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Statistical properties of type II Topp Leone inverse exponential distribution

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Abstract

In this paper a new lifetime parameters model named the type II topp leone inverse exponential (TIITLIE) model is proposed. Some fundamental properties of the TIITLIE model are calculated. The maximum likelihood (ML) method of estimation are assessed to the parameters of TIITLIE distribution. The superiority of the TIITLIE model is showed by using 63 aircraft windshields real data set.

Keywords: Inverse exponential model, type II Topp Leone-G, moments, maximum likelihood.

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1. Introduction

The inverse exponential (IE) model is proposed by Keller and Kamath [2]. If X is a non-negative random variable (rv), $X \sim E$ model then the distribution of a rv $Y = 1/X$ follows an IE distribution. The pdf and cdf of the IE model, respectively, are

$$g\left(x{:}\alpha\right) = \frac{\alpha}{x^2} e^{-\frac{\alpha}{x}}, \quad x, \alpha > 0, \tag{1.1}$$

and

$$G\left(x{:}\alpha\right) = e^{-\frac{\alpha}{x}}, \quad x, \alpha > 0. \tag{1.2}$$

Recently some authors attempts to increase the generalization of the IE model. Singh and Goel [7] proposed Beta IE distribution, Kumaraswamy IE is introduced by Oguntunde et al. [3], transmuted IE distribution is investigated by Oguntunde et al. [4], Oguntunde et al. [5] studied exponentiated generalized IE model, Weibull-IE model is proposed by Oguntunde et al. [6].

Elgarhy et al. [1] introduced *Type II Topp Leone generated (TIITL-G) distributions*. The cdf and pdf of *TIITL -G* are

$$F\left(x{:}\,\theta,\,\xi\right) = 1 - \left[1 - [G\left(x;\xi\right)]^2\right]^{\theta}, \quad x \in \mathbb{R}, \theta > 0 \tag{1.3}$$

and

$$f(x:\ \theta,\ \xi) = 2\theta g(x;\xi)\, G(x;\xi) \left[1 - [G(x;\xi)]^2\right]^{\theta-1}. \tag{1.4}$$

The rest of this article is arranged as follows. In Section 2, the TIITLIE model is proposed. Fundamental properties is calculated in Section 3. The ML method of estimation is perofrmed to show the behavior of the estimates of the TIITLIE parameters in Section 4. The application to one real data set is performed in Section 5. Finally, conclusions are mentioned in Section 6.

2. The new model

Let rv X ~ TIITLIE distribution, then, the cdf and pdf of TIITLIE model with parameters $\varphi = (\theta, \alpha)$ is obtained by inserting (1.1), (1.2) in (1.3) and (1.4) as

$$F(x;\ \varphi) = 1 - \left[1 - e^{-\frac{2\alpha}{x}}\right]^{\theta},\quad x,\theta,\alpha > 0 \tag{2.1}$$

and

$$f(x;\ \varphi) = 2\theta\alpha x^{-2} e^{-\frac{2\alpha}{x}} \left[1 - e^{-\frac{2\alpha}{x}}\right]^{\theta-1},\quad x,\theta,\alpha > 0 \tag{2.2}$$

where α is a scale parameter and θ is a shape parameter.

The survival (sf), hazard rate (hrf), and reversed hrf and cumulative hrf functions of TIITLIE model are

$$R(x;\ \varphi) = \left[1 - e^{-\frac{2\alpha}{x}}\right]^{\theta}, \qquad\qquad h(x;\ \varphi) = \frac{2\theta\alpha x^{-2} e^{-\frac{2\alpha}{x}}}{1 - e^{-\frac{2\alpha}{x}}},$$

$$\tau(x;\ \varphi) = \frac{2\theta\alpha x^{-2} e^{-\frac{2\alpha}{x}} \left[1 - e^{-\frac{2\alpha}{x}}\right]^{\theta-1}}{1 - \left[1 - e^{-\frac{2\alpha}{x}}\right]^{\theta}}, \qquad\qquad H(x;\ \varphi) = -\theta \ln\left[1 - e^{-\frac{2\alpha}{x}}\right].$$

