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Strong convergence theorems for mixed equilibrium problems and Bregman relatively nonexpansive mappings in reflexive Banach spaces

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Abstract

In this paper, we propose a new iterative method for solving the mixed equilibrium problems and the fixed point problems for a countable family of Bregman relatively nonexpansive mappings in reflexive Banach spaces. We prove that the sequence generated by the proposed iterative algorithm converges strongly to a common solution of the mentioned problems. Further, a numerical example of the iterative algorithm supporting our main result is presented.

Keywords: Mixed equilibrium problems, Bregman relatively nonexpansive mappings, reflexive Banach spaces.

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1. Introduction

Throughout this paper, let C be a nonempty closed convex subset of a reflexive Banach space E and denote the dual sapce of E by E^*. The norm and the dual pair between E and E^* are denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. We denote the set of fixed points of a mapping T on a subset C of E by $F(T) = \{x \in C : Tx = x\}$ and $\mathbb{R}$ is the set of all real numbers. Let $G : C \times C \longrightarrow \mathbb{R}$ be a bifunction and $\psi : C \longrightarrow \mathbb{R}$ be a real-valued function. We consider the following mixed equilibrium problem which is to find $x \in C$ such that

$$G(x, y) + \psi(y) - \psi(x) \geqslant 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of the problem (1.1) is denoted by $MEP(G, \psi)$ and studied by Ceng and Yao [12]. If we set ψ to be the zero mapping, then the mixed equilibrium problem (1.1) becomes the following equilibrium problem, find $x \in C$ such that

$$G(x, y) \geqslant 0, \quad \forall y \in C. \tag{1.2}$$

The solution set of the problem (1.2) is denoted by $EP(G)$ which is introduced and studied by Blum and Oettli [5]. The equilibrium problem provided a very general formulation of variational problems such as:

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(i) minimization problem: find $x \in C$ such that $h(x) \leqslant h(y)$ for all $y \in C$, where $h : C \longrightarrow \mathbb{R}$ is a functional, in this case, we define $G(x,y) = h(y) - h(x)$ for all $x, y \in C$;
(ii) variational inequality: find $x \in C$ such that $\langle A(x), y - x\rangle \geqslant 0$ for all $y \in C$, where $A : C \longrightarrow E^*$ is a mapping, in this case, we define $G(x,y) = \langle A(x), y - x\rangle$ for all $x, y \in C$.

In 2008, Ceng and Yao [12] investigated the problem of finding a common element of the set of solutions of the mixed equilibrium problem (1.1) and the set of common fixed points of finitely many nonexpansive mappings in real Hilbert spaces.

Whenever the researchers attempted to extend this theory to generalized Banach spaces, they discovered some difficulties and there are a lot of ways to overpower these barriers, for instant, using the Bregman distance in place of the norm, Bregman (quasi-) nonexpansive mappings in place of the (quasi-) nonexpansive mappings and the Bregman projection in place of the metric projection.

In 1967, Bregman [6] discovered an elegant and effective technique using the Bregman distance function $D_f(\cdot,\cdot)$ in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which the Bregman's technique has been applied in various ways in order to design and analyze iterative algorithms for solving the feasibility and optimization problems, for approximating the variational inequalities and equilibrium problems, for computing the fixed points of nonlinear mappings and so on (see, e.g., [7, 14–16, 20, 22, 27] and the references therein).

In 2013, Agarwal et al. [1] proved the strong convergence theorems for finding the common solutions of the equilibrium problem (1.2) and the fixed point problem of a weak Bregman relatively nonexpansive mapping in real reflexive Banach spaces. Recently, Kazmi et al. [18] introduced the following algorithm:

$$\begin{cases} x_1, z_1 \in C, \\ u_n = \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(Tx_n)), \\ z_{n+1} = \mathrm{Res}^f_{G,\phi} u_n, \\ C_n = \{z \in C : D_f(z, z_{n+1}) \leqslant \alpha_n D_f(z, z_n) + (1-\alpha_n) D_f(z, x_n)\}, \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n\rangle \leqslant 0\}, \\ x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n} x_1, \quad \forall n \geqslant 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. They proved a strong convergence theorem for finding a common solution of a generalized equilibrium problem and a fixed point problem for a Bregman relatively nonexpansive mapping in reflexive Banach spaces.

Recall the generalized equilibrium problem which is to find $x \in C$ such that

$$G(x,y) + \phi(x,y) - \phi(x,x) \geqslant 0, \quad \forall y \in C, \tag{1.4}$$

where $\phi : C \times C \longrightarrow \mathbb{R}$ is a bifunction. The solution set of the problem (1.4) is denoted by $\mathrm{GEP}(G,\phi)$.

Motivated and inspired by above works, the purpose of this paper is to establish a new iterative method for finding a common solution of the mixed equilibrium problems and the fixed point problems for a countable family of Bregman relatively nonexpansive mappings in reflexive Banach spaces. The strong convergence theorems under suitable control conditions are proven and a numerical example of the iterative algorithm supporting our main result is also illustrated.

2. Preliminaries

Throughout this paper, we let E be a reflexive Banach space and with dual E^*, $f : E \to (-\infty, +\infty]$ be a proper lower semicontinuous convex function. We denote the domain of f by $\mathrm{dom} f$, that is $\mathrm{dom} f = \{x \in E : f(x) < +\infty\}$. The subdifferential of f at $x \in \mathrm{int}(\mathrm{dom} f)$ is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leqslant f(y), \quad \forall y \in E\},$$

and the Fenchel conjugate of f is the function $f^* : E^* \longrightarrow (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

We know that the Young-Fenchel inequality holds:

$$\langle x^*, x\rangle \leqslant f(x) + f^*(x^*), \quad \forall x \in E,\ x^* \in E^*.$$

Furthermore, we know that $x^* \in \partial f(x)$ if and only if $f(x) + f^*(x^*) = \langle x^*, x\rangle$ for all $x \in E$. It is not difficult to check that f^* is a proper convex and lower semicontinuous function. A function f on E is said to be strong coercive if

$$\lim_{\|x\|\to+\infty}\left(\frac{f(x)}{\|x\|}\right) = +\infty.$$

For any $x \in \mathrm{int}(\mathrm{dom} f)$ and $y \in E$, the right-hand derivative of f at x in the direction y is defined by

$$f^0(x, y) := \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{2.1}$$

The function f is said to be Gâteaux differentiable at x if the limit as $t \longrightarrow 0^+$ in (2.1) exists for any y. In this case, the gradient of f at x is the linear function $\nabla f(x)$, which is defined by $\langle y, \nabla f(x)\rangle := f^0(x, y)$ for all $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in \mathrm{int}(\mathrm{dom} f)$. When the limit as $t \longrightarrow 0^+$ in (2.1) is attained uniformly $\|y\| = 1$, we say that f is Frêchet differentiable at x. Finally f is said to be uniform Frêchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

The Legendre function f is defined from a general Banach space E into $(-\infty, +\infty]$, see [4]. It is well known that in reflexive spaces, f is the Legendre function if and only if it satisfies the following conditions:

(L_1) $\mathrm{int}(\mathrm{dom} f) \neq \emptyset$, f is Gâteaux differentiable on $\mathrm{int}(\mathrm{dom} f)$ and $\mathrm{dom}\nabla f = \mathrm{int}(\mathrm{dom} f)$;
(L_2) $\mathrm{int}(\mathrm{dom} f^*) \neq \emptyset$, f^* is Gâteaux differentiable on $\mathrm{int}(\mathrm{dom} f^*)$ and $\mathrm{dom}\nabla f^* = \mathrm{int}(\mathrm{dom} f^*)$.

