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Generalized Suzuki type α-$\mathcal{Z}$-contraction in b-metric space

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Abstract

In this paper, we introduce the concept of generalized Suzuki type α-$\mathcal{Z}$-contraction concerning a simulation function ζ in b-metric space and prove the existence of fixed point results for this contraction. Our result extend the fixed point result of [A. Padcharoen, P. Kumam, P. Saipara, P. Chaipunya, Kragujevac J. Math., **42** (2018), 419–430].

Keywords: Simulation function, triangular α-admissible mapping with respect to ζ, b-metric space, generalized Suzuki type α-$\mathcal{Z}$-contraction mapping.

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1. Introduction and Preliminaries

In 1993, Czerwik [4] generalized the concept of metric space by introducing a real number $s \geqslant 1$ in the triangle inequality of metric space and give the notion of b-metric spaces. Since then several papers have been published on the fixed point theory of various classes of single-valued and multi-valued operators in b-metric spaces (see, [3, 5, 11, 14]).

Definition 1.1 ([4])**.** Let X be a non-empty set and $s \geqslant 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric space if, for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x, y) = 0$ iff $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leqslant s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

It should be noted that, every metric space is a b-metric space with $s = 1$ and hence the class of b-metric spaces is larger than the class of metric spaces. But a metric space does not need to be b-metric space (see [13, example 1.4]).

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Definition 1.2 ([3])**.** Let (X, d) be a b-metric space.

(i) A sequence $\{x_n\}$ in X is called b-convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$, as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n \to x$.
(ii) $\{x_n\}$ in X is said to be b-Cauchy if and only if $d(x_n, x_m) \to 0$, as $n, m \to \infty$.
(iii) The b-metric space (X, d) is said to be b-complete if every b-Cauchy sequence $\{x_n\}$ in X is convergent.

In 2012, Samet et al. [15] introduced the concept of α-admissible mapping.

Definition 1.3 ([15])**.** Let T be a self mapping on X and $\alpha : X \times X \to [0, \infty)$ be a function. We say that T is α-admissible, if $x, y \in X$,

$$\alpha(x, y) \geqslant 1 \Longrightarrow \alpha(Tx, Ty) \geqslant 1.$$

The concept of α-admissible mappings has been used by several researchers (see for example [1, 10]). Later, Karapinar et al. [7] introduced the notion of triangular α-admissible mappings.

Definition 1.4 ([7])**.** Let $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}$. Then T is said to be triangular α-admissible if

(T_1) T is α-admissible;
(T_2) $\alpha(x, y) \geqslant 1$ and $\alpha(y, z) \geqslant 1 \Longrightarrow \alpha(x, z) \geqslant 1$, $x, y, z \in X$.

Lemma 1.5 ([7])**.** *Let* T *be a triangular* α*-admissible mapping. Assume that there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$. *Define sequence* $\{x_n\}$ *by* $x_n = T^n x_0$. *Then* $\alpha(x_m, x_n) \geqslant 1$ *for all* $m, n \in \mathbb{N}$ *with* $m < n$.

Recently, in 2015, Khojasteh et al. [8] introduced the notion of simulation function with a view to consider a new class of contractions, called $\mathcal{Z}$-contraction with respect to a simulation function. Such family generalized the Banach contraction and unified some known nonlinear contractions.

Definition 1.6 ([8])**.** A simulation function is a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$, satisfying the following conditions:

(i) (ζ_1) $\zeta(0, 0) = 0$;
(ii) (ζ_2) $\zeta(t, s) < s - t$, for all $s, t > 0$;
(iii) (ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$,

then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$. We denote the set of all simulation functions by $\mathcal{Z}$.