Figure 1 shows some graphs of the pdf and hrf for TIITLIE with various values of parameters.
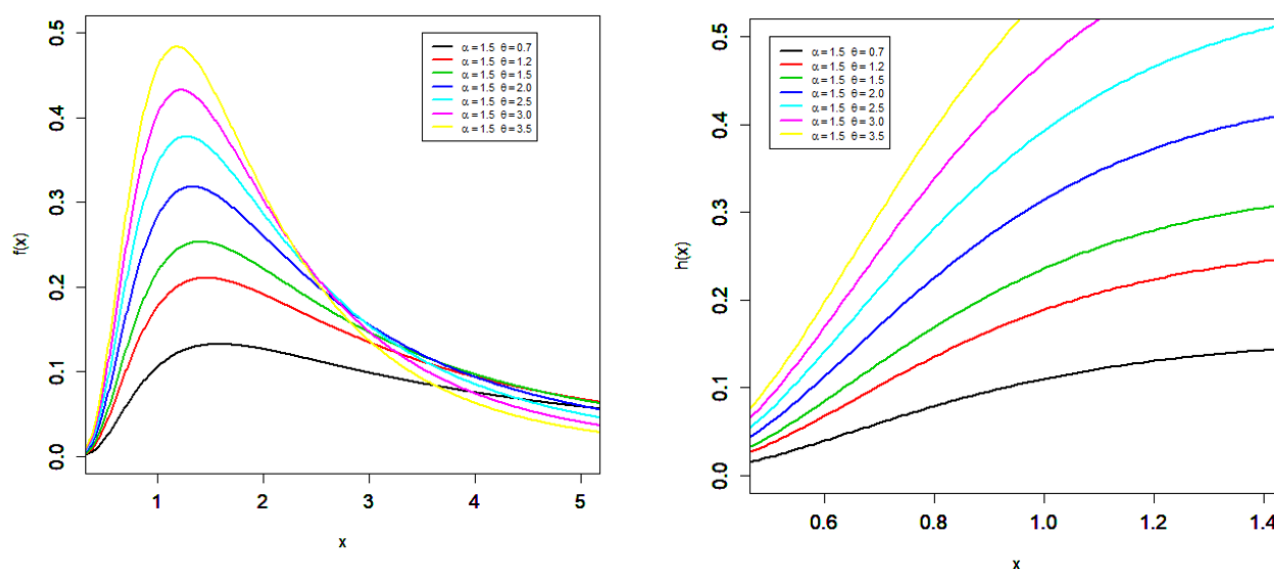

Figure 1: Plots of the pdf and hrf of the TIITLIE model.

From Figure 1, we notice that pdf of TIITLIE model can be right skewed. Also the hrf can be increasing.

3. Fundamental properties

Some fundamental properties of the TIITLIE model are calculated in this section.

3.1. Useful representation

The two useful expansions of pdf and cdf for TIITLIE distribution is derived in this subsection. Using the next binomial expansions

$$(1-a)^n = \sum_{i=0}^{\infty} (-1)^i \binom{a}{i} a^i. \tag{3.1}$$

By using (3.1), the following term in (2.2), we get

$$\left(1 - e^{-\frac{2\alpha}{x}}\right)^{\theta-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\theta-1}{i} e^{-\frac{2\alpha}{x}}. \tag{3.2}$$

Inserting (3.2) in (2.2), then, the pdf of TIITLIE is

$$f(x) = \alpha \sum_{i=0}^{\infty} W_i x^{-2} e^{-2(i+1)\frac{\alpha}{x}}, \tag{3.3}$$

where $w_i = 2(-1)^i \binom{\theta-1}{i} \theta$.

To get the expansion of $[F(x)]^h$, where h is an integer,

$$[F(x)]^h = \left[1 - \left(1 - e^{-2\frac{\alpha}{x}}\right)^\theta\right]^h.$$

Then,

$$[F(x)]^h = \sum_{k=0}^{\infty} W_k e^{-2k\frac{\alpha}{x}}, \tag{3.4}$$

where, $w_k = \sum_{j=0}^{h} (-1)^{j+k} \binom{h}{j} \binom{\theta j}{k}$.

3.2. Quantile function

If X has the cdf (2.1), then the quantile function is

$$Q(u) = \alpha \left(-\ln \sqrt{1 - (1-u)^{\frac{1}{\theta}}}\right)^{-1}, \quad 0 < u < 1.$$

The median (M) is given by

$$M = \alpha \left(-\ln \sqrt{1 - (0.5)^{\frac{1}{\theta}}}\right)^{-1}.$$

3.3. Moments

If $X \sim$ TIITLIE model then, the rth moment is

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x; \varphi) dx. \tag{3.5}$$

Inserting (3.3) in (3.5), then

$$\mu_r^{\backslash} = \alpha \sum_{i=0}^{\infty} W_i \int_0^{\infty} x^{r-2} e^{-2(i+1)\frac{\alpha}{x_i}} \, dx.$$

Let $y = \frac{\alpha}{x}$, then

$$\mu_r^{\backslash} = \alpha^r \sum_{i=0}^{\infty} W_i \int_0^{\infty} y e^{-2(i+1)y} \, dy.$$

Then, μ'_r becomes

$$\mu_r^{\backslash} = \alpha^r \sum_{i=0}^{\infty} W_i \frac{\Gamma(1-r)}{(i+1)^{1-r}}, \quad r < 1.$$

