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A generalization of Lim's lemma

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Abstract

It follows from [A. L. Dontchev, R. T. Rockafellar, Springer, New York, (2014), Theorem 5I.3] that the distance from a point x to the set of fixed points of a set-valued contraction mapping Φ is bounded by a constant times the distance from x to $\Phi(x)$. In this paper, we generalize both this result and Lim's lemma for a larger class of set-valued mappings instead of the class of set-valued contraction mappings. As consequence, we obtain some known fixed points theorems.

Keywords: Fixed point, Lim's lemma, Nadler's fixed point theorem, contraction mappings, Hardy-Rogers mappings.

2020 MSC: 47H10, 54H25.

1. Introduction and preliminaries

Let (X, d) be a metric space. We denote by $\mathcal{C}(X)$ the set of nonempty and closed subsets of X. The *extended*[1] *Hausdorff distance* between two elements A and B of $\mathcal{C}(X)$ is

$$h(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup_{a \in A} d(x, B)$ is the *excess* from the set A to the set B. The pair $(\mathcal{C}(X), h)$ is an extended metric space, see for instance [1].

Let $\Phi : X \to \mathcal{C}(X)$ be a set-valued mapping.

(i) Φ is said to be *Lipschitzian* if there exists a nonnegative constant α such that

$$h(\Phi(x), \Phi(y)) \leqslant \alpha d(x, y), \quad \forall x, y \in X.$$

The constant α is called *Lipschitz constant* of Φ. If $\alpha < 1$, Φ is said to be a *set-valued contraction mapping*.

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[1] The word "extended" refers to the possibility of the distance being ∞.

(ii) Φ is said to be a *set-valued Kannan mapping* if there exists a constant $\beta \in [0, 1/2)$ such that

$$h(\Phi(x), \Phi(y)) \leqslant \beta[d(x, \Phi(x)) + d(y, \Phi(y))], \quad \forall x, y \in X.$$

(iii) Φ is said to be a *set-valued Chatterjea mapping* if there exists a constant $\gamma \in [0, 1/2)$ such that

$$h(\Phi(x), \Phi(y)) \leqslant \gamma[d(x, \Phi(y)) + d(y, \Phi(x))] \quad \forall x, y \in X.$$

(iv) Φ is said to be a *set-valued Reich mapping* if there exists two nonnegative constants α, β with $\alpha + 2\beta < 1$ such that

$$h(\Phi(x), \Phi(y)) \leqslant \alpha d(x, y) + \beta[d(x, \Phi(x)) + d(y, \Phi(y))] \quad \forall x, y \in X.$$

(v) Φ is said to be a *set-valued Hardy-Rogers mapping* if there exists three nonnegative constants α, β, γ with $\alpha + 2\beta + 2\gamma < 1$ such that

$$h(\Phi(x), \Phi(y)) \leqslant \alpha d(x, y) + \beta[d(x, \Phi(x)) + d(y, \Phi(y))] + \gamma[d(x, \Phi(y)) + d(y, \Phi(x))] \quad \forall x, y \in X.$$

It is easy to see that the set-valued contraction mappings and the set-valued Kannan mappings are set-valued Reich mappings. It was shown by two examples in [6] that the set-valued contraction mappings and the set-valued Kannan mappings are independent. An example in [9] shows that the set-valued Reich mapping is a proper generalization of the set-velued contaction mappings and the set-valued Kannan mappings. Observe also that Hardy-Rogers set-valued mapping cover both of Reich and Chatterjea mappings set-valued.

A point $x \in X$ is said to be a *fixed point* of a set-valued mapping $\Phi : X \to \mathcal{C}(X)$ if $x \in \Phi(x)$. The set of fixed points of Φ will be denoted by $\mathrm{Fix}(\Phi)$. It follows from [3, Theorem 5I.3] that the distance from a point x to the set of fixed points of a set-valued contraction mapping Φ is bounded by a constant times the distance from x to $\Phi(x)$. Precisely, we have the following result:

Theorem 1.1 ([3]). *Let (X, d) be a complete metric space, and let $\Phi : X \to \mathcal{C}(X)$ be a set-valued contraction mapping with Lipschitz constant α. Then, for every $x \in X$,*

$$d(x, \mathrm{Fix}(\Phi)) \leqslant \frac{1}{1-\alpha} d(x, \Phi(x)).$$

Our first goal in this paper is to give a generalization of this theorem (Theorem 1.1) for set-valued Hardy-Rogers mappings instead of set-valued contaction mappings.