Example 1.7 ([8])**.** Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$, be defined by

(i) $\zeta(t, s) = \psi(s) - \phi(t)$ for all $t, s \in [0, \infty)$, where $\phi, \psi : [0, \infty) \to [0, \infty)$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leqslant \phi(t)$ for all $t > 0$;
(ii) $\zeta(t, s) = s - \frac{f(t, s)}{g(t, s)} t$ for all $t, s \in [0, \infty)$, where $f, g : [0, \infty) \to [0, \infty)$ are two continuous functions with respect to each variable such that $f(t, s) > g(t, s)$ for all $t, s > 0$;
(iii) $\zeta(t, s) = s - \phi(s) - t$ for all $t, s \in [0, \infty)$, where $\phi : [0, \infty) \to [0, \infty)$ is a continuous functions such that $\phi(t) = 0$ if and only if $t = 0$.

These are simulation functions.

Definition 1.8 ([8])**.** Let (X, d) be a metric space, $T : X \to X$ be a mapping and $\zeta \in \mathcal{Z}$. Then T is called a $\mathcal{Z}$-contraction with respect to ζ if the following condition is satisfied

$$\zeta(d(Tx, Ty), d(x, y)) \geqslant 0,$$

for all $x, y \in X$.

Later, in 2017, Kumam et al. [9] introduce the notion of Suzuki type $\mathcal{Z}$-contraction as follows.

Definition 1.9 ([9])**.** Let (X, d) be a metric space, $T : X \to X$ be a mapping and $\zeta \in \mathcal{Z}$. Then T is called a Suzuki type $\mathcal{Z}$-contraction with respect to ζ, if the following condition is satisfied

$$\frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow \zeta(d(Tx, Ty), d(x, y)) \geqslant 0,$$

for all $x, y \in X$, with $x \neq y$.

Remark 1.10 ([9]). It is clear from the definition of simulation function that $\zeta(t, s) < s - t \leqslant 0$, for all $t \geqslant s > 0$. Therefore if T is a Suzuki type $\mathcal{Z}$-contraction with respect to ζ, then

$$\frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow d(Tx, Ty) < d(x, y),$$

for all distinct $x, y \in X$.

Theorem 1.11 ([9])**.** *Let (X, d) be a metric space and $T : X \to X$ be a Suzuki type $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Then T has at most one fixed point.*

In 2018, Padcharoen et al. [12] introduced the generalized Suzuki type $\mathcal{Z}$-contraction in metric space as follows.

Definition 1.12 ([12])**.** Let (X, d) be a metric space, $T : X \to X$ a mapping and $\zeta \in \mathcal{Z}$. Then T is called a generalized Suzuki type $\mathcal{Z}$-contraction with respect to ζ, if the following condition is satisfied

$$\frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow \zeta(d(Tx, Ty), M(x, y)) \geqslant 0,$$

for all distinct $x, y \in X$, where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Motivated and inspired by Definition 1.12 and the work of Babu et al. [2], we introduced the definition of generalized Suzuki type α-$\mathcal{Z}$-contraction with respect to ζ in b-metric space.

Definition 1.13. Let (X, d) be a b-metric space with coefficient $s \geqslant 1$ and $\alpha : X \times X \to \mathbb{R}$ be a function. A mapping $T : X \to X$ is said to be a generalized Suzuki type α-$\mathcal{Z}$ contraction with respect to ζ if there exists a simulation function $\zeta \in \mathcal{Z}$ such that

$$\frac{1}{2s}d(x, Tx) < d(x, y) \Rightarrow \zeta(s^4\alpha(x, y)d(Tx, Ty), M_T(x, y)) \geqslant 0, \tag{1.1}$$

for all distinct $x, y \in X$, where

$$M_T(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.$$

Remark 1.14. It is clear from the definition of simulation function that $\zeta(t, s) < s - t \leqslant 0$, for all $t \geqslant s > 0$. Therefore if T is a generalized Suzuki type α-$\mathcal{Z}$-contraction with respect to ζ, then

$$\frac{1}{2s}d(x, Tx) < d(x, y) \Rightarrow s^4\alpha(x, y)d(Tx, Ty) < M(x, y)),$$

for all distinct $x, y \in X$.

