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Existence of integrable solutions for integro-differential inclusions of fractional order; coupled system approach

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Abstract

In this article, we establish the existence of solutions for a functional integral equation of fractional order. The study upholds the case when the set-valued function has L^1-Caratheòdory selections, we reformulate the functional integral inclusion according to these selections via a classical fixed point theorem of Schauder and present theorem for the existence of integrable solutions. As an application, the existence of solutions of nonlinear functional integro-differential inclusion with an initial value, and the initial value problem for the arbitrary-order differential inclusion will be studied.

Keywords: Fractional calculus, integro-differential inclusion, L^1-Caratheòdory selections, Schauder fixed point principle, Kolmogorov compactness criterion.

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1. Introduction

The topic of differential and integral inclusions is of much interest in the subject of set-valued analysis. Differential equations and control processes, the existence theorems for the inclusions problems are generally obtained under the assumption that the set-valued function is either lower or upper semicontinuous on the domain of its definitions (see [2, 21]) and for the discontinuity of the set-valued function (see [8]).

Indeed set-valued functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 10–15, 18]).

In this paper we discuss the existence of integrable solutions to the following fractional order functional integral inclusion

$$x(t) \in F_1(t, I^{\alpha} f_2(t, x(t))), \quad \in [0, T], \tag{1.1}$$

where $\alpha \in (0, 1)$ and $F_1 : [0, T] \times R^+ \to P(R)$ is a set-valued mapping and $P(R)$ denotes the family of nonempty subsets of R under a set of several suitable assumptions on the function F_1.

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Our study is based on the selections of the set-valued function F_1 by reformulating the functional integral inclusion (1.1) into a coupled system. We present the existence of integrable solution under the assumption that a set-valued function F_1 has L^1-Carathèodory selection and with the classical Schauder fixed point principle and Kolmogorov compactness criterion.

As an application we study the existence of solutions of integro- differential inclusion of fractional order

$$x(t) \in \int_0^t F_1(s, I^{\alpha} f_2(s, x'(s)))\, ds, \quad t \in [0, T], \quad \text{with } x(0) = x_\circ. \tag{1.2}$$

Also, the initial-value problem for the arbitrary (fractional) order differential inclusion

$$\frac{dx(t)}{dt} \in F_1(t, D^{\beta} x(t)), \text{ a.e. } t \in (0, T], \quad \beta \in (0, 1], \quad x(0) = x_\circ, \tag{1.3}$$

where $F_1(t, x(t))$ L^1-Carathèodory set-valued function defined on $(0, T] \times R^+$ will be studied.

2. Preliminaries

In this section, we introduce notations, definitions and preliminary facts from set-valued analysis which are used throughout this paper. Denote by $L^1(I)$ the class of Lebesgue integrable functions on the interval $I = [0, T]$, endowed with the usual norm

$$\|x\| = \int_0^T |x(t)|\, dt.$$

Definition 2.1. The Riemann-Liouville of a fractional integral of the function $f \in L^1(I)$ of order $\alpha \in R^+$ is defined by

$$I_a^{\alpha} f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds$$

and when $a = 0$, we have $I^{\alpha} f(t) = I_0^{\alpha} f(t)$.

Definition 2.2. The (Caputo) fractional-order derivative D^{α}, $\alpha \in (0, 1]$ of the absolutely continuous function g is defined as

$$D_a^{\alpha} g(t) = I_a^{1-\alpha} \frac{d}{dt} g(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{ds} g(s) ds, \quad t \in [a, b].$$

For further properties of fractional calculus operator see [20, 22–24].

Definition 2.3. Let X and Y be two nonempty sets, a set-valued (multi-valued) map $F : X \to Y$ is a function that associates to any element $x \in X$ a subset $F(x)$ of Y, called the (image) valued of F at x.

Definition 2.4. Let F be a strict set-valued map (we say F is strict if the domain of F is X itself), f is called a selection of F if $f(x) \in F(x)$, for every $x \in X$, we denote by $S_F = \{f : f(x) \in F(x), x \in X\}$ the set of all selections of F (for the properties of the selection of F see [6, 10, 17, 19]).

