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# Impulsive-integral inequalities for attracting and quasi-invariant sets of neutral stochastic partial functional integrodifferential equations with impulsive effects



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## Abstract

In this article, we investigate a class of neutral stochastic partial functional integrodifferential equations with impulsive effects. The results are obtained by using the new integral inequalities, the attracting and quasi-invariant sets combined with theories of resolvent operators. In the end, one example is given to illustrate the feasibility and effectiveness of results obtained.

**Keywords:** Impulsive integral inequality, attracting set, quasi-invariant set, stochastic integrodifferential equations, resolvent operator.

**2010 MSC:** 35B35, 35B40, 39B82, 60H15.



## 1. Introduction

The attracting set and invariant set of dynamical systems have been extensively studied over the past few decades and various results are reported. For discrete systems, see [6, 22]. For deterministic differential systems with or without delays, see [12, 13, 18–20, 25, 26]. For partial differential systems, see [21]. For stochastic or random systems, see [9, 16].

Stochastic partial differential equations in Hilbert spaces were studied by some authors and many valuable results on the existence, uniqueness and stability of the solutions were established, refer to [1–5, 7, 10, 14, 17]. However, under impulsive perturbation, the equilibrium point sometimes does not exist in many real physical systems, especially in nonlinear dynamical systems. Therefore, an interesting subject is to discuss the attracting set and the invariant set of impulsive systems. Some significant progress has been made in the techniques and methods of determining the invariant set and the attracting set for impulsive differential systems including impulsive functional differential equations, impulsive stochastic functional differential equations and so on [23, 24]. It should be pointed out that there are only a few works about

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the attracting and invariant set of impulsive stochastic partial differential equations (see [15] and the reference therein). Unfortunately, the corresponding problems for neutral stochastic partial functional integrodifferential equations with impulsive effects have not been considered prior to this work. Due to the wide application of fractional Brownian motion (fBm) in hydrology, economics, telecommunications and medicine, much interesting work has been carried out on stochastic differential equations driven by fBm. Neutral stochastic functional differential equations (NSFDEs) have been also widely discussed by many researchers because of potential applications in control theory, mechanics, engineering, economics, etc..

Motivated by the above discussion, our objective in this paper is to determine an attracting and quasi-invariant sets of neutral stochastic partial functional integrodifferential equations with impulsive effects of the form:

$$\begin{aligned} d\left[x(t)+h(t,x_t)\right] &= A\left[x(t)+h(t,x_t)\right]+\left[\int_0^t B(t-s)\left[x(s)+h(s,x_s)\right]ds+f(t,x_t)\right]dt \\ &\quad + g(t,x_t)dw(t), \quad t \geqslant 0,\ t \neq t_k, \\ \Delta x(t_k) &= x(t_k^+)-x(t_k^-) = I_k(x(t_k)), \quad t=t_k, \quad k=1,2,\ldots, \\ x(t) &= \varphi(t) \in \mathscr{PC}^{\mathscr{B}}_{\mathscr{F}_0}((-\tau,0];\mathbb{X}), \end{aligned} \tag{1.1}$$

where $A$ is the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geqslant 0}$ on $\mathbb{X}$, $B(t)$ is a closed linear operator with domain $\mathcal{D}(B(t)) \supset \mathcal{D}(A)$; $f,h : [0,\infty)\times\mathscr{PC} \to \mathbb{X}$ and $g : [0,\infty)\times\mathscr{PC} \to \mathcal{L}_2^0$ are jointly continuous functions; the segment $x_t : [-\tau,0] \to \mathbb{X}$ is defined by $x_t(\theta)=x(t+\theta)$ for $t \geqslant 0$ belongs to the phase space $\mathscr{PC}$; the fixed moments of time $t_k$ satisfies $0=t_0<t_1<t_2<\cdots<t_m<\cdots$, and $\lim_{k\to\infty} t_k=\infty$; $x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of $x(t)$ at time $t_k$; $x(t)$ at each impulsive point $t_k$ is right continuous. And $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$ represents the jump in the state $x$ at time $t_k$, where $I_k$ determines the size of the jump.

## 2. Preliminaries

Let $\mathbb{X}$ and $\mathbb{Y}$ be two real separable Hilbert spaces and let $\mathcal{L}(\mathbb{Y},\mathbb{X})$ be the space of bounded linear operator from $\mathbb{Y}$ to $\mathbb{X}$. Let $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geqslant 0},\mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). Let $\{\mathcal{W}(t), t\geqslant 0\}$ denote a $\mathbb{Y}$-valued $\{\mathscr{F}_t\}_{t\geqslant 0}$-Wiener process defined on $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geqslant 0},\mathbb{P})$ with covariance operator $Q$, i.e.,