2. Main result

Theorem 2.1. *Let (X, d) be a complete b-metric space with coefficient $s \geqslant 1$ and $\alpha : X \times X \to \mathbb{R}$ be a function. Let $T : X \to X$ be a self mapping and $\zeta \in \mathcal{Z}$. Suppose that the following conditions are satisfied:*

(i) *T is generalized Suzuki type α-$\mathcal{Z}$-contraction with respect to ζ;*

(ii) T *is a triangular* α*- admissible;*
(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$;
(iv) *either* T *is continuous or for any sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n, x_{n+1}) \geqslant 1$ *for all* $n \in \mathbb{N}_0$ *such that* $x_n \to x \in X$ *as* $n \to \infty$, *we have* $\alpha(x_n, x) \geqslant 1$ *for all* $n \in \mathbb{N}_0$.

Then T *has a fixed point* $x^* \in X$.

Proof. By assumption (iii), there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$. Define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}_0$ (where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}$ = set of natural numbers). If there exists an n_0 such that $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}_0$, then x_{n_0} is a fixed point of T, which completes the proof. Therefore we assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}_0$. Hence we have

$$\frac{1}{2s} d(x_n, Tx_n) < d(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N}_0.$$

The mapping T is triangular α-admissible by Lemma 1.5, we have

$$\alpha(x_n, x_{n+1}) \geqslant 1, \quad \text{for all } n \in \mathbb{N}_0.$$

Then by (1.1), we have

$$0 \leqslant \zeta(s^4\alpha(x_n, x_{n+1})d(Tx_n, Tx_{n+1}), M_T(x_n, x_{n+1}) < M_T(x_n, x_{n+1}) - s^4\alpha(x_n, x_{n+1})d(Tx_n, Tx_{n+1}).$$

Consequently, we drive that

$$d(x_{n+1}, x_{n+2}) \leqslant s^4\alpha(x_n, x_{n+1})d(Tx_n, Tx_{n+1}) < M_T(x_n, x_{n+1}).$$

Thus we have

$$d(x_{n+1}, x_{n+2}) < M_T(x_n, x_{n+1}), \tag{2.1}$$

where

$$\begin{aligned} M_T(x_n, x_{n+1}) &= \max\left\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \frac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)}{2s}\right\} \\ &= \max\left\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2s}\right\} \\ &= \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2})}{2s}\right\}. \end{aligned}$$

Since

$$\frac{d(x_n, x_{n+2})}{2s} \leqslant \frac{s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]}{2s} \leqslant \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\},$$

then we get

$$M_T(x_n, x_{n+1}) \leqslant \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}.$$

If $d(x_n, x_{n+1}) < d(x_{n+1}, x_{n+2})$, then

$$\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2}).$$

Then (2.1) becomes

$$d(x_{n+1}, x_{n+2}) < d(x_{n+1}, x_{n+2}),$$

which is a contradiction. Thus we conclude that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}). \tag{2.2}$$

Which implies that $d(x_n, x_{n+1})$ is monotonically decreasing sequence of non negative real numbers. Thus there exists $r \geqslant 0$, such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. We shall prove that $r = 0$. Suppose on the contrary that $r > 0$. Letting $t_n = \alpha(x_n, x_{n+1})d(x_{n+1}, x_{n+2})$ and $s_n = d(x_n, x_{n+1})$ and using (ζ_3), we get

$$0 \leqslant \limsup_{n\to\infty} \zeta(s^4\alpha(x_n, x_{n+1})d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})) < 0,$$

which is a contradiction. Thus we conclude that $r = 0$, i.e.,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.3}$$

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. Suppose on the contrary that $\{x_n\}$ is not a Cauchy sequence. Thus there exist $\epsilon > 0$ and the sequences $\{u(n)\}_{n=1}^{\infty}$ and $\{v(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$u(n) > v(n) > n,\ d(x_{u(n)}, x_{v(n)}) \geqslant \epsilon. \tag{2.4}$$

Moreover, corresponding to $v(n)$, we can choose the smallest $u(n)$ satisfying (2.4). Then

$$d(x_{u(n)-1}, x_{v(n)}) < \epsilon. \tag{2.5}$$

By using (2.4), (2.5), and the triangle inequality, we get

$$\epsilon \leqslant d(x_{u(n)}, x_{v(n)}) \leqslant s[d(x_{u(n)}, x_{u(n)-1}) + d(x_{u(n)-1}, x_{v(n)})] \leqslant sd(x_{u(n)}, x_{u(n)-1}) + s\epsilon.$$