Remark 2.1 ([4]). If E is a reflexive Banach space and $f : E \longrightarrow (-\infty, +\infty]$ is the Legendre function, then all of the following conditions are true:

(a) f is the Legendre function if and only if f^* is the Legendre function;
(b) $(\partial f)^{-1} = \partial f^*$;
(c) $\nabla f = (\nabla f^*)^{-1}$, $\mathrm{ran}\nabla f = \mathrm{dom}\nabla f^* = \mathrm{int}(\mathrm{dom} f^*)$, $\mathrm{ran}\nabla f^* = \mathrm{dom}\nabla f = \mathrm{int}(\mathrm{dom} f)$;
(d) the functions f and f^* are strictly convex on the interior of respective domains.

Example 2.2 ([4])**.** Let E be a smooth and strictly convex Banach space. One important and interesting Legendre function is $\frac{1}{p}\|\cdot\|^p (1 < p < \infty)$. In this case, the gradient ∇f of f is coincident with the generalized duality mapping of E, i.e., $\nabla f = J_p$ $(1 < p < \infty)$. In particular, $\nabla f = I$ the identity mapping in Hilbert spaces.

Definition 2.3 ([6])**.** Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable and convex function. The Bregman distance with respect to f is the bifunction $D_f : \mathrm{dom} f \times \mathrm{int}(\mathrm{dom} f) \longrightarrow [0, +\infty)$ defined by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle. \tag{2.2}$$

Remark 2.4 ([23]). The Bregman distance D_f is not a distance in the usual sense because D_f is not symmetric and does not satisfy the triangle inequality. However, D_f satisfies the three point identity:

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle,$$

for any $x \in \mathrm{dom} f$ and $y, z \in \mathrm{int}(\mathrm{dom} f)$.

Definition 2.5 ([6])**.** Let C be a nonempty closed convex subset of $\mathrm{int}(\mathrm{dom} f)$, $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable and convex function. The Bregman projection with respect to f of $x \in \mathrm{int}(\mathrm{dom} f)$ onto C is defined as the necessarily unique vector $\mathrm{proj}^f_C(x) \in C$, which satisfies

$$D_f(\mathrm{proj}^f_C(x), x) = \inf\{D_f(y, x) : y \in C\}. \tag{2.3}$$

Remark 2.6 ([1]). In Example 2.2, if $f(x) = \frac{1}{2}\|x\|^2$, $\forall x \in E$, then we have $\nabla f = J$, where J is the normalized duality mapping from E to 2^{E^*}, and hence $D_f(x, y)$ is reduced to the Lyapunov function defined by $\Phi(x, y) = \|y\|^2 - 2\langle Jx, y\rangle + \|x\|^2$, $\forall x, y \in E$, which is introduced by Alber [2], and so we obtain that the Bregman projection $\mathrm{proj}_C^f(x)$ is reduced to the generalized projection $\prod_C(x)$, which is defined by

$$\Phi(\prod_C(x), x) = \min_{y \in C} \Phi(y, x).$$

Moreover, in Hilbert spaces, the Bregman projection $\mathrm{proj}_C^f(x)$ is reduced to the metric projection of x onto C.

Definition 2.7 ([8]). Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable and convex function, $\nu_f : \mathrm{int}(\mathrm{dom}f) \times [0, +\infty) \longrightarrow [0, +\infty)$, define the modulus of total convexity of the function f at x by

$$\nu_f(x, t) := \inf\{D_f(y, x) : y \in \mathrm{dom}f, \|y - x\| = t\}.$$

Then the function f is called to be

(a) totally convex at a point $x \in \mathrm{int}(\mathrm{dom}f)$, if the modulus of total convexity of the function f at x is positive, $\nu_f(x, t) > 0$ whenever $t > 0$;

(b) totally convex, if it is totally convex at every point $x \in \mathrm{int}(\mathrm{dom}f)$, let B be a nonempty bounded subset of E, define the modulus of total convexity of the function f on the set B by

$$\nu_f(B, t) := \inf\{\nu_f(x, t) : x \in B \cap \mathrm{dom}f\};$$

(c) totally convex on bounded sets, if the modulus of total convexity of the function f on the set B is positive, $\nu_f(B, t) > 0$ for any nonempty bounded subset B of E and $t > 0$.

Lemma 2.8 ([9]). *Let $f : E \longrightarrow (-\infty, +\infty]$ be a Legendre function. Then, the function f is totally convex on bounded sets if and only if f is uniformly convex on bounded subsets of E.*

Lemma 2.9 ([29]). *Let $f : E \longrightarrow \mathbb{R}$ be a strong coercive and uniformly convex on bounded subsets of E, then f^* is bounded and uniformly Frêchet differentiable on bounded subsets of E^*.*

Lemma 2.10 ([21]). *Let C be a bounded subset of a reflexive Banach space E and $f : E \to (-\infty, +\infty]$ be uniformly Frêchet differentiable and bounded on $C \subset E$. Then, f is uniformly continuous on $C \subset E$ and ∇f is uniformly continuous on a bounded subset C from the strong topology of E to the strong topology of E^*.*

Definition 2.11 ([22]). The function $f : E \longrightarrow (-\infty, +\infty]$ is called sequentially consistent, if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathrm{int}(\mathrm{dom}f)$ and $\mathrm{dom}f$, respectively such that sequence $\{x_n\}$ is bounded, then

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \ \text{ implies } \ \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 2.12 ([11]). *If $f : E \longrightarrow (-\infty, +\infty]$ is a convex function whose domain contains at least two points, then, f is totally convex on bounded sets if and only if it is sequentially consistent.*

Let $f : E \longrightarrow \mathbb{R}$ be a Legendre and Gâteaux differentiable function. We make use of the function $V_f : E \times E^* \longrightarrow [0, +\infty)$ associated with f, which is defined by

$$V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \quad \forall x \in E, x^* \in E^*.$$

Then V_f is nonnegative and

$$V_f(x, x^*) = D_f(x, \nabla f(x^*)), \quad \forall x \in E, x^* \in E^*.$$

Moreover, by the subdifferential inequality,

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \leqslant V_f(x, x^* + y^*), \quad \forall x \in E, x^*, y^* \in E^*,$$

(for more details see [2]).