The moment generating function of X is

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) = \sum_{r,i=0}^{\infty} \frac{t^r}{r!} \frac{w_i \Gamma(1-r)}{[i+1]^{1-r}}, \quad r < 1.$$

3.4. Probability weighted moments

The PMWs is given in the next equation

$$\tau_{r,s} = E[X^r F(x)^s] = \int_{-\infty}^{\infty} x^r f(x)(F(x))^s dx. \tag{3.6}$$

By inserting (3.3) and (3.4) into (3.6), replacing h with s, leads to:

$$\tau_{r,s} = \alpha \sum_{i,k=0}^{\infty} W_i W_k \int_0^{\infty} x^{r-2} e^{-2(i+k+1)\frac{\alpha}{x}} dx.$$

Then,

$$\tau_{r,s} = \alpha^r \sum_{i,k=0}^{\infty} W_i W_k \frac{\Gamma(1-r)}{(i+k+1)^{1-r}}, \quad r < 1.$$

3.5. Rényi entropy

The Rényi entropy is defined by

$$I_\delta(X) = \frac{1}{1-\delta} \log \int_{-\infty}^{\infty} f(x; \varphi)^\delta dx, \quad \delta > 0 \text{ and } \delta \neq 1.$$

By applying the binomial theory (3.1) in the pdf (2.2), then the $f(x; \varphi)^\delta$ can be written as

$$f(x; \varphi)^\delta = \sum_{i=0}^{\infty} W_i x^{-2\delta} e^{-\frac{2\alpha[i+\delta]}{x}},$$

where

$$W_i = (2\theta\alpha)^\delta (-1)^i \begin{pmatrix} \delta(\theta-1) \\ i \end{pmatrix}.$$

Therefore, the Rényi entropy of TIITLIE distribution is

$$I_\delta(X) = \frac{1}{1-\delta} \log \left[\sum_{i=0}^{\infty} W_i \int_0^\infty x^{-2\delta} e^{-\frac{2\alpha[i+\delta]}{x}} dx \right],$$

then,

$$I_\delta(X) = \frac{1}{1-\delta} \log \left[\sum_{i=0}^{\infty} \frac{W_i \Gamma(1-2\delta)}{[2\alpha[i+\delta]]^{2\delta-1}} \right].$$

4. ML estimation

Let $X_1, \ldots, X_n$ be observed values from the TIITLIE model with parameters $\varphi = (\alpha, \theta)^\mathsf{T}$. The total log-likelihood function is

$$\ln L(\varphi) = n \ln 2\theta + n \ln \alpha - 2 \sum_{i=1}^{n} \ln x_i - 2 \sum_{i=1}^{n} \left(\frac{\alpha}{x_i} \right) + (\theta-1) \sum_{i=1}^{n} \ln \left[1 - e^{-2\left(\frac{\alpha}{x_i}\right)} \right].$$

The elements of the score function $U(\varphi) = (U_\alpha, U_\theta)$ are

$$U_\alpha = \frac{n}{\alpha} - \sum_{i=1}^{n} x_i^{-1} + 2(\theta-1) \sum_{i=1}^{n} \frac{x_i^{-1} e^{-2\left(\frac{\alpha}{x_i}\right)}}{1 - e^{-2\left(\frac{\alpha}{x_i}\right)}}, \quad \text{and} \quad U_\theta = \frac{n}{\theta} + \sum_{i=1}^{n} \ln \left[1 - e^{-2\left(\frac{\alpha}{x_i}\right)} \right].$$

Then the ML estimators of the parameters φ are investigated by putting U_φ to be zero and solving them. The numerical results is showed in Tables 1-3.

Table 1: The parameter estimation from TIITLIE model.

n	$\alpha = 0.5, \theta = 0.5$		$\alpha = 1.5, \theta = 0.5$	
	MLE	MSE	MLE	MSE
50	0.519	0.010	0.651	0.574
	0.529	0.018	1.591	0.176
100	0.511	0.006	0.559	0.077
	0.516	0.010	1.546	0.095
200	0.506	0.003	0.530	0.020
	0.509	0.005	1.530	0.044
500	0.502	0.001	0.508	0.005
	0.503	0.002	1.508	0.014

Table 2: The parameter estimation from TIITLIE model.

n	$\alpha = 1.5, \theta = 1.5$		$\alpha = 0.5, \theta = 1$	
	MLE	MSE	MLE	MSE
50	1.898	5.603	1.035	0.037
	1.592	0.171	0.531	0.020
100	1.709	0.637	1.021	0.024
	1.550	0.097	0.516	0.011
200	1.585	0.164	1.011	0.012
	1.526	0.043	0.508	0.005
500	1.522	0.043	1.004	0.004
	1.507	0.014	0.504	0.002

Table 3: The parameter estimation from TIITLIE model.

n	$\alpha = 0.5$, $\theta = 1.5$		$\alpha = 1$, $\theta = 0.5$	
	MLE	MLE	MLE	MSE
50	1.561	0.552	0.552	0.036
	0.535	1.066	1.066	0.083
100	1.536	0.529	0.529	0.017
	0.520	1.039	1.039	0.041
200	1.512	0.512	0.512	0.008
	0.508	1.018	1.018	0.020
500	1.506	0.506	0.506	0.002
	0.504	1.007	1.007	0.006

5. Application

In this section, one application to a real data set is performed to show the superiority of the TIITLIE model against IE model. The data set is taken from Tahir et al. [8] represents Failure times of 63 aircraft windshields.