Taking the upper and lower limits as $n \to \infty$ and using (2.3), we get

$$\epsilon \leqslant \liminf_{n\to\infty} d(x_{u(n)}, x_{v(n)}) \leqslant \limsup_{n\to\infty} d(x_{u(n)}, x_{v(n)}) \leqslant s\epsilon. \tag{2.6}$$

Again by the triangle inequality, we have

$$\epsilon \leqslant d(x_{u(n)}, x_{v(n)}) \leqslant s[d(x_{u(n)}, x_{v(n)+1}) + d(x_{v(n)+1}, x_{v(n)})] \tag{2.7}$$

and

$$d(x_{u(n)}, x_{v(n)+1}) \leqslant s[d(x_{u(n)}, x_{v(n)}) + d(x_{v(n)}, x_{v(n)+1})]. \tag{2.8}$$

So from (2.3), (2.6), (2.7), and (2.8), we have

$$\frac{\epsilon}{s} \leqslant \limsup_{n\to\infty} d(x_{u(n)}, x_{v(n)+1}) \leqslant s^2\epsilon. \tag{2.9}$$

Again, using above process we get

$$\frac{\epsilon}{s} \leqslant \limsup_{n\to\infty} d(x_{u(n)+1}, x_{v(n)}) \leqslant s^2\epsilon. \tag{2.10}$$

By the triangle inequality

$$d(x_{u(n)}, x_{v(n)+1}) \leqslant s[d(x_{u(n)}, x_{u(n)+1}) + d(x_{u(n)+1}, x_{v(n)+1})].$$

Now using (2.3) and (2.9)

$$\frac{\epsilon}{s^2} \leqslant \limsup_{n\to\infty} d(x_{u(n)+1}, x_{v(n)+1}). \tag{2.11}$$

By the triangle inequality

$$\begin{aligned} d(x_{u(n)+1}, x_{v(n)+1}) &\leqslant s[d(x_{u(n)+1}, x_{v(n)}) + d(x_{v(n)}, x_{v(n)+1})] \\ &\leqslant s^2[d(x_{u(n)+1}, x_{u(n)}) + d(x_{u(n)}, x_{v(n)})] + sd(x_{v(n)}, x_{v(n)+1}). \end{aligned}$$

Using (2.6)

$$\limsup_{n\to\infty} d(x_{u(n)+1}, x_{v(n)+1}) \leqslant s^3\epsilon. \tag{2.12}$$

So from (2.11) and (2.12), we have

$$\frac{\epsilon}{s^2} \leqslant \limsup_{n\to\infty} d(x_{u(n)+1}, x_{v(n)+1}) \leqslant s^3\epsilon. \tag{2.13}$$

Similarly, we can obtain

$$\frac{\epsilon}{s^2} \leqslant \liminf_{n\to\infty} d(x_{u(n)+1}, x_{v(n)+1}) \leqslant s^3\epsilon. \tag{2.14}$$

Using (2.13) and (2.14), we have

$$\frac{\epsilon}{s^2} \leqslant \liminf_{n\to\infty} d(x_{u(n)+1}, x_{v(n)+1}) \leqslant \limsup_{n\to\infty} d(x_{u(n)+1}, x_{v(n)+1}) \leqslant s^3\epsilon. \tag{2.15}$$

Now from (2.3), (2.4), and (2.5), we can choose a positive integer $n_1 \in \mathbb{N}$ such that

$$\frac{1}{2s} d(x_{u(n)}, Tx_{u(n)}) < \frac{\epsilon}{2s} < d(x_{u(n)}, x_{v(n)}), \quad \forall n \geqslant n_1.$$