As application, we provide a generalization of Lim's lemma, see [7, Lemma 1], which says that the extended Hausdorff distance between the sets of fixed points of two set valued contraction mappings is bounded by a constant times the uniform extended Hausdorff distance between the mappings.

2. Main result and consequences

In this section, we present our main result, see Theorem 2.1 below, from which we will derive a list of consequences, among them Theorem 1.1. The key idea of the proof of our main result is based on an iteration procedure similar to that used in proving [4, Theorem 2.1].

Theorem 2.1. *Let (X, d) be a complete metric space, and let $\Phi : X \to \mathcal{C}(X)$ be a set-valued Hardy-Rogers mapping with constants α, β and γ. Then, for every $x \in X$,*

$$d(x, \mathrm{Fix}(\Phi)) \leqslant \frac{1-(\beta+\gamma)}{1-(\alpha+2\beta+2\gamma)} d(x, \Phi(x)). \tag{2.1}$$

Proof. By assumption, we have $\alpha + 2\beta + 2\gamma < 1$ and

$$h(\Phi(x), \Phi(y)) \leqslant \alpha d(x, y) + \beta[d(x, \Phi(x)) + d(y, \Phi(y))] + \gamma[d(x, \Phi(y)) + d(y, \Phi(x))], \tag{2.2}$$

for all $x, y \in X$.

Let $x \in X$ and $y \in \Phi(x)$. If $y = x$, then $x \in \Phi(x)$, hence $x \in \mathrm{Fix}(\Phi)$. Therefore the left side of (2.1) is 0 and the estimation (2.1) holds automatically.

Assume next that $y \neq x$. Consider a real $\alpha' > \alpha$ such that $\alpha' + 2\beta + 2\gamma < 1$ and define

$$r := \frac{\alpha' + \beta + \gamma}{1 - (\beta + \gamma)}.$$

By induction, we will construct a sequence (x_n) of elements of X, with $x_0 = x$, such that for all $n \in \mathbb{N}$

$$x_{n+1} \in \Phi(x_n) \quad \text{and} \quad d(x_n, x_{n+1}) \leqslant r^n d(x, y). \tag{2.3}$$

By taking $x_1 = y$ we obtain (2.3) for $n = 0$.

Now, suppose that we have already found $x_0, x_1, \cdots, x_m$, satisfying (2.3) for $n = 0, 1, \cdots, m-1$, for some $m \in \mathbb{N}^*$.

By assumption of induction, $x_m \in \Phi(x_{m-1})$. Then, using (2.2), we obtain

$$\begin{aligned} d(x_m, \Phi(x_m)) &\leqslant h(\Phi(x_{m-1}), \Phi(x_m)) \\ &\leqslant \alpha d(x_{m-1}, x_m) + \beta[d(x_{m-1}, \Phi(x_{m-1})) + d(x_m, \Phi(x_m))] \\ &\quad + \gamma[d(x_{m-1}, \Phi(x_m)) + d(x_m, \Phi(x_{m-1}))] \\ &\leqslant \alpha d(x_{m-1}, x_m) + \beta d(x_{m-1}, x_m) + \beta d(x_m, \Phi(x_m)) \\ &\quad + \gamma d(x_{m-1}, x_m) + \gamma d(x_m, \Phi(x_m)). \end{aligned}$$

Thus, by vertu of the assumption of induction,

$$d(x_m, \Phi(x_m)) \leqslant \frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} d(x_m, x_{m-1}) < r^m d(x, y).$$

Hence, there exists $x_{m+1} \in \Phi(x_m)$ such that

$$d(x_m, x_{m+1}) \leqslant r^m d(x, y).$$

In this moment, we have completely finished the induction step, hence (2.3) holds for every $n \in \mathbb{N}$.