The MLEs, standard errors (SEs) of the TIITLIE parameters, the minus *two log-likelihood* $(-2 \log L)$ and Kolmogorov-smirnov (k-s) are mentioned in Table 4.

Table 4: MLEs and SEs, $-2 \log L$ and k-s for the data set.

Model	MLE and SE		$-2 \log L$	k-s
TIITLIE(α, θ)	0.365 (0.067)	0.912 (0.161)	270.299	0.27504
IE(α)	0.782 (0.002)	-	270.575	0.28158

Table 4 compares the fits of the TIITLIE model against the IE model. Also, show the superiority of the TIITLIE model against IE model. The estimated cdf and estimated sf plots for the competitive models are mentioned in Figures 2 and 3.
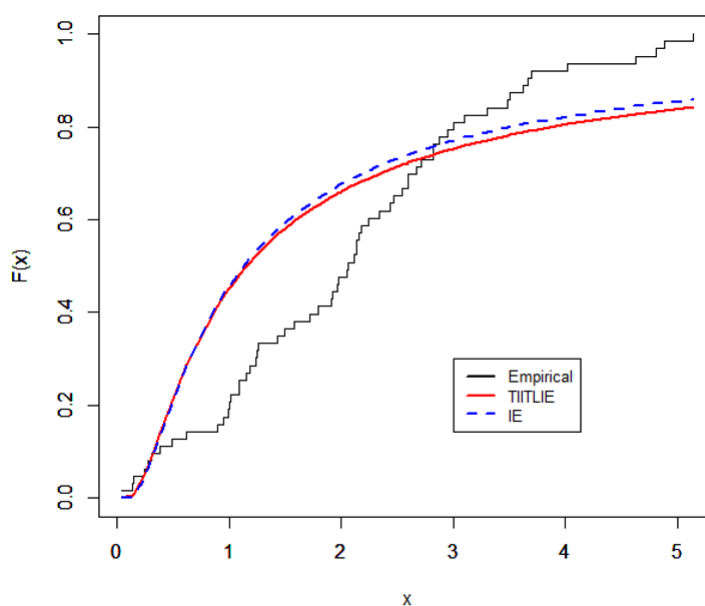

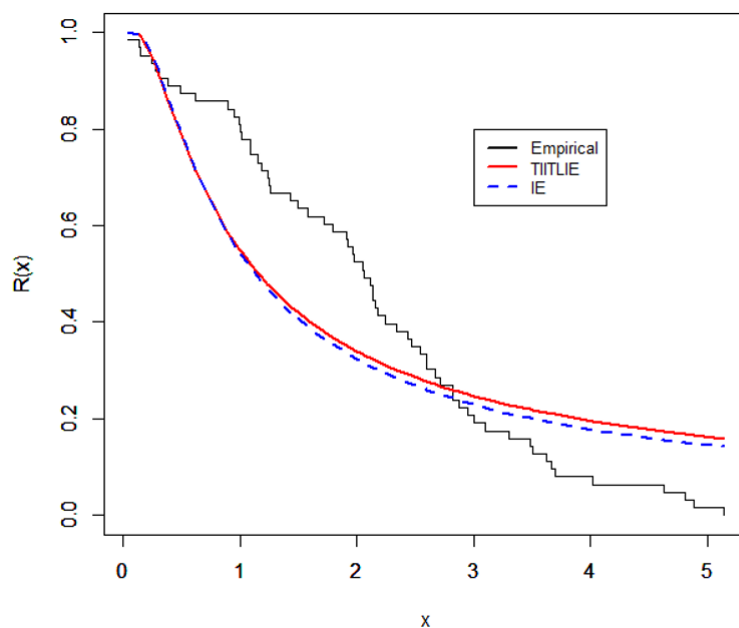Figure 2: The estimated cdf of the fitted models.

Figure 3: The estimated sf of the fitted models.

6. Conclusions

In this article, we introduce and study a new lifetime model, which called the TIITLIE model. We investigate some of its fundamental properties of the TIITLIE model. ML method of estimation is assessed to the parameters of the new model. The superiority of the TIITLIE model against IE model is showed by using the 63 aircraft windshields real data set.

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