Then by assumption of the theorem for every $n \geqslant n_1$ and by Lemma 1.5, we have $\alpha(x_{u(n)}, x_{v(n)}) \geqslant 1$. Then from (1.1), we have

$$\begin{aligned} 0 &\leqslant \zeta(s^4\alpha(x_{u(n)}, x_{v(n)})d(x_{u(n)+1}, x_{v(n)+1}), M_T(x_{u(n)}, x_{v(n)})) \\ &< M_T(x_{u(n)}, x_{v(n)}) - s^4\alpha(x_{u(n)}, x_{v(n)})d(x_{u(n)+1}, x_{v(n)+1}), \end{aligned} \tag{2.16}$$

which is equivalent to

$$d(x_{u(n)+1}, x_{v(n)+1}) \leqslant s^4\alpha(x_{u(n)}, x_{v(n)})d(x_{u(n)+1}, x_{v(n)+1}) < M_T(x_{u(n)}, x_{v(n)}),$$

where

$$\begin{aligned} M_T(x_{u(n)}, x_{v(n)}) &= \max\Big\{d(x_{u(n)}, x_{v(n)}), d(x_{u(n)}, Tx_{u(n)}), d(x_{v(n)}, Tx_{v(n)}), \\ &\qquad \frac{d(x_{u(n)}, Tx_{v(n)}) + d(x_{v(n)}, Tx_{u(n)})}{2s}\Big\} \\ &= \max\Big\{d(x_{u(n)}, x_{v(n)}), d(x_{u(n)}, x_{u(n)+1}), d(x_{v(n)}, x_{v(n)+1}), \\ &\qquad \frac{d(x_{u(n)}, x_{v(n)+1}) + d(x_{v(n)}, x_{u(n)+1})}{2s}\Big\}. \end{aligned}$$

Taking the upper limit as $n \to \infty$ on each side of the above inequality and using (2.6), (2.9), and (2.10), we have

$$\limsup_{n\to\infty} M_T(x_{u(n)}, x_{v(n)}) = \limsup_{n\to\infty}[\max\Big\{s\epsilon, 0, 0, \frac{s^2\epsilon + s^2\epsilon}{2s}\Big\}] = s\epsilon.$$

Therefore from (2.16) taking upper limit and using (2.15), we get

$$\begin{aligned} 0 &\leqslant \limsup_{n\to\infty} \zeta\left(s^4\alpha(x_{u(n)},x_{v(n)})d(x_{u(n)+1},x_{v(n)+1}), M_T(x_{u(n)},x_{v(n)})\right) \\ &< \limsup_{n\to\infty}\left[M_T(x_{u(n)},x_{v(n)}) - s^4\alpha(x_{u(n)},x_{v(n)})d(x_{u(n)+1},x_{v(n)+1})\right] \\ &\leqslant \limsup_{n\to\infty} M_T(x_{u(n)},x_{v(n)}) - s^4\alpha(x_{u(n)},x_{v(n)})\liminf_{n\to\infty} d(x_{u(n)+1},x_{v(n)+1}) \\ &\leqslant s\epsilon - s^4\alpha(x_{u(n)},x_{v(n)})\left(\frac{\epsilon}{s^2}\right) < 0, \end{aligned}$$

which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in (X, d). Since X is complete b-metric space, then there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*. \tag{2.17}$$

Now, we show that x^* is a fixed point of T. Assume that (iv) holds, then $\alpha(x_n, x^*) \geqslant 1$. We claim that, for every $n \in \mathbb{N}$,

$$\frac{1}{2s}d(x_n, Tx_n) < d(x_n, x^*) \text{ or } \frac{1}{2s}d(Tx_n, T^2x_n) < d(Tx_n, x^*). \tag{2.18}$$

Suppose on the contrary that there exists $m \in \mathbb{N}$, such that

$$\frac{1}{2s}d(x_m, Tx_m) \geqslant d(x_m, x^*) \quad\text{and}\quad \frac{1}{2s}d(Tx_m, T^2x_m) \geqslant d(Tx_m, x^*). \tag{2.19}$$

Therefore

$$2sd(x_m, x^*) \leqslant d(x_m, Tx_m) \leqslant s[d(x_m, x^*) + d(x^*, Tx_m)].$$