Let us now prove that $(x_n)_n$ is a Cauchy sequence. Let $(n, m) \in \mathbb{N}^2$ such that $n < m$. With the triangle inequality and the use of (2.3), it follows that

$$d(x_n, x_m) \leqslant \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leqslant \sum_{k=n}^{m-1} r^k d(x, y) \leqslant \frac{r^n}{1 - r} d(x, y). \tag{2.4}$$

Since $r \in [0, 1)$ the right hand of (2.4) converges to 0 when n goes to ∞. This proves that $(x_n)_n$ is a Cauchy sequence of elements of X. By completeness of X we deduce that the sequence (x_n) converges to some $\bar{x} \in X$.

On the other hand, we have

$$\begin{aligned} d(\bar{x}, \Phi(\bar{x})) &\leqslant d(\bar{x}, x_{n+1}) + d(x_{n+1}, \Phi(\bar{x})) \\ &\leqslant d(\bar{x}, x_{n+1}) + h(\Phi(x_n), \Phi(\bar{x})) \\ &\leqslant d(\bar{x}, x_{n+1}) + \alpha d(x_n, \bar{x}) + \beta[d(x_n, \Phi(x_n)) + d(\bar{x}, \Phi(\bar{x}))] \\ &\quad + \gamma[d(x_n, \Phi(\bar{x})) + d(\bar{x}, \Phi(x_n))] \end{aligned}$$

$$\leqslant d(\bar{x}, x_{n+1}) + \alpha d(x_n, \bar{x}) + \beta[d(x_n, x_{n+1}) + d(\bar{x}, \Phi(\bar{x}))]$$
$$+\gamma[d(x_n, \bar{x}) + d(\bar{x}, \Phi(\bar{x})) + d(\bar{x}, x_{n+1})].$$

Passing to the limit in this latter when n goes to $+\infty$, we get

$$d(\bar{x}, \Phi(\bar{x})) \leqslant (\beta + \gamma) d(\bar{x}, \Phi(\bar{x})).$$

Since $\beta + \gamma < 1$, it follows that $d(\bar{x}, \Phi(\bar{x})) = 0$. Therefore, $\bar{x} \in \mathrm{Fix}(\Phi)$ because $\Phi(\bar{x})$ is closed. By taking $n = 0$ and letting m goes to ∞ in (2.4), we obtain

$$d(x, \mathrm{Fix}(\Phi)) \leqslant d(x, \bar{x}) \leqslant \frac{1}{1-r} d(x, y) = \frac{1-(\beta+\gamma)}{1-(\alpha'+2\beta+2\gamma)} d(x, y).$$

Also, by letting α' goes to α in this latter, we get

$$d(x, \mathrm{Fix}(\Phi)) \leqslant \frac{1-(\beta+\gamma)}{1-(\alpha+2\beta+2\gamma)} d(x, y).$$

By taking the infimum over $y \in \Phi(x)$, we conclude the required estimation

$$d(x, \mathrm{Fix}(\Phi)) \leqslant \frac{1-(\beta+\gamma)}{1-(\alpha+2\beta+2\gamma)} d(x, \Phi(x)).$$

□

As a first consequence of our main result (taking $\beta = \gamma = 0$) we obtain Theorem 1.1. A second consequence is [4, Theorem 2.1], its statement is given in the following corollary, which generalizes both Nadler's fixed point theorem [8, Theorem 5] and the fixed point theorem of Hardy and Rogers [5, Theorem 1].

Corollary 2.2. *Let (X, d) be a complete metric space and let $\Phi : X \to \mathcal{C}(X)$ be a set-valued Hardy-Rogers mapping with constants α, β and γ. Then Φ has a fixed point.*

In the following corollaries, we give some other consequences of our main result.