$$\mathbf{E}\left\langle \mathcal{W}(t),x\right\rangle_{\mathbb{Y}}\left\langle \mathcal{W}(t),y\right\rangle_{\mathbb{Y}} = (t\wedge s)\left\langle Qx,y\right\rangle_{\mathbb{Y}} \quad \text{for all } x,y\in\mathbb{Y},$$

where $Q$ is a positive, self-adjoint, trace class operator on $\mathbb{Y}$. In particular, we shall call such $\{\mathcal{W}(t), t\geqslant 0\}$, a $\mathbb{Y}$-valued $\{\mathscr{F}_t\}_{t\geqslant 0}$-Wiener process with respect to $\{\mathscr{F}_t\}_{t\geqslant 0}$. In order to define stochastic integrals with respect to the Q-Wiener process $\mathcal{W}(t)$, we introduce the subspace $\mathbb{Y}_0=Q^{\frac{1}{2}}(\mathbb{Y})$ which, endowed with the inner product

$$\langle u,v\rangle_{\mathbb{Y}_0} = \left\langle Q^{\frac{-1}{2}}u, Q^{\frac{-1}{2}}v\right\rangle_{\mathbb{Y}},$$

is a Hilbert space. Let $\mathcal{L}_2^0=\mathcal{L}_2(\mathbb{Y}_0,\mathbb{X})$ denote the space of all Hilbert-Schmidt operators from $\mathbb{Y}_0$ into $\mathbb{X}$. It turns out to be a separable Hilbert space equipped with the norm

$$\|\varphi\|^2_{\mathcal{L}_2^0} = \operatorname{tr}\left(\left(\varphi Q^{\frac{1}{2}}\right)\left(\varphi Q^{\frac{1}{2}}\right)^*\right) \quad \text{for all } \varphi\in\mathcal{L}_2^0.$$

Clearly, for any bounded operators $\varphi\in\mathcal{L}(\mathbb{Y},\mathbb{X})$, this norm reduces to $\|\varphi\|^2_{\mathcal{L}_2^0}=\operatorname{tr}(\phi Q\varphi^*)$. Let $\mathbb{R}^+=[0,\infty)$ and $\mathscr{C}(\mathbb{X},\mathbb{Y})$ denotes the space of continuous mappings from the topological space $\mathbb{X}$ to the topological space $\mathbb{Y}$. Especially, $\mathscr{C}=\mathscr{C}([-\tau,0];\mathbb{R})$ denotes the family of all continuous $\mathbb{R}$-valued functions $\varphi$

defined on $[-\tau, 0]$ with the norm $\|\varphi\|_\tau = \sup_{-\tau\leqslant\theta\leqslant 0}\|\varphi(\theta)\|$ denote by $\mathscr{C} = \mathscr{C}([-\tau, 0]; \mathbb{X})$ equipped with the norm

$$\|\varphi\|_{\mathscr{C}} = \sup_{-\tau\leqslant\theta\leqslant 0}\|\varphi(\theta)\|.$$

$\mathscr{PC}(J, \mathbb{R}^n) = \{\varphi : J \to \mathbb{R}^n$ is continuous for all but at most a finite number of point $t \in J$ and at these points $t \in J$, $\varphi(t^+)$ and $\varphi(t^-)$ exist, $\varphi(t^+) = \varphi(t)\}$, where $J \subset \mathbb{R}$ is a bounded interval, $\varphi(t^+)$ and $\varphi(t^-)$ denote the right-hand and left-hand limits of the function $\varphi(t)$, respectively. Especially, let $\mathscr{PC} = \mathscr{PC}([-\tau, 0]; \mathbb{X})$. Let $\mathscr{PC} = \mathscr{PC}([-\tau, 0]; \mathbb{X})$ denotes the family of all bounded $\mathscr{F}_0$-measurable, $\mathscr{PC}([-\tau, 0]; \mathbb{X})$-value random variables $\varphi$, satisfying

$$\|\varphi\|^p = \sup_{-\tau<\theta<0}\mathbf{E}\|\varphi(\theta)\|^p < \infty \quad \text{for } p > 0.$$

### 2.1. *Partial integro-differential equations in Banach spaces*

In this subsection, we recall some knowledge on partial integrodifferential equations and the related resolvent operators. Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces such that

$$\|y\| = \|Ay\| + \|y\|, \;\; y \in \mathbb{Y}.$$

$A$ and $B(t)$ are closed linear operator on $\mathbb{X}$. Let $\mathscr{C}([0,\infty); \mathbb{Y})$, $\mathscr{B}(\mathbb{Y}, \mathbb{X})$ stand for the space of all continuous functions from $[0,\infty)$ into $\mathbb{Y}$ and the set of all bounded linear operators from $\mathbb{Y}$ into $\mathbb{X}$, respectively. In what follows, we suppose the following assumptions.