Lemma 2.13 ([19]). *Let $f : E \longrightarrow (-\infty, +\infty]$ be a proper lower semicontinuous and convex function, then $f^* : E^* \longrightarrow (-\infty, +\infty]$ is proper weak* lower semicntinuous and convex. Hence, V_f is convex in the second variable. Thus, for all $z \in E$, we have*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \leqslant \sum_{i=1}^{N} t_i D_f(z, x_i), \tag{2.4}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.

Lemma 2.14 ([22]). *Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex function. If $x_1 \in E$ and the sequence $\{D_f(x_n, x_1)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 2.15 ([25]). *Let $f : E \longrightarrow (-\infty, +\infty]$ be a Legendre function such that ∇f^* is bounded on bounded subsets of $\text{int}(\text{dom}f)$. If $x_1 \in E$ and $\{D_f(x_1, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

Lemma 2.16 ([11]). *Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex function on $\text{int}(\text{dom}f)$. Let $x \in \text{int}(\text{dom}f)$ and $C \subset \text{int}(\text{dom}f)$ be a nonempty closed convex set. If $z \in C$, then the following conditions are equivalent:*

(i) *the vector $z \in C$ is the Bregman projection of x onto C with respect to f, i.e., $z = \text{proj}_C^f(x)$;*
(ii) *the vector $z \in C$ is the unique solution of the variational inequality:*

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \geqslant 0, \quad \forall y \in C;$$

(iii) *the vector z is the unique solution of the inequality:*

$$D_f(y, z) + D_f(z, x) \leqslant D_f(y, x), \quad \forall y \in C. \tag{2.5}$$

Definition 2.17 ([20]). Let T be a mapping from C into itself. A point $\hat{x} \in C$ is said to be an asymptotic fixed point of T if there exists a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup \hat{x}$ and $\|x_n - Tx_n\| \longrightarrow 0$. We denote the set of asymptotic fixed points of T by $\hat{F}(T)$.

Definition 2.18 ([13]). Let $T : C \longrightarrow \text{int}(\text{dom}f)$ be a mapping. Then

(a) T is said to be Bregman quasi-nonexpansive if

$$F(T) \neq \emptyset \ \text{ and } \ D_f(p, Tx) \leqslant D_f(p, x), \quad \forall x \in C, p \in F(T);$$

(b) T is said to be Bregman relatively nonexpansive if

$$\hat{F}(T) = F(T) \neq \emptyset \ \text{ and } \ D_f(p, Tx) \leqslant D_f(p, x), \quad \forall x \in C, p \in F(T);$$

(c) T is said to be Bregman firmly nonexpansive if

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leqslant D_f(Tx, y) + D_f(Ty, x), \quad \forall x, y \in C.$$

Assumption 2.19. Let $G : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying the following assumptions:

(i) $G(x, x) = 0$ for all $x \in C$;
(ii) G is monotone, i.e., $G(x, y) + G(y, x) \leqslant 0$ for all $x, y \in C$;
(iii) for each $x, y, z \in C$, $\limsup_{t \to 0^+} G(tz + (1-t)x, y) \leqslant G(x, y)$;
(iv) for each $x \in C$, $G(x, \cdot)$ is convex and lower semicontinuous.

Assumption 2.20. The function $\psi : C \longrightarrow \mathbb{R}$ satisfies the following assumptions:

(i) ψ is lower semicontinuous;

(ii) ψ is convex.

Lemma 2.21 ([17]). *Let $f : E \to (-\infty, +\infty]$ be a strong coercive Legendre function and C be a nonempty closed convex subset of $\operatorname{int}(\operatorname{dom} f)$. Let $G : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 2.19 and $\psi : C \longrightarrow \mathbb{R}$ satisfying Assumption 2.20. For $x \in E$ and define a mapping $\operatorname{Res}^f_{G,\psi} : E \to 2^C$ as follows:*

$$\operatorname{Res}^f_{G,\psi}(x) = \{z \in C : G(z,y) + \psi(y) - \psi(z) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \geqslant 0, \forall y \in C\}.$$

Then the following statements are true:

(1) $\operatorname{Res}^f_{G,\psi}$ is single-valued and $\operatorname{dom}(\operatorname{Res}^f_{G,\psi}) = E$;
(2) $\operatorname{Res}^f_{G,\psi}$ is Bregman firmly nonexpansive;
(3) $\operatorname{MEP}(G,\psi)$ is a closed convex subset of C and $\operatorname{MEP}(G,\psi) = F(\operatorname{Res}^f_{G,\psi})$;
(4) for all $x \in E, u \in F(\operatorname{Res}^f_{G,\psi})$,

$$D_f(u, \operatorname{Res}^f_{G,\psi}x) + D_f(\operatorname{Res}^f_{G,\psi}x, x) \leqslant D_f(u,x). \tag{2.6}$$

Let $CB(C)$ denote the family of nonempty closed bounded subsets of C.

Lemma 2.22 ([26]). *Let E be a reflexive Banach space, and let $f : E \longrightarrow \mathbb{R}$ be uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty closed and convex subset of $\operatorname{int}(\operatorname{dom} f)$ and $T : C \longrightarrow CB(C)$ be a Bregman relatively nonexpansive mapping. Then $F(T)$ is closed and convex.*

Lemma 2.23 ([22]). *Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex function, x_1 be an element in E and C be a nonempty closed convex subset of E. Suppose that the sequence $\{x_n\}$ is bounded and the weak limits of any subsequence of a sequence $\{x_n\}$ belong to $C \subset E$. If $D_f(x_n, x_1) \leqslant D_f(\operatorname{proj}^f_C(x_1), x_1)$ for any $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $\operatorname{proj}^f_C(x_1)$.*

3. Main Result

In this section, we prove the strong convergence theorems for the common solutions of the mixed equilibrium problems and the common fixed points for a countable family of Bregman relatively nonexpansive mappings in reflexive Banach spaces.

Theorem 3.1. *Let E be a reflexive Banach space with dual E^* and C be a nonempty closed convex subset of E such that $C \subset \operatorname{int}(\operatorname{dom} f)$. Let $f : E \longrightarrow (-\infty, +\infty]$ be a strong coercive Legendre function which is bounded uniformly Fréchet differentiable and totally convex on bounded subsets of E, $G : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying the Assumption 2.19 and $\psi : C \longrightarrow \mathbb{R}$ satisfy the Assumption 2.20. Let $\{T_i : C \longrightarrow C\}_{i=1}^N$ be a countable family of Bregman relatively nonexpansive mappings. Assume that $\Omega := \bigcap_{i=1}^N F(T_i) \cap \operatorname{MEP}(G,\psi) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by the iterative scheme:*

$$\begin{cases} x_1 \in C, T_i x_1 = z_1^i \in C; \\ u_n^i = \nabla f^*(\alpha_n \nabla f(z_n^i) + (1-\alpha_n)\nabla f(T_i x_n)); \\ z_{n+1}^i = \operatorname{Res}^f_{G,\psi}(u_n^i); \\ C_n^i = \{z \in C : D_f(z, z_{n+1}^i) \leqslant \alpha_n D_f(z, z_n^i) + (1-\alpha_n) D_f(z, x_n)\}; \\ C_n = \displaystyle\bigcap_{i=1}^N C_n^i; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n\rangle \leqslant 0\}; \\ x_{n+1} = \operatorname{proj}^f_{C_n \cap Q_n} x_1, \quad \forall n \geqslant 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $\operatorname{proj}^f_\Omega x_1$ where $\operatorname{proj}^f_\Omega x_1$ is the Bregman projection of C onto Ω.

