{\frac{\alpha}{2}-2}e^{-|x|\sqrt{i}s^{\frac{\alpha}{2}}}.$$

By Lemma 2.6, we obtain the solution of (3.1) as follows

$$\phi(x,t) = \frac{1}{2}\sqrt{i}r(-\sqrt{i}|x|,t) = \frac{1}{2}\sqrt{i}t^{-\frac{\alpha}{2}+1}W_{-\frac{\alpha}{2},2-\frac{\alpha}{2}}(-\sqrt{i}|x|t^{-\frac{\alpha}{2}}). \tag{3.6}$$

If we put $\alpha = 2$ in (3.1) and (3.6), respectively, the equation reduces to the Wave-Schrodinger equation

$$\frac{\partial^2}{\partial t^2}\phi(x,t) + i\frac{\partial^2}{\partial x^2}\phi(x,t) = 0, \tag{3.7}$$

and the solution of (3.7) is given by

$$\phi(x,t) = \frac{1}{2}\sqrt{i}W_{-1,1}(-\sqrt{i}|x|t^{-1}).$$

This completes the proof. □

References

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