**(H1)** $A$ is the infinitesimal generator of a strongly continuous semigroup on $\mathbb{X}$.

**(H2)** For all $t \geqslant 0$, $B(t)$ is a continuous linear operator from $(\mathbb{Y}, |\cdot|_{\mathbb{Y}})$ into $(\mathbb{X}, |\cdot|_{\mathbb{X}})$. Moreover, there exists an integrable function $\mathscr{C} : [0,\infty) \to \mathbb{R}^+$ such that for any $y \in \mathbb{Y}$, $t \mapsto B(t)y$ belongs to $\mathcal{W}^{1,1}([0,\infty), \mathbb{X})$ and
$$\left|\frac{d}{dt}B(t)(t)y\right|_{\mathbb{X}} \leqslant \mathscr{C}(t)|y|_{\mathbb{Y}} \;\text{ for }\; y \in \mathbb{Y} \;\text{ and }\; t \geqslant 0.$$

By Grimmer [11], under the assumptions **(H1)** and **(H2)**, the following Cauchy problem

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds \;\text{ for }\; t \geqslant 0, \\ v(0) = v_0 \in \mathbb{X}, \end{cases} \tag{2.1}$$

has an associated resolvent operator of bounded linear operator valued function $R(t) \in \mathcal{L}(\mathbb{X})$ for $t \geqslant 0$.

**Definition 2.1.** A resolvent operator associated with (2.1) is a bounded linear operator valued function $R(t) \in \mathcal{L}(\mathbb{X})$ for $t \geqslant 0$, satisfying the following properties:

(i) $R(0) = I$ and $\|R(t)\| \leqslant Me^{\beta t}$ for some constants $M$ and $\beta$;

(ii) for each $x \in \mathbb{X}$, $R(t)x$ is strongly continuous for $t \geqslant 0$;

(iii) for $x \in \mathbb{Y}$, $R(\cdot)x \in \mathscr{C}^1([0,\infty); \mathbb{X}) \cap \mathscr{C}([0,\infty); \mathbb{Y})$ and

$$R'(t)x = AR(t)x + \int_0^t B(t-s)R(s)xds = R(t)Ax + \int_0^t R(t-s)B(s)xds \qquad \text{for } t \geqslant 0.$$

By Grimmer [11], we can establish the existence and uniqueness of the mild solution to the following integrodifferential equation

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds + q(t) \;\text{ for }\; t \geqslant 0, \\ v(0) = v_0 \in \mathbb{X}, \end{cases} \tag{2.2}$$

where $q : [0,\infty) \to \mathbb{X}$ is a continuous function.

The resolvent operators play an important role to study the existence of solutions and to give a variation of constants formula for nonlinear systems. We need to know when the linear system (2.1) has a resolvent operator. For more details on resolvent operators, we refer the reader to [11].

Next, we shall get the global attracting and quasi-invariant set and exponential p-th stability of system (1.1), we firstly introduce the concept of the global attracting and quasi-invariant set and exponential p-stability.

**Definition 2.2.** The set $S \subset \mathbb{X}$ is called a quasi-invariant set of (1.1), if there exist positive constants $k$ and $b$, such that for any initial value $\varphi \in \mathscr{PC}^{\mathscr{B}}_{\mathscr{F}_0}((-\tau, 0]; \mathbb{X})$, the solution $kx(t, \varphi) + b \in S$, $t \geqslant 0$.

**Definition 2.3.** The set $S \subset \mathbb{X}$ is called a global attracting set of (1.1), if for any initial value $\varphi \in \mathscr{PC}^{\mathscr{B}}_{\mathscr{F}_0}((-\tau, 0]; \mathbb{X})$, the solution $kx(t, \varphi)$ converges to $S$ as $t \to \infty$. This is, $\operatorname{dist}(x_t(0, \varphi), S) \to 0$ as $t \to \infty$, where $\operatorname{dist}(x, S) = \inf_{y \in S} \mathbf{E}\,\|x - y\|$.

**Definition 2.4.** The mild solution of system (1.1) is said to be exponentially stable in $p(p \geqslant 2)$th moment if there exists a pair of positive constants $\lambda > 0$ and $M \geqslant 1$ such that for any solution $x(t, \varphi)$ with the initial condition $\varphi \in \hat{D}$,

$$\mathbf{E}\,\|x(t, \varphi)\|^p \leqslant M\,\|\varphi\|_{\mathcal{L}_p}\, e^{-\lambda t}, \quad t \geqslant 0,\ p \geqslant 2.$$

Especially, system (1.1) is said to be exponentially stable in mean square when $p = 2$.