Definition 2.5. A single-valued function $f : I \to R \times R$ is called L^1-Carathèodory if:

(1) $t \to f(t, x)$ is measurable in $t \in I$ for all $x \in R$;
(2) $x \to f(t, x)$ is continuous in $x \in R$ for almost all $t \in I$;
(3) there exists $h \in L^1(I)$ such that $|f(t, x)| \leqslant h(t)$ for almost all $t \in I$.

Theorem 2.6 ([7]). *Let $F : I \times R \to P(R)$ be an L^1-Carathèodory multi-function, the set $S_1 F(., x(.))$ is nonempty (i.e., there exists a selector f of F which belongs to $L^1(I)$).*

Theorem 2.7 ([7], Schauder fixed point theorem)**.** *Let* Q *be a convex subset of a Banach space* X, $T : Q \to Q$ *be a compact, continuous map. Then* T *has at least one fixed point in* Q.

Theorem 2.8 ([9], Kolmogorov compactness criterion)**.** *Let* $\Omega \subseteq L^p(I)$, $1 \leqslant P \leqslant \infty$. *If*

(i) Ω *is bounded in* $L^p(I)$*; and*
(ii) $x_h \to x$ *as* $h \to 0$ *uniformly with respect to* $x \in \Omega$,

then Ω *is relatively compact in* $L^p(I)$, *where*

$$x_h(t) = \frac{1}{h}\int_0^{t+h} x(s)\, ds.$$

3. Existence of integrable solutions

In this section, we present our main result by proving the existence of at least one solution $x \in L^1(I)$ of the functional integral inclusion (1.1) under the following assumptions:

(H_1) Let $F_1(t, x(t)) : I \times R^+ \to 2^{R^+}$ satisfy the following assumptions:
  (i) the set $F_1(t, x)$ is non-empty, closed and convex subset for all $(t, x) \in I \times R^+$;
  (ii) $F_1(t, .)$ is upper semi-continuous in $x \in R^+$ for each $t \in I$;
  (iii) $F_1(., x)$ is measurable in $t \in I$ for each $x \in R^+$;
  (iv) there exists an integrable function $h(t) \in L^1(I)$, such that $|F_1(t, x)| = \sup\{|f_1| : f_1 \in F_1(t, x)\} \leqslant h(t)$, for almost al $t \in I$;

(H_2) $f_2 : I \times R^+ \to R^+$, satisfies Carathèodory condition, i.e., f_2 is measurable in t for any $x \in R^+$ and continuous in x for almost all $t \in I$. There exists a function $a \in L^1$ and a constant $b > 0$ such that

$$|f_2(t, x)| \leqslant\ a(t) +\ b\,|x|, \quad \forall\ t \in I \text{ and } x \in R^+.$$

Now, let

$$y(t) = I^\alpha f_2(t, x(t)), \qquad t \in I.$$

Then the nonlinear functional integral inclusion (1.1) can be written in the form of the coupled system of functional inclusion and functional integral equation

$$x(t) \in F_1(t, y(t)), \quad t \in I. \tag{3.1}$$

$$y(t) = I^\alpha f_2(t, x(t)), \quad t \in I. \tag{3.2}$$

Definition 3.1. Let X be the class of all ordered pairs (u, v), $u, v \in C[0, T]$, with the norm

$$\|(u; v)\|_X = \|u\| + \|v\|.$$

Definition 3.2. By a solution of the coupled system (3.1), (3.2) we mean the functions $x, y \in L^1(I)$ satisfying (3.1), (3.2).

Now for the existence of integrable solution $U = (x, y)$, $x, y \in L^1(I)$ of the coupled system (3.1), (3.2) we have the following theorem.

Theorem 3.3. *Let the assumptions* (H_1)-(H_2) *be satisfied. Then there exists at least one integrable solution* $U = (x, y)$, $x, y \in L^1(I)$ *of the coupled system* (3.1), (3.2).