Proof. The proof is separated into seven steps.

Step 1: We will show that Ω is closed and convex. By the result of Lemma 2.22, we obtain that $F(T_i)$ is closed and convex for all $i = 1, 2, \ldots, N$ which implies that $\bigcap_{i=1}^{N} F(T_i)$ is also and follows from Lemma 2.21 (3), we have $MEP(G, \psi)$ is closed and convex and hence $\Omega := \bigcap_{i=1}^{N} F(T_i) \cap MEP(G, \psi)$ is closed and convex.

Step 2: We will prove that $C_n \cap Q_n$ is closed and convex for all n. First, we will show that Q_n is convex for all $n \geqslant 1$. Let $a, b \in Q_n$ and $t \in [0, 1]$, setting $w = ta + (1-t)b$. Then

$$\langle \nabla f(x_1) - \nabla f(x_n), a - x_n \rangle \leqslant 0 \tag{3.2}$$

and

$$\langle \nabla f(x_1) - \nabla f(x_n), b - x_n \rangle \leqslant 0. \tag{3.3}$$

Multiplying t and $(1-t)$ on both sides of (3.2) and (3.3), respectively, we obtain that

$$\langle \nabla f(x_1) - \nabla f(x_n), ta + (1-t)b - x_n \rangle \leqslant 0,$$

implies that

$$\langle \nabla f(x_1) - \nabla f(x_n), w - x_n \rangle \leqslant 0.$$

Therefore, $w \in Q_n$ and so Q_n is convex. Let $\{v_m\}$ be a sequence in Q_n with $v_m \longrightarrow v$ as $m \longrightarrow \infty$. From the definition of Q_n, we have

$$\langle \nabla f(x_1) - \nabla f(x_n), v_m - x_n \rangle \leqslant 0,$$

implies that

$$\langle \nabla f(x_1) - \nabla f(x_n), v_m - v \rangle + \langle \nabla f(x_1) - \nabla f(x_n), v - x_n \rangle \leqslant 0.$$

Taking $m \longrightarrow \infty$, we obtain

$$\langle \nabla f(x_1) - \nabla f(x_n), v - x_n \rangle \leqslant 0.$$

Hence $v \in Q_n$, this shows that Q_n is closed for all $n \geqslant 1$. Next, we will show that C_n is closed for all $n \geqslant 1$. Let $\{s_m\}$ be a sequence in C_n with $s_m \longrightarrow s$ as $m \longrightarrow \infty$. Then $\{s_m\}$ is a sequence in C_n^i for all $i = 1, 2, \ldots, N$, by the definition of C_n^i, we have

$$D_f(s_m, z_{n+1}^i) \leqslant \alpha_n D_f(s_m, z_n^i) + (1-\alpha_n) D_f(s_m, x_n), \quad \forall i = 1, 2, \ldots, N. \tag{3.4}$$

By the equation (2.2), definition of the Bregman distance $D_f(\cdot, \cdot)$, we obtain that

$$\begin{aligned} f(s_m) - f(z_{n+1}^i) - \langle \nabla f(z_{n+1}^i), s_m - z_{n+1}^i \rangle \leqslant \alpha_n \left(f(s_m) - f(z_{n+1}^i) - \langle \nabla f(z_n^i), s_m - z_n^i \rangle \right) \\ + (1-\alpha_n) \left(f(s_m) - f(x_n) - \langle \nabla f(x_n), s_m - x_n \rangle \right), \end{aligned} \tag{3.5}$$

it follows that

$$\begin{aligned} \alpha_n \langle \nabla f(z_n^i), s_m - z_n^i \rangle + (1-\alpha_n) \langle \nabla f(x_n), s_m - x_n \rangle - \langle \nabla f(z_{n+1}^i), s_m - z_{n+1}^i \rangle \\ \leqslant f(z_{n+1}^i) - \alpha_n f(z_n^i) - (1-\alpha_n) f(x_n). \end{aligned} \tag{3.6}$$

This implies that

$$\begin{aligned} \alpha_n (\langle f(z_n^i), s_m - s \rangle + \langle \nabla f(z_n^i), s - z_n^i \rangle) + (1-\alpha_n) \left(\langle \nabla f(x_n), s_m - s \rangle + \langle \nabla f(x_n), s - x_n \rangle \right) \\ - \langle \nabla f(z_{n+1}^i), s_m - s \rangle - \langle \nabla f(z_{n+1}^i), s - z_{n+1}^i \rangle \leqslant f(z_{n+1}^i) - \alpha_n f(z_n^i) - (1-\alpha_n) f(x_n). \end{aligned}$$

Taking $m \longrightarrow \infty$, we obtain that

$$\alpha_n \langle \nabla f(z_n^i), s - z_n^i \rangle + (1-\alpha_n) \langle \nabla f(x_n), s - x_n \rangle - \langle \nabla f(z_{n+1}^i), s - z_{n+1}^i \rangle$$

$$\leqslant f(z^i_{n+1}) - \alpha_n f(z^i_n) - (1-\alpha_n) f(x_n), \quad \forall i = 1,2,\ldots,N,$$

which implies that $s \in C^i_n$ for all $i = 1,2,\ldots,N$. Therefore, $s \in C_n$ and C_n is closed. For any $a, b \in C_n$, we have $a, b \in C^i_n$ for all $i = 1,2,\ldots,N$ and $a, b \in C$. Since C is convex, $w = ta + (1-t)b \in C$ for $t \in [0,1]$. By the definition of C_n, we have

$$D_f(a, z^i_{n+1}) \leqslant \alpha_n D_f(a, z^i_n) + (1-\alpha_n) D_f(a, x_n)$$

and

$$D_f(b, z^i_{n+1}) \leqslant \alpha_n D_f(b, z^i_n) + (1-\alpha_n) D_f(b, x_n).$$

It follows from (3.4), (3.5), and (3.6), we observe that the above two inequalities are equivalent to

$$\alpha_n \langle \nabla f(z^i_n), a - z^i_n \rangle + (1-\alpha_n)\langle \nabla f(x_n), a - x_n \rangle - \langle \nabla f(z^i_{n+1}), a - z^i_{n+1} \rangle \leqslant f(z^i_{n+1}) - f(z^i_n) - (1-\alpha_n) f(x_n) \tag{3.7}$$

and

$$\alpha_n \langle \nabla f(z^i_n), b - z^i_n \rangle + (1-\alpha_n)\langle \nabla f(x_n), b - x_n \rangle - \langle \nabla f(z^i_{n+1}), b - z^i_{n+1} \rangle \leqslant f(z^i_{n+1}) - f(z^i_n) - (1-\alpha_n) f(x_n). \tag{3.8}$$