**Lemma 2.5** ([7])**.** *Let $y : \mathbb{R}^+ \to \mathbb{R}^+$ be Borel measurable. If $y(t)$ is a solution of the delay integral inequality*

$$y(t) = \begin{cases} \|\varphi\|_\tau\, e^{-r(t-t_0)} + b_1\,\|y_t\|_\tau + b_2 \displaystyle\int_{t_0}^{t} e^{-r(t-s)}\,\|y_s\|_\tau\, ds \\ \qquad + \sum_{0<t_k<t} c_k e^{-r(t-t_k)} y(t_k^-) + J,\ t \geqslant t_0, \\ \varphi(t), \quad t \in [t_0 - \tau, t_0], \end{cases} \tag{2.3}$$

*where $\varphi(t) \in \mathscr{PC}([t_0 - \tau, t_0]; \mathbb{R}^+)$, $r > 0$, $b_1, b_2, c_k$ $(k = 1, 2, \ldots)$ and $J$ are nonnegative constants and if $\|\varphi\|_\tau \leqslant K$ for some constant $K > 0$ and*

$$b_1 + \frac{b_2}{r} + \sum_{k=1}^{\infty} c_k = \rho < 1,$$

*then there are constants $\lambda \in (0, r)$ and $N \geqslant K$ such that*

$$y(t) \leqslant N e^{-\lambda(t-t_0)} + \frac{J}{1-\rho}, \quad \forall t \geqslant t_0,$$

*where $\lambda$ and $N$ satisfy that*

$$\rho_\lambda = b_1 e^{\lambda\tau} + \frac{b_2 e^{\lambda t}}{r - \lambda} + \sum_{k=1}^{\infty} c_k < 1 \ \text{ and } \ N \geqslant \frac{K}{1 - \rho_\lambda},$$

*or $b_2 \neq 0$ and*

$$\rho_\lambda \leqslant 1 \ \text{ and } \ N \geqslant \frac{(r-\lambda)\left[K - \frac{b_2 J}{r(1-\rho)}\right]}{b_2 e^{\lambda\tau}}.$$

*Remark* 2.6. Let $t_0 = 0$, $b_1 = 0$, $b_2 \neq 0$, $J = 0$ in Lemma 2.5, then we get the Lemma 3.1 in Chen [7].

**Definition 2.7.** A stochastic process $\{x(t), t \in [0, T]\}$, $0 \leqslant T \leqslant \infty$, is a mild solution of (1.1) if

(i) $x(t)$ is $\mathscr{F}_t$-adapted, $t \geqslant 0$;

(ii) $x(t)$ satisfies the integral equation

$$\begin{aligned} x(t) = R(t)\,[\varphi(0) + h(0,\varphi)] - h(t,x_t) + \int_0^t R(t-s) f(s,x_s)\,ds + \int_0^t R(t-s) g(s,x_s)\,dw(s) \\ + \sum_{0<t_k<t} R(t-t_k) I_k(x(t_k)). \end{aligned} \tag{2.4}$$

In the sequel, we impose the following assumptions.

**(H3)** There exist constants $\lambda > 0$ and $M \geqslant 1$ such that $\|R(t)\| \leqslant Me^{-\lambda t}$.

**(H4)** There exists constants $L_f > 0$, $L_g > 0$, $b_f > 0$ and $b_g > 0$ such that for any $x, y \in \mathscr{PC}$ and $t \geqslant 0$,

$$\|f(t,x_t) - f(t,y_t)\| \leqslant L_f \|x-y\|_{\mathscr{PC}}, \quad \|f(t,0)\|_{\mathbb{X}} \leqslant b_f$$
$$\|g(t,x_t) - g(t,y_t)\|_{\mathcal{L}_2^0} \leqslant L_g \|x-y\|_{\mathscr{PC}}, \quad \|g(t,0)\|_{\mathcal{L}_2^0} \leqslant b_g.$$

**(H5)** There exists constants $L_h > 0$ such that the function $h$ is $\mathbb{Y}$-valued and satisfies for any $x, y \in \mathscr{PC}$ and $t \geqslant 0$,

$$\|h(t,x_t) - h(t,y_t)\| \leqslant L_h \|x-y\|_{\mathscr{PC}}, \quad h(t,0) = 0.$$

**(H6)** There exists some positive constants $d_k$ such that for any $x, y \in \mathbb{X}$ and $\sum_{k=1}^{\infty} d_k < \infty$,

$$\|I_k(x(t_k)) - I_k(y(t_k))\| \leqslant d_k \|x-y\| \quad \text{and} \quad I_k(0) = 0, k = 1, 2, \ldots.$$