Which implies that

$$d(x_m, x^*) \leqslant d(x^*, Tx_m). \tag{2.20}$$

Now, from (2.2) and (2.20) we have

$$d(Tx_m, T^2x_m) < d(x_m, Tx_m) \leqslant s[d(x_m, x^*) + d(x^*, Tx_m)] \leqslant 2sd(x^*, Tx_m). \tag{2.21}$$

It follows from (2.19) and (2.21) that

$$d(Tx_m, T^2x_m) < d(Tx_m, T^2x_m).$$

This is a contradiction. Hence (2.18) holds. If part (i) of (2.18) is true, by generalized Suzuki type α-$\mathcal{Z}$-contraction with respect to ζ, we have

$$0 \leqslant \zeta(s^4\alpha(x_n, x^*)d(Tx_n, Tx^*), M_T(x_n, x^*)) < M_T(x_n, x^*) - s^4\alpha(x_n, x^*)d(Tx_n, Tx^*),$$

which is equivalent to

$$d(Tx_n, Tx^*) \leqslant s^4\alpha(x_n, x^*)d(Tx_n, Tx^*) < M_T(x_n, x^*),$$

where

$$M_T(x_n, x^*) = \max\left\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), \frac{d(x_n, Tx^*) + d(x^*, Tx_n)}{2s}\right\}.$$

Letting $n \to \infty$ and by using (2.17), we obtain

$$\lim_{n\to\infty} M_T(x_n, x^*) = d(x^*, Tx^*). \tag{2.22}$$

By using (2.21), (2.22), (iv), and (ζ_3), we have

$$\begin{aligned} 0 &\leqslant \zeta\big(s^4\alpha(x_n,x^*)d(Tx_n,Tx^*), M_T(x_n,x^*)\big) \\ &\leqslant \limsup_{n\to\infty} \zeta\big(s^4\alpha(x_n,x^*)d(Tx_n,Tx^*), M_T(x_n,x^*)\big) \\ &\leqslant \limsup_{n\to\infty} \Big[M_T(x_n,x^*) - s^4\alpha(x_n,x^*)d(Tx_n,Tx^*)\Big]. \end{aligned}$$

According to property (ζ_3) from Definition 1.6, since the both sequences $d(Tx_n, Tx^*), M_T(x_n, x^*)$ converge to the $d(x^*, Tx^*) > 0$. By assumption it is clear that

$$0 \leqslant \limsup_{n\to\infty} \zeta\big(s^4\alpha(x_n,x^*)d(Tx_n,Tx^*), M_T(x_n,x^*)\big) < 0,$$

which is a contradiction. Hence $x^* = Tx^*$, i.e., x^* is a fixed point of T. If part (ii) of (2.18) is true, using a similar method to the above, we get $x^* = Tx^*$. Hence x^* is a fixed point of T. □

Now, we prove the uniqueness of the fixed point result. We need the following additional condition.

(A) For all $x^*, y^* \in \mathrm{Fix}(T)$, there exists $z \in X$ such that $\alpha(x^*, z) \geqslant 1$ and $\alpha(y^*, z) \geqslant 1$, where $\mathrm{Fix}(T)$ denotes the set of fixed points of T.

Theorem 2.2. *By adding condition* (A) *to the hypothesis of Theorem* 2.1*, we obtain that* x^* *is the unique fixed point of T.*

Proof. We argue by contradiction, i.e., if $x^*, y^* \in X$ are two fixed points of T, such that $x^* \neq y^*$. Since T is triangular α-admissible and by assumption (A), we have $\alpha(x^*, y^*) \geqslant 1$, then we have $0 = \frac{1}{2s}d(x^*, Tx^*) < d(x^*, y^*)$ and from (1.1), we obtain

$$\zeta(s^4\alpha(x^*,y^*)d(Tx^*,Ty^*), M_T(x^*,y^*)) \geqslant 0, \tag{2.23}$$

where

$$M_T(x^*,y^*) = \max\left\{d(x^*,y^*), d(x^*,Tx^*), d(y^*,Ty^*), \frac{d(x^*,Ty^*)+d(y^*,Tx^*)}{2s}\right\} = d(x^*,y^*).$$