Multiplying t and $(1-t)$ on both sides of (3.7) and (3.8), respectively, we obtain that

$$\alpha_n \langle \nabla f(z^i_n), ta + (1-t)b - z^i_n \rangle + (1-\alpha_n)\langle \nabla f(x_n), ta + (1-t)b - x_n \rangle - \langle \nabla f(z^i_{n+1}), ta + (1-t)b - z^i_{n+1} \rangle \leqslant f(z^i_{n+1}) - f(z^i_n) - (1-\alpha_n) f(x_n), \quad \forall i = 1,2,\ldots,N.$$

From the above inequality, we can rewrite that

$$\alpha_n \langle \nabla f(z^i_n), w - z^i_n \rangle + (1-\alpha_n)\langle \nabla f(x_n), w - x_n \rangle - \langle \nabla f(z^i_{n+1}), w - z^i_{n+1} \rangle \leqslant f(z^i_{n+1}) - f(z^i_n) - (1-\alpha_n) f(x_n),$$

which implies that $w \in C^i_n$ for all $i = 1,2,\ldots,N$ and hence $w \in C_n$. It follows that C_n is closed and convex for all $n \geqslant 1$. Therefore, $C_n \cap Q_n$ is closed and convex for all $n \geqslant 1$.

Step 3: We show that $\Omega \subset C_n \cap Q_n$ for all $n \geqslant 1$. Let $p \in \Omega$ be given. Since $\mathrm{Res}^f_{G,\psi}$ is single-valued, $\mathrm{Res}^f_{G,\psi}(u^i_n) = z^i_{n+1}$ for all $i = 1,2,\ldots,N$. Then, by the results of Lemma 2.21 (3) and (2.4), we obtain that

$$\begin{aligned} D_f(p, z^i_{n+1}) &= D_f(p, \mathrm{Res}^f_{G,\psi}(u^i_n)) \\ &\leqslant D_f(p, u^i_n) - D_f(\mathrm{Res}^f_{G,\psi}(u^i_n), u^i_n) \\ &\leqslant D_f(p, u^i_n) \\ &= D_f(p, \nabla f^*(\alpha_n \nabla f(z^i_n) + (1-\alpha_n)\nabla f(T_i x_n))) \\ &\leqslant \alpha_n D_f(p, z^i_n) + (1-\alpha_n) D_f(p, T_i x_n) \\ &\leqslant \alpha_n D_f(p, z^i_n) + (1-\alpha_n) D_f(p, x_n), \quad \forall i = 1,2,\ldots,N. \end{aligned} \tag{3.9}$$

This implies that $p \in C^i_n$ for all $i = 1,2,\ldots,N$ and hence $p \in C_n = \bigcap_{i=1}^N C^i_n$. Therefore, $\Omega \subset C_n$ for all $n \geqslant 1$. Next, we show by induction that $\Omega \subset C_n \cap Q_n$ for all $n \geqslant 1$. By the definition of Q_n, we obtain that $Q_1 = C$, implies that $\Omega \subset C_1 \cap Q_1$. Suppose that $\Omega \subset C_k \cap Q_k$ for some $k > 0$. Since $C_k \cap Q_k$ is closed and convex, it follows from (2.3), definition of Bregman projection, there exists $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = \mathrm{proj}^f_{C_k \cap Q_k}(x_1)$. From Lemma 2.16 (ii), we have

$$\langle \nabla f(x_1) - \nabla f(x_{k+1}), x_{k+1} - z \rangle \geqslant 0, \quad \forall z \in C_k \cap Q_k.$$

Since $\Omega \subset C_k \cap Q_k$,

$$\langle \nabla f(x_1) - \nabla f(x_{k+1}), x_{k+1} - p \rangle \geqslant 0, \quad \forall p \in \Omega,$$

and hence $p \in Q_{k+1}$. Since $\Omega \subset C_n$ for all $n \geqslant 1$, $\Omega \subset C_{k+1} \cap Q_{k+1}$. Therefore, we have $\Omega \subset C_n \cap Q_n$, for all $n \geqslant 1$ and hence $x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n}(x_1)$ is well-defined for all $n \geqslant 1$. This means that $\{x_n\}$ is well-defined.

Step 4: We will prove that the sequences $\{x_n\}$, $\{z_n^i\}_{n=1}^{\infty}$ and $\{T_i x_n\}_{n=1}^{\infty}$ are bounded for all $i = 1, 2, \ldots, N$. It follows from the definition of Q_n and Lemma 2.6 that $x_n = \text{proj}_{Q_n}^f(x_1)$. By using (2.5), we have

$$D_f(x_n, x_1) = D_f(\text{proj}_{Q_n}^f(x_1), x_1) \leqslant D_f(p, x_1) - D_f(p, \text{proj}_{Q_n}^f(x_1)) \leqslant D_f(p, x_1), \quad \forall p \in \Omega \subset Q_n.$$

Hence $\{D_f(x_n, x_1)\}$ is bounded. Therefore by Lemma 2.14, $\{x_n\}$ is bounded. On the orther hand, we have

$$D_f(p, x_n) = D_f(p, \text{proj}_{C_{n-1} \cap Q_{n-1}}^f(x_1)) \leqslant D_f(p, x_1) - D_f(x_n, x_1) \leqslant D_f(p, x_1),$$

implies that $\{D_f(p, x_n)\}$ is bounded. Now, it follows from the fact $D_f(p, T_i x_n) \leqslant D_f(p, x_n)$ for all $p \in \Omega$, $i = 1, 2, \ldots, N$, which implies that $\{D_f(p, T_i x_n)\}_{n=1}^{\infty}$ is bounded for all $i = 1, 2, \ldots, N$. Since f is strong coercive, f^* and ∇f^* are bounded on bounded subsets. It follows from Lemma 2.15, we obtain that $\{T_i x_n\}_{n=1}^{\infty}$ is bounded for all $i = 1, 2, \ldots, N$. Since $\{D_f(p, x_n)\}$ is bounded, there exists $M > 0$ such that $D_f(p, x_n) \leqslant M$. It follows from (3.9), we obtain that

$$D_f(p, z_{n+1}^i) \leqslant \alpha_n D_f(p, z_n^i) + (1 - \alpha_n) M.$$

Let $K = \max\{D_f(p, z_1^i), M\}$. Clearly that $D_f(p, z_1^i) \leqslant K$ for all $i = 1, 2, \ldots, N$. Let $D_f(p, z_n^i) \leqslant K$ for some n, then it follows from above inequality, we get that

$$D_f(p, z_{n+1}^i) \leqslant \alpha_n K + (1 - \alpha_n) K \leqslant K, \quad \forall i = 1, 2, \ldots, N.$$

It follows that $\{D_f(p, z_n^i)\}_{n=1}^{\infty}$ is bounded, for all $i = 1, 2, \ldots, N$. Again, by Lemma 2.15, we have $\{z_n^i\}_{n=1}^{\infty}$ is also bounded for all $i = 1, 2, \ldots, N$.