Proof. It is clear that from Theorem 2.6 and assumption (H_2), the set of L^1-Carathèodory selection of F_1 is non empty. So, the solution of the single-valued integral equation

$$x(t) = f_1(t, I^\alpha f_2(t, x(t))), \quad t \in I, \tag{3.3}$$

where $f_1 \in S_{F_1}$, is a solution to the inclusion (1.1). It must be noted that the Carathèodory selection $f_1 : I \times R^+ \to R^+$ satisfies the following assumptions:

(I) $f_1(x, .)$ is continuous in $x \in R^+$ for almost all $t \in I$;

(II) $f_1(., t)$ is measurable in $t \in I$ for any $x \in R^+$;

(III) there exists an integrable function $h(t) \in L^1(I)$ such that $|f_1(t; x)| \leqslant h(t), \quad t \in I$.

Then the nonlinear functional integral equation (3.3) can be written in the form

$$x(t) = f_1(t, y(t)), \quad t \in I. \tag{3.4}$$

Hence, the functional integral equation (3.3) is equivalent to the coupled system (3.2) and (3.4). Let

$$U(t) = (x(t), y(t)) = (f_1(t, y(t)), I^\alpha f_2(t, x(t))), \quad t \in I.$$

Let A be any operator defined by

$$AU(t) = A(x(t), y(t)) = (A_1 y(t), A_2 x(t)),$$

where $A_1 y(t) = f_1(t; y(t)), \quad t \in I, \ A_2 x(t) = I^\alpha f_2(t, x(t)), \quad t \in I$. Let the set Q_r be defined as

$$Q_r = \{U = (x, y) \in X \ : x, y \in L^1[0, T], \|U\| \leqslant r\}, \quad r = \|h\| + \frac{(\|a\| \ + b\|x\|) \ T^\alpha}{\Gamma(\alpha + 1)}.$$

Then, it is clear that it is a nonempty, bounded, closed, and convex. Let $U \in Q_r$ be an arbitrary ordered pair, then

$$|A_1 y(t)| = |f_1(t; y(t))|, \quad t \in I.$$

From the properties (1)-(3) of Definition 2.5, and by integration, we get

$$\int_0^t |A_1 y(s)| ds = \int_0^t |f_1(s; y(s))| ds \leqslant \int_0^t |f_1(s; y(s))| ds \leqslant \int_0^t h(s) ds \leqslant \int_0^t |h(s)| ds.$$

Then $\|A_1 y\| \leqslant \|h\|$ and

$$\begin{aligned}
|A_2 x(t)| &= |I^\alpha f_2(t, x(t))|, \\
\|A_2 x\| &\leqslant \int_0^t |I^\alpha f_2(s, x(s))| ds \\
&\leqslant \int_0^t \Big| \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f_2(\tau, x(\tau)) \ d\tau \ \Big| ds \\
&\leqslant \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |f_2(\tau, x(\tau))| \ \ d\tau \ ds \\
&\leqslant \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |a(\tau)| d\tau \ ds + \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} b \ |x(\tau)| d\tau \ ds \\
&\leqslant \int_0^t |a(\tau)| \int_\tau^t \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} ds \ d\tau + b \int_0^t |x(\tau)| \int_\tau^t \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} ds \ d\tau \\
&\leqslant \int_0^t |a(\tau)| \frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)} d\tau + b \int_0^t |x(\tau)| \frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)} d\tau \\
&\leqslant \frac{\|a\| \ T^\alpha}{\Gamma(\alpha+1)} + \frac{b \ T^\alpha}{\Gamma(\alpha+1)} \int_0^t |x(\tau)| d\tau \\
&\leqslant \frac{\|a\| \ T^\alpha}{\Gamma(\alpha+1)} + \frac{b\|x\| \ T^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

Now

$$\|AU\|_X = \|A_1 y\| + \|A_2 x\| \leqslant \|h\| + \frac{(\|a\| + b\|x\|) \ T^\alpha}{\Gamma(\alpha+1)}.$$

Hence $AU \in Q_r$, which proves that $AQ_r \subset Q_r$, i.e., $A : Qr \to Qr$. The estimate shows that the operator A maps $l^1(I)$ into itself.