So, by (2.23), we have

$$\begin{aligned} 0 \leqslant \zeta(s^4\alpha(x^*,y^*)d(Tx^*,Ty^*), M_T(x^*,y^*)) &= \zeta(s^4\alpha(x^*,y^*)d(x^*,y^*), d(x^*,y^*)) \\ &< d(x^*,y^*) - s^4\alpha(x^*,y^*)d(x^*,y^*) \leqslant 0, \end{aligned}$$

which is a contradiction. Hence, $x^* = y^*$. □

Example 2.3. Let $X = \{1,2,3,4,5\}$ and $d : X \times X \to [0,\infty)$ be defined as follows: $d(1,2) = d(2,4) = d(3,5) = 1$, $d(1,5) = 1.02$, $d(1,3) = d(3,4) = 1.5$, $d(1,4) = d(2,5) = d(4,5) = 2.4$, $d(2,3) = 3$, $d(1,1) = d(2,2) = d(3,3) = d(4,4) = d(5,5) = 0$, and $d(x,y) = d(y,x)$ for all $x, y \in X$. As $3 = d(2,3) \nleqslant d(2,1) + d(1,3) = 2.5$, d is not a metric on X. Clearly (X, d) is a complete b-metric space with parameter $s = \frac{6}{5}$. We define $T : X \to X$ such that

$$T(1) = T(2) = T(5) = 2,\ T(3) = 5,\ \text{and } T(4) = 1.$$

Let $A = \Big\{(1,1),(1,2),(2,1),(2,2),(2,5),(5,2),(5,5),(1,5),(5,1),(3,4),(4,3),(3,3),(4,4)\Big\}$, and $\alpha : X \times X \to \mathbb{R}$ by

$$\alpha(x,y)=\begin{cases}1, & \text{if } (x,y)\in A,\\ 0, & \text{otherwise.}\end{cases}$$

Let $\zeta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ defined by $\zeta(t,s)=\frac{11}{12}s-t$ for all $s,t\in[0,\infty)$. Now we show that T is α-admissible. If $x,y\in\{1,2,5\}$, then $\alpha(x,y)=1$ implies that $\alpha(Tx,Ty)=\alpha(2,2)=1$. If $x,y\in\{3,4\}$ then, $\alpha(3,4)=1$ implies that $\alpha(T3,T4)=\alpha(5,1)=1$. Thus for any $x,y\in X$, $\alpha(x,y)=1$ implies that $\alpha(Tx,Ty)=1$. Therefore T is α-admissible. If $x,y,z\in\{1,2,5\}$, then $\alpha(x,y)=1$ and $\alpha(y,z)=1$ implies that $\alpha(x,z)=1$. If $x,y\in\{3,4\}$, then $\alpha(x,z)=1$ and $\alpha(y,z)=1$ implies that $\alpha(x,y)=1$. Thus for any $x,y,z\in X$, $\alpha(x,z)=1$ and $\alpha(z,y)=1$ implies that $\alpha(x,y)=1$. Therefore T is triangular α-admissible mapping. Now we verify the inequality (1.1) for all distinct $x,y\in X$. Note that for all distinct $x,y\in X$ and for $s=\frac{6}{5}$ the inequalities $\frac{5}{12}d(x,Tx)<d(x,y)$ and $\alpha(x,y)\geqslant 1$, give

$$(x,y)\in\Big\{(1,2),(2,1),(2,5),(5,2),(1,5),(5,1),(3,4),(4,3)\Big\}.$$

So, this implies that

$$\begin{aligned}\zeta(s^4\alpha(x,y)d(Tx,Ty),M_T(x,y))&=\zeta(\Big(\frac{6}{5}\Big)^4\alpha(x,y)d(Tx,Ty),M_T(x,y))\\ &=\frac{11}{12}M_T(x,y)-\Big(\frac{6}{5}\Big)^4\alpha(x,y)d(Tx,Ty)\geqslant 0,\end{aligned}$$

implies that

$$\Big(\frac{6}{5}\Big)^4\alpha(x,y)d(Tx,Ty)\leqslant\frac{11}{12}M_T(x,y)=\frac{11}{12}\Big[\max\Big\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{12/5}\Big\}\Big].$$

Now, we consider the following cases.