Step 5: We will show that $\lim_{n\to\infty} \|x_n - z_{n+1}^i\| = 0$, $\lim_{n\to\infty} \|x_n - u_n^i\| = 0$ and $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \ldots, N$. We know that $x_{n+1} = \text{proj}_{C_n \cap Q_n}^f(x_1)$ and $x_n = \text{proj}_{Q_n}^f(x_1)$, we have

$$D_f(x_n, x_1) \leqslant D_f(x_{n+1}, x_1), \quad \forall n \geqslant 1.$$

It follows that $\{D_f(x_n, x_1)\}$ is nondecreasing. Sine $\{D_f(x_n, x_1)\}$ is bounded, $\lim_{n\to\infty} D_f(x_n, x_1)$ exists. Further, the inequality

$$D_f(x_{n+1}, x_n) = D_f(x_{n+1}, \text{proj}_{Q_n}^f(x_1)) \leqslant D_f(x_{n+1}, x_1) - D_f(\text{proj}_{Q_n}^f(x_1), x_1) = D_f(x_{n+1}, x_1) - D_f(x_n, x_1),$$

implies that

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0. \tag{3.10}$$

Since f is totally convex on bounded sets, f is sequentially consistent. It follows from Lemma 2.11 and above equality, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.11}$$

It follows from the three point identity of the Bregman distance, we have

$$D_f(x_{n+1}, z_n^i) = \langle \nabla f(z_n^i) - \nabla f(x_{n+1}), p - x_{n+1} \rangle + D_f(p, z_n^i) - D_f(p, x_{n+1}).$$

Since f is bounded on bounded subsets of E, ∇f is also bounded on bounded subsets of E. It follows from boundedness of $\{x_n\}$, $\{z_n^i\}_{n=1}^{\infty}$ and $\{T_i x_n\}_{n=1}^{\infty}$, we obtain that the sequences $\{\nabla f(x_n)\}$, $\{\nabla f(z_n^i)\}_{n=1}^{\infty}$ and $\{\nabla f(T_i x_n)\}_{n=1}^{\infty}$ are bounded in E^* for all $i = 1, 2, \ldots, N$, which implies that $\{D_f(x_{n+1}, z_n^i)\}_{n=1}^{\infty}$ is bounded. It follows from $x_{n+1} = \text{proj}_{C_n \cap Q_n}^f(x_1) \in C_n$ and the definition of C_n, we have

$$D_f(x_{n+1}, z_{n+1}^i) \leqslant \alpha_n D_f(x_{n+1}, z_n^i) + (1 - \alpha_n) D_f(x_{n+1}, x_n), \quad \forall i = 1, 2, \ldots, N.$$

Since $\{D_f(x_{n+1}, z_n^i)\}_{n=1}^{\infty}$ is bounded and $\lim_{n\to\infty} \alpha_n = 0$, it follows from the above inequality and (3.10), we obtain that

$$\lim_{n\to\infty} D_f(x_{n+1}, z_{n+1}^i) = 0, \quad \forall i = 1, 2, \dots, N.$$

Since f is totally convex on bounded subsets, again using Lemma 2.11, we have

$$\lim_{n\to\infty} \|x_{n+1} - z_{n+1}^i\| = 0, \quad \forall i = 1, 2, \dots, N. \tag{3.12}$$

Taking into account

$$\|x_n - z_{n+1}^i\| \leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - z_{n+1}^i\|,$$

it follows from (3.11) and (3.12), we get

$$\lim_{n\to\infty} \|x_n - z_{n+1}^i\| = 0, \quad \forall i = 1, 2, \dots, N. \tag{3.13}$$

It follows from Lemma 2.10, we have f and ∇f are uniformly continuous since f is uniformly Frêchet differentiable on bounded subsets. Therefore,

$$\lim_{n\to\infty} |f(x_n) - f(z_{n+1}^i)| = 0, \quad \forall i = 1, 2, \dots, N \tag{3.14}$$

and

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(z_{n+1}^i)\|, \quad \forall i = 1, 2, \dots, N. \tag{3.15}$$

We next consider the following inequality, for each $i = 1, 2, \dots, N$,

$$\begin{aligned} &D_f(p, x_n) - D_f(p, z_{n+1}^i) \\ &\quad = f(p) - f(x_n) - \langle \nabla f(x_n), p - x_n \rangle - \big(f(p) - f(z_{n+1}^i) - \langle \nabla f(z_{n+1}^i), p - z_{n+1}^i \rangle\big) \\ &\quad = f(z_{n+1}^i) - f(x_n) + \langle \nabla f(z_{n+1}^i), p - x_n \rangle + \langle \nabla f(z_{n+1}^i), x_n - z_{n+1}^i \rangle - \langle \nabla f(x_n), p - x_n \rangle \\ &\quad = f(z_{n+1}^i) - f(x_n) + \langle \nabla f(z_{n+1}^i) - \nabla f(x_n), p - x_n \rangle + \langle \nabla f(z_{n+1}^i), x_n - z_{n+1}^i \rangle. \end{aligned} \tag{3.16}$$

Since $\{z_{n+1}^i\}_{n=1}^{\infty}$ and $\{\nabla f(z_{n+1}^i)\}_{n=1}^{\infty}$ are bounded for all $i = 1, 2, \dots, N$, it follows from (3.13), (3.14), (3.15), and (3.16) that

$$\lim_{n\to\infty} \|D_f(p, x_n) - D_f(p, z_{n+1}^i)\| = 0, \quad \forall i = 1, 2, \dots, N. \tag{3.17}$$

Moreover, it follows from (2.6) and Lemma 2.13, we obtain that, for each $i = 1, 2, \dots, N$,

$$\begin{aligned} D_f(z_{n+1}^i, u_n^i) &\leqslant D_f(p, u_n^i) - D_f(p, z_{n+1}^i) \\ &= D_f(p, \nabla f^*(\alpha_n \nabla f(z_n^i) + (1 - \alpha_n)\nabla f(T_i x_n))) - D_f(p, z_{n+1}^i) \\ &\leqslant \alpha_n D_f(p, z_n^i) + (1 - \alpha_n) D_f(p, T_i x_n) - D_f(p, z_{n+1}^i) \\ &\leqslant \alpha_n D_f(p, z_n^i) + (1 - \alpha_n) D_f(p, x_n) - D_f(p, z_{n+1}^i) \\ &= \alpha_n \big(D_f(p, z_n^i) - D_f(p, x_n)\big) + D_f(p, x_n) - D_f(p, z_{n+1}^i). \end{aligned} \tag{3.18}$$