Corollary 2.3. *Let (X, d) be a complete metric space, and let $\Phi : X \to \mathcal{C}(X)$ be a set-valued mapping such that*

$$h(\Phi(x), \Phi(y)) \leqslant a_1 d(x, y) + a_2 d(x, \Phi(x)) + a_3 d(y, \Phi(y)) + a_4 d(x, \Phi(y)) + a_5 d(y, \Phi(x)),$$

for all $x, y \in X$, where $a_i \geqslant 0$ for each $i \in \{1, 2, \cdots, 5\}$ and $\sum_{i=1}^{5} a_i < 1$. Then, for every $x \in X$,

$$d(x, \mathrm{Fix}(\Phi)) \leqslant \frac{2 - \sum_{i=2}^{5} a_i}{2 - 2\sum_{i=1}^{5} a_i} d(x, \Phi(x)).$$

Corollary 2.4. *Let (X, d) be a complete metric space, and let $\Phi : X \to \mathcal{C}(X)$ be a set-valued Kannan mapping with constant β. Then, for every $x \in X$,*

$$d(x, \mathrm{Fix}(\Phi)) \leqslant \frac{1-\beta}{1-2\beta} d(x, \Phi(x)).$$

Kannan's fixed point theorem [6] is straight from this corollary.

Corollary 2.5. *Let (X, d) be a complete metric space, and let $\Phi : X \to \mathcal{C}(X)$ be a set-valued Chatterjea mapping with constant γ. Then, for every $x \in X$,*

$$d(x, \mathrm{Fix}(\Phi)) \leqslant \frac{1-\gamma}{1-2\gamma} d(x, \Phi(x)).$$

Chatterjea's fixed point theorem [2] can be directly derived from this corollary.

Corollary 2.6. *Let (X, d) be a complete metric space, and let $\Phi : X \to \mathcal{C}(X)$ be a set-valued Reich mapping with constants α and β. Then, for every $x \in X$,*

$$d(x, \mathrm{Fix}(\Phi)) \leqslant \frac{1-\beta}{1-(\alpha+2\beta)} d(x, \Phi(x)).$$

Riech's fixed point theorem [9] easily follows from this corollary.

3. A generalization of Lim's lemma

In 1985, Lim proved the following theorem called in modern literature Lim's lemma.

Theorem 3.1 ([7, Lemma 1])**.** *Let (X, d) be a complete metric space, and let T_1 and T_2 be two set-valued contraction mappings from X into $\mathcal{C}(X)$ with the same Lipschitz constant α. Then*

$$h(\mathrm{Fix}(T_1), \mathrm{Fix}(T_2)) \leqslant \frac{1}{1-\alpha} \sup_{x \in X} h(T_1(x), T_2(x)).$$

A variant of this result has been recently appeared in [3] as follows.

Theorem 3.2 ([3, Theorem 5I.4])**.** *Let (X, d) be a complete metric space and let T_1 and T_2 be two set-valued contraction mappings from X into $\mathcal{C}(X)$ with the same Lipschitz constant α. Then*

$$e(\mathrm{Fix}(T_1), \mathrm{Fix}(T_2)) \leqslant \frac{1}{1-\alpha} \sup_{x \in X} e(T_1(x), T_2(x)).$$

In the following theorem we extend this result (Theorem 3.2) for set-valued Hardy-Rogers mappings instead of set-valued contaction mappings.

Theorem 3.3. *Let (X, d) be a complete metric space, and let T_1 and T_2 be two set-valued Hardy-Rogers mappings from X into $\mathcal{C}(X)$ with the same constants α, β and γ. Then*

$$e(\mathrm{Fix}(T_1), \mathrm{Fix}(T_2)) \leqslant \frac{1-(\beta+\gamma)}{1-(\alpha+2\beta+2\gamma)} \sup_{x \in X} e(T_1(x), T_2(x)).$$