## 3. Main results

**Theorem 3.1.** *Assume that* **(H1)-(H6)** *are satisfied, then* $S = \left\{\varphi \in \mathscr{PC}^{\mathscr{B}}_{\mathscr{F}_0}([-\tau,0],\mathbb{X}) : \|\varphi\|^p \leqslant (1-\hat{\rho})^{-1}\hat{J}\right\}$ *is a global attracting set of the mild solution of* (1.1) *and* $S_1 = \left\{\varphi \in \mathscr{PC}^{\mathscr{B}}_{\mathscr{F}_0}([-\tau,0],\mathbb{X}) : \|\varphi\|^p \leqslant r, r > 0\right\}$ *is a quasi-invariant set of the mild solution of* (1.1) *if the following inequality*

$$\begin{aligned} \hat{\rho} = {} & 5^{p-1}L_h^p + 5^{p-1}2^{p-1}M^p\lambda^{-p}L_f^p \\ & + 5^{p-1}2^{p-1}M^pL_g^p \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} \left(\frac{p-2}{2\lambda(p-1)}\right)^{p/2-1} \lambda^{-p/2} \\ & + 5^{p-1}M^p \left(\sum_{0<t_k<t} d_k\right)^{p/q+1} < 1 \end{aligned}$$

*holds for* $p \geqslant 2$, *and* $\hat{J} := 10^{p-1}M^p\lambda^{-p}L_f^p + 10^{p-1}M^pL_g^p \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} \left(\frac{p-2}{2(p-1)}\right)^{p/2-1} \lambda^{-p/2}b_g^p$, $1/p + 1/q = 1$.

*Proof.* From (2.4), we can get

$$\begin{aligned} \mathbf{E}\|x(t)\|_{\mathbb{X}}^p = {} & 5^{p-1}\mathbf{E}\|R(t)\,[\varphi(0) + h(t,x_t)]\|_{\mathbb{X}}^p + 5^{p-1}\mathbf{E}\|h(t,x_t)\|_{\mathbb{X}}^p \\ & + 5^{p-1}\mathbf{E}\left\|\int_0^t R(t-s)f(s,x_s)\,ds\right\|_{\mathbb{X}}^p + 5^{p-1}\mathbf{E}\left\|\int_0^t R(t-s)g(s,x_s)\,dw(s)\right\|_{\mathbb{X}}^p \\ & + 5^{p-1}\mathbf{E}\left\|\sum_{0<t_k<t} R(t-t_k)I_k(x(t_k))\right\|_{\mathbb{X}}^p = 5^{p-1}\sum_{i=1}^{5} Q_i. \end{aligned} \tag{3.1}$$

We first evaluate the first term of the right-hand side

$$\begin{aligned} Q_1(t) &= \mathbf{E}\|R(t)\,[\varphi(0) + h(t,x_t)]\|_{\mathbb{X}}^p \\ &\leqslant M^pe^{-p\lambda t}\mathbf{E}\,[\|\varphi(0)\| + \|h(0,\varphi)\|]^p \\ &\leqslant M^pe^{-p\lambda t}\mathbf{E}\,[\|\varphi(0)\| + L_h\|\varphi\|_{\mathscr{PC}}]^p \\ &\leqslant M^*\|\varphi\|_{\mathcal{L}_p}^p e^{-\lambda t}, \end{aligned} \tag{3.2}$$

where $M^* \geqslant 1$ is an appropriate constant. From **(H5)**, we can obtain

$$Q_2(t) = \mathbf{E}\left\|h(t,x_t)\right\|_{\mathbb{X}}^p \leqslant L_h^p \mathbf{E}\left\|x_t\right\|_{\mathscr{C}}^p. \tag{3.3}$$

From **(H4)** and Holder's inequality, we get

$$\begin{aligned} Q_3(t) &= \mathbf{E}\left\|\int_0^t R(t-s)f(s,x_s)ds\right\|_{\mathbb{X}}^p \\ &\leqslant \mathbf{E}\left(\int_0^t Me^{-\lambda(t-s)}\left[L_f\left\|x_s\right\|_{\mathscr{C}} + \left\|f(s,0)\right\|\right]ds\right)^p \\ &\leqslant 2^{p-1}M^p\lambda^{1-p}L_f^p\int_0^t e^{-\lambda(t-s)}\mathbf{E}\left\|x_s\right\|_{\mathscr{C}}^p ds + 2^{p-1}M^p\lambda^{-p}L_f^p \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} Q_4(t) &= \mathbf{E}\left\|\int_0^t R(t-s)g(s,x_s)dw(s)\right\|_{\mathbb{X}}^p \\ &\leqslant M^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\int_0^t e^{-\lambda p(t-s)}\mathbf{E}\left\|g(s,x_s)\right\|_{\mathcal{L}_2^0}^p\right)^{\frac{p}{2}} \\ &\leqslant M^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\int_0^t e^{2\lambda(p-1)}/(p-2)(t-s)ds\right)^{p/2-1}\int_0^t e^{-\lambda(t-s)}\mathbf{E}\left\|g(s,x_s)\right\|_{\mathcal{L}_2^0}^p ds \\ &\leqslant 2^{p-1}M^pL_g^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p-2}{2\lambda(p-1)}\right)^{p/2-1}\int_0^t e^{-\lambda(t-s)}\mathbf{E}\left\|x_s\right\|_{\mathscr{C}}^p ds \\ &\quad + 2^{p-1}M^pL_g^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p-2}{2(p-1)}\right)^{p/2-1}\lambda^{-p/2}b_g^p. \end{aligned} \tag{3.5}$$