Now, let us observe that the assumptions (I)-(III) imply that A_1 is continuous on the set Q_r (see [3, 5]), and from the assumptions (H_3), f_2 is continuous in x and I^α maps $L^1(I)$ continuously into itself, then $I^\alpha f(t, x(t))$ is continuous in x, and the operator A_2 is continuous on the set Q_r (see [4, 16]). Hence, we deduce that the operator A is continuous on Q_r.

Finally, we will show that A is compact, to prove this we will apply Kolmogorov compactness criterion. Let Ω be a bounded subset of Q_r. Then $(A\Omega)$ is bounded in $L^1(I)$, i.e, condition (i) of Theorem 2.8 is satisfied.

It remains to show that $(AU)_h \to AU$ in L^1 as $h \to 0$ uniformly with respect to $AU \in \Omega$. We have the following.

Let $U \in \Omega \subset Q_r$, that is $y, x \in \Omega \subset Q_r$, $\{A_1\Omega\}, \{A_2\Omega\} \subset Q_r \subset L^1(I)$, then

$$(A_1y)_h(t) - (A_1y)(t) = \frac{1}{h}\int_t^{t+h} A_1y(\tau)d\tau - A_1y(t) = \frac{1}{h}\int_t^{t+h}(A_1y(\tau) - A_1y(t))d\tau,$$

and

$$|(A_1y)_h(t) - (A_1y)(t)| \leqslant \frac{1}{h}\int_t^{t+h} |A_1y(\tau) - A_1y(t)|d\tau,$$

then

$$\begin{aligned}
\|(A_1y)_h(t) - (A_1y)(t)\| &= \int_0^T |(A_1y)_h(t) - A_1y(t)|dt \\
&\leqslant \int_0^T \frac{1}{h}\int_t^{t+h} |A_1y(\tau) - A_1y(t)|d\tau\, dt \\
&\leqslant \int_0^T \frac{1}{h}\int_t^{t+h} |p(\tau) - p(t)|d\tau\, dt + \int_0^T \frac{1}{h}\int_t^{t+h} |f_1(\tau, y(\tau)) - f_1(t, y(t))|d\tau dt.
\end{aligned}$$

Now $f_1 \in L^1(I)$, then ([25])

$$\frac{1}{h}\int_t^{t+h} |f_1(\tau, y(\tau)) - f_1(t, y(t))|d\tau \to 0.$$

Therefore $(A_1y)_h \to (A_1y)$, uniformly as $h \to 0$,

$$\begin{aligned}
\|(A_2x)_h(t) - (A_2x)(t)\| &= \int_0^T |(A_2x)_h(\tau) - A_2x(t)|dt \\
&\leqslant \int_0^T \frac{1}{h}\int_t^{t+h} |A_2x(\tau) - A_2x(t)|d\tau dt \\
&\leqslant \int_0^T \frac{1}{h}\int_t^{t+h} |I^\alpha f_2(\tau, x(\tau)) - I^\alpha f_2(t, x(t))|d\tau dt.
\end{aligned}$$

Now $f_2 \in L^1(I)$ and $I^\alpha f_2 \in L^1(I)$, then following ([25]) we have

$$\frac{1}{h}\int_t^{t+h} |I^\alpha f_2(\tau, x(\tau)) - I^\alpha f_2(t, x(t))|d\tau\, dt \to 0.$$

Therefore

$$(A_2x)_h \to (A_2x), \text{ uniformly as } h \to 0.$$

Now

$$A(x, y)_h(t) - A(x, y)(t) = \frac{1}{h}\int_t^{t+h} A(x, y)(\tau)\, d\tau - A(x, y)(t)$$

$$= \frac{1}{h}\int_t^{t+h} (A(x,y)(\tau) - A(x,y)(t))\, d\tau$$
$$= \frac{1}{h}\int_t^{t+h} ((A_2x(\tau), A_1y(\tau)) - (A_2x(t), A_1y(t)))\, d\tau.$$