Since $\{D_f(p, x_n)\}$ and $\{D_f(p, z_n^i)\}_{n=1}^{\infty}$ are bounded for all $i = 1, 2, \dots, N$, it follows from (3.17), (3.18), and $\lim_{n\to\infty} \alpha_n = 0$,

$$\lim_{n\to\infty} D_f(z_{n+1}^i, u_n^i) = 0, \quad \forall i = 1, 2, \dots, N,$$

so, we have

$$\lim_{n\to\infty} \|z_{n+1}^i - u_n^i\| = 0, \quad \forall i = 1, 2, \dots, N. \tag{3.19}$$

Taking into account

$$\|x_n - u_n^i\| \leqslant \|x_n - z_{n+1}^i\| + \|z_{n+1}^i - u_n^i\|,$$

and using (3.13) and (3.19), we get that

$$\lim_{n\to\infty}\|x_n - u_n^i\| = 0, \quad \forall i = 1,2,\dots,N. \tag{3.20}$$

Since f is uniformly Frêchet differentiable and by Lemma 2.10, ∇f is uniformly continuous on bounded sets. It follows from (3.19) and (3.20), we obtain that

$$\lim_{n\to\infty}\|\nabla f(z_{n+1}^i) - \nabla f(u_n^i)\| = 0, \quad \forall i = 1,2,\dots,N, \tag{3.21}$$

and

$$\lim_{n\to\infty}\|\nabla f(x_n) - \nabla f(u_n^i)\| = 0, \quad \forall i = 1,2,\dots,N. \tag{3.22}$$

Furthermore, for each $i = 1,2,\dots,N$, we now consider the following inequality

$$\begin{aligned}\|\nabla f(x_n) - \nabla f(u_n^i)\| &= \|\nabla f(x_n) - \nabla f\left(\nabla f^*\left(\alpha_n \nabla f(z_n^i) + (1-\alpha_n)\nabla f(T_i x_n)\right)\right)\| \\ &= \|\nabla f(x_n) - \alpha_n \nabla f(z_n^i) - (1-\alpha_n)\nabla f(T_i x_n)\| \\ &= \|\alpha_n\left(\nabla f(x_n) - \nabla f(z_n^i)\right) + (1-\alpha_n)\left(\nabla f(x_n) - \nabla f(T_i x_n)\right)\| \\ &\geqslant (1-\alpha_n)\|\nabla f(x_n) - \nabla f(T_i x_n)\| - \alpha_n\|\nabla f(x_n) - \nabla f(z_n^i)\|,\end{aligned}$$

which implies that

$$(1-\alpha_n)\|\nabla f(x_n) - \nabla f(T_i x_n)\| \leqslant \|\nabla f(x_n) - \nabla f(u_n^i)\| + \alpha_n\|\nabla f(x_n) - \nabla f(z_n^i)\|. \tag{3.23}$$

Since $\{\nabla f(x_n)\}$ and $\{\nabla f(z_n^i)\}_{n=1}^{\infty}$ are bounded for all $i = 1,2,\dots,N$, it follows from (3.22), (3.23) and $\lim_{n\to\infty}\alpha_n = 0$, we have

$$\lim_{n\to\infty}\|\nabla f(x_n) - \nabla f(T_i x_n)\| = 0, \quad \forall i = 1,2,\dots,N.$$

It follows from f is the Legendre function and f^* is uniformly Frêchet differentiable on bounded subsets, the above inequality yields that

$$\lim_{n\to\infty}\|x_n - T_i x_n\| = 0, \quad \forall i = 1,2,\dots,N. \tag{3.24}$$

Step 6: We show that $x^* \in \Omega$. By the boundedness of the sequence $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in C$ as $k \longrightarrow \infty$. It follows from (3.13) and (3.19), there exist subsequences $\{u_{n_k}^i\}$ of $\{u_n^i\}$ and $\{z_{n_k}^i\}$ of $\{z_n^i\}$ such that $u_{n_k}^i \rightharpoonup x^*$ and $z_{n_k}^i \rightharpoonup x^* x$ as $k \longrightarrow \infty$, for all $i = 1,2,\dots,N$, respectively. The consequence of (3.24) is

$$\lim_{n\to\infty}\|x_{n_k} - T_i x_{n_k}\| = 0, \quad \forall i = 1,2,\dots,N.$$

Since $x_{n_k} \rightharpoonup x^*$ and using the above equality, it follows from the definition of asymptotic fixed points, we have $x^* \in \hat{F}(T_i)$ for all $i = 1,2,\dots,N$. Since $\{T_i\}_{i=1}^{N}$ is a countable family of Bregman relatively nonexpansive mappings, $x^* \in F(T_i)$ for all $i = 1,2,\dots,N$, implies that $x^* \in \bigcap_{i=1}^{N} F(T_i)$. Next, we show that x^* is the solution of the mixed equilibrium problem. Since $z_{n+1}^i = \mathrm{Res}_{G,\psi}^f(u_n^i)$, for each $i = 1,2,\dots,N$

$$G(z_{n_k+1}^i, y) + \psi(y) - \psi(z_{n_k+1}^i) + \langle \nabla f(z_{n_k+1}^i) - \nabla f(u_{n_k}^i), y - z_{n_k+1}^i\rangle \geqslant 0, \quad \forall y \in C.$$

Using the Assumption 2.19 (ii), we obtain that

$$\begin{aligned}\psi(y) - \psi(z_{n_k+1}^i) + \langle \nabla f(z_{n_k+1}^i) - \nabla f(u_{n_k}^i), y - z_{n_k+1}^i\rangle &\geqslant -G(z_{n_k+1}^i, y) \\ &\geqslant G(y, z_{n_k+1}^i), \quad \forall y \in C, \quad i = 1,2,\dots,N.\end{aligned}$$

For any $y \in C$ and $t \in (0,1]$, we let $y_t = ty + (1-t)x^* \in C$. This implies that

$$\psi(y_t) - \psi(u^i_{n_k}) + \langle \nabla f(z^i_{n_k+1}) - \nabla f(u^i_{n_k}), y_t - z^i_{n_k+1}\rangle \geqslant G(y_t, z^i_{n_k+1}).$$

Using the Assumption 2.19 (iv) and the Assumption 2.20 (i), $G(x,\cdot)$ and ψ are lower semicontinuous, it follows from (3.21) and above inequality, this yields

$$\liminf_{k\to\infty}(G(y_t, z^i_{n_k+1}) - \psi(y_t) + \psi(z^i_{n_k+1})) \leqslant \liminf_{k\to\infty}\langle \nabla f(z^i_{n_k+1}) - \nabla f(u^i_{n_k}), y_t - z^i_{n_k+1}\rangle, \quad \forall i = 1,2,\dots,N.$$