From **(H6)** and Holder's inequality, we get

$$\begin{aligned} Q_5(t) &= \mathbf{E}\left\|\sum_{0<t_k<t} R(t-t_k)I_k(x(t_k))\right\|_{\mathbb{X}}^p \\ &\leqslant M^p\mathbf{E}\left(\sum_{0<t_k<t} d_k e^{-\lambda(t-t_k)}\left\|x(t_k^-)\right\|\right)^2 \\ &\leqslant M^p\left(\sum_{0<t_k<t} d_k\right)^{\frac{p}{q}}\sum_{0<t_k<t} d_k e^{-\lambda(t-t_k)}\mathbf{E}\left\|x(t_k^-)\right\|^p. \end{aligned} \tag{3.6}$$

Substituting (3.2)-(3.6) into (3.1), we obtain

$$\begin{aligned} \mathbf{E}\left\|x(t)\right\|^p &\leqslant 5^{p-1}M^*\left\|\varphi\right\|_{\mathcal{L}_p}^p e^{-\lambda t} + 5^{p-1}L_h^p\mathbf{E}\left\|x_t\right\|_{\mathscr{C}}^p + 5^{p-1}2^{p-1}M^p\lambda^{1-p}L_f^p\int_0^t e^{-\lambda(t-s)}\mathbf{E}\left\|x_s\right\|_{\mathscr{C}}^p ds \\ &\quad + 5^{p-1}2^{p-1}M^pL_g^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p-2}{2\lambda(p-1)}\right)^{p/2-1}\int_0^t e^{-\lambda(t-s)}\mathbf{E}\left\|x_s\right\|_{\mathscr{C}}^p ds \\ &\quad + 5^{p-1}M^p\left(\sum_{0<t_k<t} d_k\right)^{\frac{p}{q}}\sum_{0<t_k<t} d_k e^{-\lambda(t-t_k)}\mathbf{E}\left\|x(t_k^-)\right\|^p + \hat{J}. \end{aligned} \tag{3.7}$$

Let $\hat{a} := 5^{p-1}M^*$, $\hat{b_1} := 5^{p-1}L_h^p$, $\hat{b_2} := 5^{p-1}2^{p-1}M^p\lambda^{1-p}L_f^p + 5^{p-1}2^{p-1}M^pL_g^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p-2}{2\lambda(p-1)}\right)^{p/2-1}$, $\delta_k := 5^{p-1}M^p\left(\sum_{0<t_k<t} d_k\right)^{\frac{p}{q}} d_k$. From (2.3), we know $\hat{\rho} := \hat{b_1} + \hat{b_2}/r + \sum_{k=1}^{\infty}\delta_k < 1$. Since $\varphi \in$

$\mathscr{PC}^{\mathscr{B}}_{\mathscr{F}_0}([-\tau,0];\mathbb{X})$, so there exist $\hat{K} \geqslant 0$, $\hat{N} > 0$, $\gamma \in (0,\lambda)$ such that $\hat{a}\,\|\varphi\|^p_{\mathcal{L}_p} \leqslant \hat{K}$, $\hat{\rho}_\lambda := \hat{b_1}e^{\lambda\tau} + \frac{\hat{b_2}e^{\lambda\tau}}{\lambda-\gamma} + \sum_{k=1}^{\infty}\delta_k \leqslant 1$ and $\frac{(\lambda-\gamma)[\hat{K}-\frac{\hat{b_2}}{\lambda(1-\rho)}]}{\hat{b_2}e^{\lambda\tau}} \leqslant \hat{N}$. It follows from Lemma 2.5 that

$$\mathbf{E}\,\|x(t)\|^p \leqslant \hat{N}e^{-\lambda t} + \frac{\hat{J}}{1-\hat{\rho}}.$$

So, by Definition 2.3 we know that S is a attracting set of the mild solution to (1.1). When $\varphi \in S_1 = \left\{\varphi \in \mathscr{PC}^{\mathscr{B}}_{\mathscr{F}_0}([-\tau,0],\mathbb{X}) : \|\varphi\|^p \leqslant r, r > 0\right\}$, then we can write (3.7) in the following form:

$$\begin{aligned}\mathbf{E}\,\|x(t)\|^p \leqslant{}& 5^{p-1}M^* r + 5^{p-1}L_h^p\mathbf{E}\,\|x_t\|^p_{\mathscr{C}} + 5^{p-1}2^{p-1}M^p\lambda^{1-p}L_f^p\int_0^t e^{-\lambda(t-s)}\mathbf{E}\,\|x_s\|^p_{\mathscr{C}}\,ds\\ &+5^{p-1}2^{p-1}M^pL_g^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p-2}{2\lambda(p-1)}\right)^{p/2-1}\int_0^t e^{-\lambda(t-s)}\mathbf{E}\,\|x_s\|^p_{\mathscr{C}}\,ds\\ &+5^{p-1}M^p\left(\sum_{0<t_k<t}d_k\right)^{\frac{p}{q}}\sum_{0<t_k<t}d_ke^{-\lambda(t-t_k)}\mathbf{E}\left\|x(t_k^-)\right\|^p + \hat{J}.\end{aligned}$$