(i) If $x,y\in\{1,2,5\}$, then

$$\Big(\frac{6}{5}\Big)^4\alpha(x,y)d(Tx,Ty)=0\leqslant\frac{11}{12}M_T(x,y).$$

(ii) If $x=3$ and $y=4$, then

$$\Big(\frac{6}{5}\Big)^4\alpha(3,4)d(T3,T4)=2.11\leqslant\frac{11}{12}M_T(3,4)=2.20.$$

That is $\frac{5}{12}d(x,Tx)<d(x,y)$ and $\alpha(x,y)\geqslant 1$ implies that $\zeta(\Big(\frac{6}{5}\Big)^4\alpha(x,y)d(Tx,Ty),M_T(x,y))\geqslant 0$ for all distinct $x,y\in X$. Moreover, there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geqslant 1$. In fact, for $x_0=1$, we have $\alpha(1,T1)=\alpha(1,2)=1$. Here all conditions of Theorem 2.1 hold, therefore T has a fixed point. Here, $x=2$ is a fixed point of T.

Remark 2.4. In b-metric space defined as above, Theorem 3.4 in [6] fails. By choosing $x=2$ and $y=4$, we have $\zeta(\frac{6}{5}d(T2,T4),d(2,4))<0$. Thus it is not a b-simulation function.

Corollary 2.5. *Let* (X,d) *be a complete* b*-metric space with coefficient* $s\geqslant 1$ *and* $\alpha:X\times X\to\mathbb{R}$ *be a function. A mapping* $T:X\to X$ *be a self mapping and* $\zeta\in\mathcal{Z}$*. Suppose that the following conditions are satisfied:*

(i) T *is Suzuki type* α-$\mathcal{Z}$ *contraction with respect to* ζ*, i.e.,*

$$\frac{1}{2s}d(x,Tx)<d(x,y)\Rightarrow\zeta(s^4\alpha(x,y)d(Tx,Ty),d(x,y))\geqslant 0,$$

for all distinct $x,y\in X$*;*

(ii) T *is a triangular* α*- admissible;*

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$;
(iv) *either* T *is continuous or for any sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n, x_{n+1}) \geqslant 1$ *for all* $n \in \mathbb{N}_0$ *such that* $x_n \to x \in X$ *as* $n \to \infty$, *we have* $\alpha(x_n, x) \geqslant 1$ *for all* $n \in \mathbb{N}_0$.

Then T *has a fixed point* $x^* \in X$.

By setting $s = 1$ in Theorem 2.1, we deduce the following result.

Corollary 2.6. *Let* (X, d) *be a complete metric space and* $\alpha : X \times X \to \mathbb{R}$ *be a function. Let* $T : X \to X$ *be a self mapping and* $\zeta \in \mathcal{Z}$. *Suppose that the following conditions are satisfied:*

(i) T *is generalized Suzuki type* α-$\mathcal{Z}$*-contraction with respect to* ζ, *i.e.*,

$$\frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow \zeta(\alpha(x, y)d(Tx, Ty), M(x, y)) \geqslant 0,$$

for all distinct $x, y \in X$, *where*

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\};$$

(ii) T *is a triangular* α*- admissible;*
(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$;
(iv) *either* T *is continuous or for any sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n, x_{n+1}) \geqslant 1$ *for all* $n \in \mathbb{N}_0$ *such that* $x_n \to x \in X$ *as* $n \to \infty$, *we have* $\alpha(x_n, x) \geqslant 1$ *for all* $n \in \mathbb{N}_0$.

Then T *has a fixed point* $x^* \in X$.

Corollary 2.7. *Adding condition* (A) *to the hypotheses of Corollary* 2.5 *(resp. Corollary* 2.6*), we obtain that* x^* *is the unique fixed point of* T.

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