Proof. By assumption $T_2 : X \to \mathcal{C}(X)$ is a set-valued Hardy-Rogers mapping with constants α, β and γ. Then, applying Theorem 2.1 whith $\Phi = T_2$, we have for any $x \in X$

$$d(x, \mathrm{Fix}(T_2)) \leqslant \frac{1-(\beta+\gamma)}{1-(\alpha+2\beta+2\gamma)} d(x, T_2(x)). \tag{3.1}$$

Passing to the supremum in (3.1) with respect to $x \in \mathrm{Fix}(T_1)$ we have

$$\begin{aligned} e(\mathrm{Fix}(T_1), \mathrm{Fix}(T_2)) &\leqslant \frac{1-(\beta+\gamma)}{1-(\alpha+2\beta+2\gamma)} \sup_{x \in \mathrm{Fix}(T_1)} d(x, T_2(x)) \\ &\leqslant \frac{1-(\beta+\gamma)}{1-(\alpha+2\beta+2\gamma)} \sup_{x \in \mathrm{Fix}(T_1)} e(T_1(x), T_2(x)) \\ &\leqslant \frac{1-(\beta+\gamma)}{1-(\alpha+2\beta+2\gamma)} \sup_{x \in X} e(T_1(x), T_2(x)). \end{aligned}$$ □

As a corollary of Theorem 3.3, we obtain the following generalization of Lim's lemma (Theorem 3.1).

Corollary 3.4. *Let (X, d) be a complete metric space, and let T_1 and T_2 be two set-valued Hardy-Rogers mappings from X into $\mathcal{C}(X)$ with the same constants α, β and γ. Then*

$$h(\mathrm{Fix}(T_1), \mathrm{Fix}(T_2)) \leqslant \frac{1-(\beta+\gamma)}{1-(\alpha+2\beta+2\gamma)} \sup_{x \in X} h(T_1(x), T_2(x)).$$

We deduce immediately from Corollary 3.4 the following result, which can be regarded as an extension of [3, Corollary 5I.6].

Corollary 3.5. *Let (X, d) be a complete metric space, and let (Y, δ) be a metric space. Consider a mapping $M : Y \times X \to \mathcal{C}(X)$ having the following properties:*

(i) $M(y, \cdot)$ *is a set-valued Hardy-Rogers mapping with* α, β *and* γ *uniformly in* $y \in Y$*;*

(ii) $M(\cdot, x)$ *is Lipschitzian with a constant* λ *uniformly in* $x \in X$*.*

Then, the mapping $y \mapsto \mathrm{Fix}(M(y, \cdot))$ *is Lipschitzian with constant* $\lambda \frac{1-(\beta+\gamma)}{1-(\alpha+2\beta+2\gamma)}$.

Next, we give in the following corollaries some particular cases of Theorem 3.3.

Corollary 3.6. *Let* (X, d) *be a complete metric space, and* T_1 *and* T_2 *be two set-valued Kannan mappings from* X *into* $\mathcal{C}(X)$ *with the same constant* β*. Then*

$$e(\mathrm{Fix}(T_1), \mathrm{Fix}(T_2)) \leqslant \frac{1-\beta}{1-2\beta} \sup_{x \in X} e(T_1(x), T_2(x)).$$

Corollary 3.7. *Let* (X, d) *be a complete metric space, and let* T_1 *and* T_2 *be two set-valued Chatterjea mappings from* X *into* $\mathcal{C}(X)$ *with the same constant* γ*. Then*

$$e(\mathrm{Fix}(T_1), \mathrm{Fix}(T_2)) \leqslant \frac{1-\gamma}{1-2\gamma} \sup_{x \in X} e(T_1(x), T_2(x)).$$

Corollary 3.8. *Let* (X, d) *be a complete metric space, and let* T_1 *and* T_2 *be two set-valued Reich mappings from* X *into* $\mathcal{C}(X)$ *with the same constants* α *and* β*. Then*

$$e(\mathrm{Fix}(T_1), \mathrm{Fix}(T_2)) \leqslant \frac{1-\beta}{1-(\alpha+2\beta)} \sup_{x \in X} e(T_1(x), T_2(x)).$$

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