Thus, from Lemma 2.5, Remark 2.6, and Definition 2.2, we have

$$\mathbf{E}\,\|x(t)\|^p \leqslant (1-\hat{\rho})^{-1}(5^{p-1}M^*r+\hat{J}) = 5^{p-1}M^*(1-\hat{\rho})^{-1}r + (1-\hat{\rho})^{-1}\hat{J}.$$

So, by Definition 2.2 we know that $S_1$ is a quasi-invariant set of the mild solution to (1.1). The proof is complete. □

**Theorem 3.2.** *Assume that* **(H1)-(H6)** *hold and* $b_f = b_g = 0$ *are satisfied. Then the mild solution of system* (1.1) *is exponentially stable in pth moment, where* $p \geqslant 2$, $1/p+1/q=1$ *and the following inequality holds:*

$$\begin{aligned}\hat{\rho} ={}& 5^{p-1}L_h^p + 5^{p-1}2^{p-1}M^p\lambda^{-p}L_f^p\\ &+5^{p-1}2^{p-1}M^pL_g^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p-2}{2\lambda(p-1)}\right)^{p/2-1}\lambda^{-p/2}\\ &+5^{p-1}M^p\left(\sum_{0<t_k<t}d_k\right)^{p/q+1} < 1.\end{aligned}\tag{3.8}$$

*Proof.* From (2.4), we can get

$$\begin{aligned}\mathbf{E}\,\|x(t)\|^p_{\mathbb{X}} ={}& 5^{p-1}\mathbf{E}\,\|R(t)\,[\varphi(0)+h(t,x_t)]\|^p_{\mathbb{X}} + 5^{p-1}\mathbf{E}\,\|h(t,x_t)\|^p_{\mathbb{X}}\\ &+5^{p-1}\mathbf{E}\left\|\int_0^t R(t-s)f(s,x_s)ds\right\|^p_{\mathbb{X}} + 5^{p-1}\mathbf{E}\left\|\int_0^t R(t-s)g(s,x_s)dw(s)\right\|^p_{\mathbb{X}}\\ &+5^{p-1}\mathbf{E}\left\|\sum_{0<t_k<t}R(t-t_k)I_k(x(t_k))\right\|^p_{\mathbb{X}} = 5^{p-1}\sum_{i=1}^{5}Q_i.\end{aligned}\tag{3.9}$$

Noticing that $b_f = b_g = 0$, substituting (3.2)-(3.6) into (3.1), we obtain

$$\begin{aligned}\mathbf{E}\,\|x(t)\|^p \leqslant{}& 5^{p-1}M^*\,\|\varphi\|^p_{\mathcal{L}_p}\,e^{-\lambda t} + 5^{p-1}L_h^p\mathbf{E}\,\|x_t\|^p_{\mathscr{C}} + 5^{p-1}2^{p-1}M^p\lambda^{1-p}L_f^p\int_0^t e^{-\lambda(t-s)}\mathbf{E}\,\|x_s\|^p_{\mathscr{C}}\,ds\\ &+5^{p-1}2^{p-1}M^pL_g^p\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p-2}{2\lambda(p-1)}\right)^{p/2-1}\int_0^t e^{-\lambda(t-s)}\mathbf{E}\,\|x_s\|^p_{\mathscr{C}}\,ds\\ &+5^{p-1}M^p\left(\sum_{0<t_k<t}d_k\right)^{\frac{p}{q}}\sum_{0<t_k<t}d_ke^{-\lambda(t-t_k)}\mathbf{E}\left\|x(t_k^-)\right\|^p.\end{aligned}\tag{3.10}$$

Let $\hat{a} := 5^{p-1}M^* \|\varphi\|^p_{\mathcal{L}_p}$, $\hat{b_1} := 5^{p-1}L^p_h$, $\hat{b_2} := 5^{p-1}2^{p-1}M^p\lambda^{1-p}L^p_f + 5^{p-1}2^{p-1}M^pL^p_g\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}$ $\times\left(\frac{p-2}{2\lambda(p-1)}\right)^{p/2-1}$, $\delta_k := 5^{p-1}M^p\left(\sum_{0<t_k<t} d_k\right)^{\frac{p}{q}} d_k$. From (2.3), we know $\hat{\rho} := \hat{b_1} + \hat{b_2}/r + \sum_{k=1}^{\infty}\delta_k < 1$. Since $\varphi \in \mathscr{PC}^{\mathscr{B}}_{\mathscr{F}_0}([-\tau,0];\mathbb{X})$, so there exist $\hat{K} \geqslant 0$, $\hat{N} > 0$, $\gamma \in (0,\lambda)$ such that $\hat{a}\|\varphi\|^p_{\mathcal{L}_p} \leqslant \hat{K}$, $\hat{\rho}_\lambda := \hat{b_1}e^{\lambda\tau} + \frac{\hat{b_2}e^{\lambda\tau}}{\lambda-\gamma} + \sum_{k=1}^{\infty}\delta_k \leqslant 1$ and $\frac{(\lambda-\gamma)\hat{K}}{\hat{b_2}e^{\lambda\tau}} \leqslant \hat{N}$.