Then

$$\begin{aligned}
\|(AU)_h(t) - (AU)(t)\| &= \|A(x,y)_h(t) - A(x,y)(t)\| \\
&= \int_0^T |\frac{1}{h}\int_t^{t+h} (A_2x(\tau), A_1y(\tau)) - (A_2x(t), A_1y(t))d\tau|\, dt \\
&\leqslant \int_0^T \frac{1}{h}\int_t^{t+h} |(A_2x(\tau), A_1y(\tau)) - (A_2x(t), A_1y(t))|d\tau dt \\
&\leqslant \int_0^T \frac{1}{h}\int_t^{t+h} |(A_2x(\tau) - A_2x(t)), (A_1y(\tau) - A_1y(t))|d\tau dt \\
&\leqslant \int_0^T \frac{1}{h}\int_t^{t+h} [|A_2x(\tau) - A_2x(t)| + |A_1y(\tau) - A_1y(t)|]d\tau dt \\
&\leqslant \int_0^T \frac{1}{h}\int_t^{t+h} |A_2x(\tau) - A_2x(t)|d\tau dt + \int_0^T \frac{1}{h}\int_t^{t+h} |A_1y(\tau) - A_1y(t)|]d\tau\, dt \to 0,
\end{aligned}$$

since from the above estimate we show that

$$(A_2x)_h \to (A_2x), \quad \text{uniformly as } \; h \to 0$$

and

$$(A_1y)_h \to (A_1y), \quad \text{uniformly as } \; h \to 0.$$

Hence

$$(AU)_h \to (AU), \quad \text{uniformly as } \; h \to 0.$$

Then, by Theorem 2.8 we deduce that $(A\Omega)$ is relatively compact, that is, A is a compact operator.

According to Schauder fixed point theorem, there exists at least one fixed point $U \in Q_r$, and then the system (3.2), (3.4) and consequently the system (3.1), (3.2) has at least one integrable solution $U = (x, y) \in Q_r$, $x; y \in L^1(I)$. Hence, there exists at least one integrable solution of the functional integral inclusion (1.1). □

4. Integro-differential inclusion

As an application of our main result we present the existence of at least one solution $x \in L^1(I)$ of the functional integro-differential inclusion (1.2).

Definition 4.1. By a solution of the problem of integro-differential inclusion (1.2) we mean a function $x \in L^1(I)$ and this function satisfies (1.2).

Theorem 4.2. *Let the assumptions of Theorem 3.3 be satisfied, then there exists at least one solution $x \in L^1(I)$ of the integro-differential inclusion (1.2).*

Proof. Differentiating both sides of (1.1), we obtain

$$x'(t) \;\in\; F_1(t, I^\alpha f_2(t, x'(t))),$$

put $x'(t) = u(t) \in L^1$, then (1.1) will be similar to (1.2), and

$$x(t) = x(0) \;+\; \int_0^t u(s)\, ds \;\in L^1[0, T],$$

and from Theorem 3.3 there exists at least one solution $x \in L^1(I)$ for the problem (1.2). □

5. Differential inclusion

Consider now the initial value problem of the differential inclusion (1.3).

Theorem 5.1. *Let the assumptions of Theorem 3.3 be satisfied, then the initial value problem (1.3) has at least one solution* $x \in L^1(I)$.

Proof. Let $y(t) = \frac{dx(t)}{dt}$, and $\alpha = \beta - 1$, then the inclusion (1.3) will be

$$y(t) \in F_1(t, I^{\beta-1}y(t)). \tag{5.1}$$

Letting $f_2(t,x) = x$ and applying Theorem 3.3 on the functional inclusion (5.1), we deduce that there exists a solution $y \in L^1(I)$ of the functional inclusion (5.1).

This implies that there exists at least one solution $x \in L^1(I)$

$$x(t) = x_\circ + \int_0^t y(s)ds$$

of the initial-value problem (1.3). This completes the proof. □

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