Combining (3.8), (3.10) with Lemma 2.5, by Definition 2.4 we know that the conclusion of Theorem 3.2 is true. The proof is complete. $\square$

## 4. Example

We consider the following neutral stochastic partial functional integrodifferential equations with impulsive effects:

$$\begin{cases} d\left[x(t)+u_1x_t\right] = \left[\frac{\partial^2}{\partial z^2}x(t)+u_1x_t+u_2x_t+\nu_1\right]dt + \displaystyle\int_0^t B(t-s)\frac{\partial^2}{\partial z^2}\left[x(s)+u_1x_s\right]ds \\ \qquad\qquad +u_3x_t+\nu_2 dw(t), \quad 0\leqslant z\leqslant\pi, \ t\geqslant 0, \ t\neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)) = \frac{\nu_3}{k^2}x(t_k^-), \qquad t=t_k, \ k=1,2,\ldots, \\ x(t)=\varphi(t)\in\mathscr{PC}^{\mathscr{B}}_{\mathscr{F}_0}([-\tau,0],\mathcal{L}^2[0,\pi]), x(t,0)=x(t,\pi)=0, \quad \tau\leqslant t\leqslant 0, \end{cases} \tag{4.1}$$

where $u_i > 0$, $\nu_i > 0$, $i = 1,2,3$ are constants. Let $\mathbb{X} = \mathcal{L}^2[0,\pi]$, $x_1 = W^{2,2}(0,\pi)\cap W^{1,2}_0(0,\pi)$. Define bounded linear operator $A : \mathbb{X} \to \mathbb{X}_1$ by

$$Ax = \frac{\partial^2 x}{\partial z^2} \in \mathbb{X}, \quad \forall x \in \mathbb{X}_1.$$

Then we get

$$Ax = \sum_{n=1}^{\infty} n^2 \langle x, e_n\rangle_{\mathbb{X}} e_n, \qquad x \in \mathbb{X}_1,$$

where $e_n(z) = \sqrt{\frac{2}{\pi}}\sin nz$, $n = 1,2,\ldots$ is the set of eigenvector of $-A$. It is well known that $A$ is the infinitesimal generator of a strongly continuous semigroup on $\mathbb{X}$, thus **(H1)** is true. Let $B : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ be the operator defined by

$$B(t)(z) = b(t)Az \quad \text{ for } t\geqslant 0 \text{ and } z\in\mathcal{D}(A).$$

Let

$$h(t,x_t) = u_1x_t, \quad f(t,x_t) = u_2x_t+\nu_1, \quad g(t,x_t) = u_3x_t+\nu_2.$$

Then we can get

$$M=1, \quad r=\pi^2, \quad L_h=u_1, \quad L_f=u_2, \quad b_f=\nu_1, \quad L_g=u_3, \quad b_g=\nu_2, \quad q_k=\frac{\nu_3}{k^2}.$$

Let $p = 2$, then we get

$$\begin{aligned}\hat{\rho} &= 5L_h^2 + 5M^2\left(\sum_{k=1}^{\infty} q_k\right)^2 + 10M^2\lambda^{-2}L_f^2 + 10M^2\lambda^{-1}L_g^2 \\ &\leqslant \frac{5u_1^2}{\pi} + \frac{\pi^4\nu_3^2}{5} + \frac{10u_2^2}{\pi^4} + \frac{10u_3^2}{\pi^2} = \hat{\rho}_0,\end{aligned}$$

$$\hat{J} := \frac{10\nu_1^2}{\pi^4} + \frac{10\nu_2^2}{\pi^2}.$$

From Theorem 3.1, we knows $S = \left\{x(t) \in \mathbb{X} | \mathbf{E}\,\|x(t)\|^2 \leqslant (1-\hat{\rho})^{-1}\hat{J}\right\}$ is a global attracting set of system (4.1) provided that $\hat{\rho}_0 < 1$. In additional, if $\nu_1 = \nu_2 = 0$ and $\hat{\rho}_0 < 1$, then by Theorem 3.2, we know the mild solution of system (4.1) is exponentially stable in 2nd moment.

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