This implies that

$$G(y_t, x^*) - \psi(y_t) + \psi(x^*) \leqslant 0.$$

Furthermore, we next consider the following inequality,

$$\begin{aligned}
0 &= G(y_t, y_t) + \psi(y_t) - \psi(y_t)\\
&= G(y_t, ty + (1-t)x^*) + \psi(ty + (1-t)x*) - \psi(y_t)\\
&\leqslant tG(y_t, y) + (1-t)G(y_t, x^*) + t\psi(y) + (1-t)\psi(x^*) - t\psi(y_t) - (1-t)\psi(y_t)\\
&= t\left(G(y_t, y) + \psi(y) - \psi(y_t)\right) + (1-t)\left(G(y_t, x^*) + \psi(x^*) - \psi(y_t)\right)\\
&\leqslant t\left(G(y_t, y) + \psi(y) - \psi(y_t)\right),
\end{aligned}$$

which implies that

$$G(y_t, y) + \psi(y) - \psi(y_t) \geqslant 0.$$

It follows from the Assumption 2.19 (iii), we have

$$\begin{aligned}
0 &\leqslant \limsup_{t\to 0^+}\left(G(y_t, y) + \psi(y) - \psi(y_t)\right)\\
&= \limsup_{t\to 0^+}\left(G(ty + (1-t)x^*, y) + \psi(y) - \psi(ty + (1-t)x^*)\right) \leqslant G(x^*, y) + \psi(y) - \psi(x^*).
\end{aligned}$$

This implies that x^* is a solution of the mixed equilibrium problem and hence $x^* \in MEP(G,\psi)$. To sum up, we have $x^* \in \Omega := \bigcap_{i=1}^{N} F(T_i) \cap MEP(G,\psi)$.

Step 7: We shall show that the sequence $\{x_n\}$ converges strongly to $x^* = \text{proj}^f_\Omega(x_1)$. Since Ω is a nonempty closed convex subset of E, $\text{proj}^f_\Omega(x_1)$ is well-defined. Let $u^* = \text{proj}^f_\Omega(x_1)$ be given. It follows from $x_{n+1} = \text{proj}^f_{C_n\cap Q_n}(x_1)$ and $\text{proj}^f_\Omega(x_1) \in \Omega \subseteq C_n \cap Q_n$, we obtain that

$$D_f(x_{n+1}, x_1) \leqslant D_f(u^*, x_1).$$

Since $\{x_{n_k}\}$ is a weak convergent subsequence of $\{x_n\}$ and follows from Lemma 2.23, we obtain that $\{x_n\}$ converges strongly to u^*. By the uniqueness of the limit, we obtain that the sequence $\{x_n\}$ converges strongly to $x^* = \text{proj}^f_\Omega(x_1)$. This completes the proof. □

If we assume that $T_i = T$ for each $i = 1,2,\dots,N$ and ψ is a zero mapping in Theorem 3.1, then we get the following corollary.

Corollary 3.2. *Let E be a reflexive Banch space with dual E^* and C be a nonempty closed convex subset of E such that $C \subset \text{int}(\text{dom} f)$. Let $f : E \longrightarrow (-\infty, +\infty]$ be a strong coercive Legendre function which is bounded uniformly Frêchet differentiable and totally convex on bounded subsets of E, $G : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying the*

Assumption 2.19. Let $T : C \longrightarrow C$ *be a Bregman relatively nonexpansive mapping. Assume that* $F(T) \cap EP(G) \neq \emptyset$*. Let* $\{x_n\}$ *be the sequence generated by the iterative scheme:*

$$\begin{cases} x_1 \in C, Tx_1 = z_1 \in C; \\ u_n = \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(Tx_n)); \\ z_{n+1} = \mathrm{Res}^f_{G,\psi}(u_n); \\ C_n = \{z \in C : D_f(z, z_{n+1}) \leqslant \alpha_n D_f(z, z_n) + (1-\alpha_n) D_f(z, x_n)\}; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n \rangle \leqslant 0\}; \\ x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n} x_1, \quad \forall n \geqslant 1, \end{cases}$$

where $\{\alpha_n\}$ *is a sequence in* $[0, 1]$ *such that* $\lim_{n\to\infty} \alpha_n = 0$*. Then,* $\{x_n\}$ *converges strongly to* $\mathrm{proj}^f_{F(T)\cap EP(G)} x_1$*.*

In Theorem 3.1, if we assume that $MEP(G, \psi) = C$ and using the facts given in Example 2.2 for the generalized duality mapping J_p, then we obtain the following corollary.

Corollary 3.3. *Let* E *be a uniformly smooth and uniformly convex Banach space and* C *be a nonempty closed convex subset of* E *such that* $C \subset \mathrm{int}(\mathrm{dom} f)$*. Let* $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$ *and* $\{T_i : C \longrightarrow C\}_{i=1}^N$ *be a countable family of relatively nonexpansive mappings. Assume that* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$*. Let* $\{x_n\}$ *be the sequence generated by the iterative scheme:*

$$\begin{cases} x_1 \in C, T_i x_1 = z_1^i \in C; \\ z_{n+1}^i = J_p^{-1}(\alpha_n J_p(z_n^i) + (1-\alpha_n) J_p(T_i x_n)); \\ C_n^i = \{z \in C : V(z, z_{n+1}^i) \leqslant \alpha_n V(z, z_n^i) + (1-\alpha_n) V(z, x_n)\}; \\ C_n = \bigcap_{i=1}^N C_n^i; \\ Q_n = \{z \in C : \langle J_p(x_1) - J_p(x_n), z - x_n \rangle \leqslant 0\}; \\ x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n} x_1, \quad \forall n \geqslant 1, \end{cases}$$

where $\{\alpha_n\}$ *is a sequence in* $[0, 1]$ *such that* $\lim_{n\to\infty} \alpha_n = 0$*. Then,* $\{x_n\}$ *converges strongly to* $\mathrm{proj}^f_{\bigcap_{i=1}^N F(T_i)} x_1$*.*

4. Applications

Zeros of maximal monotone operators

Let $A : E \longrightarrow 2^{E^*}$ be a set-valued mapping. Denote $G(A)$ by the graph of A, that is $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. A multi-valued operator A is said to be monotone if $\langle x^* - y^*, x - y \rangle > 0$ for each $(x, x^*), (y, y^*) \in G(A)$. A monotone operator A is said to be maximal if its graph, $G(A)$ is not contained in the graph of any other monotone operators on E. Let $f : E \longrightarrow (-\infty, +\infty]$, then the resolvent of A, $\mathrm{Res}^f_{\lambda A} : E \longrightarrow 2^E$ is defined as follows:

$$\mathrm{Res}^f_{\lambda A}(x) = (\nabla f + \lambda A)^{-1} \circ \nabla f(x), \quad \lambda > 0.$$

In 2003, Bauschke et al. [3] proved that $\mathrm{Res}^f_{\lambda A}$ is a single-valued and Bregman firmly nonexpansive mapping and $F(\mathrm{Res}^f_{\lambda A}) = A^{-1}(0^*) = \{x \in E : 0^* \in Ax\}$. It is known that if A is maximal monotone, then the set $A^{-1}(0^*)$ is closed and convex. We also define the Yosida approximation $A_\lambda : E \longrightarrow E$ by

$$A_\lambda(x) = \frac{1}{\lambda}(\nabla f - \nabla f \circ \mathrm{Res}^f_{\lambda A})(x), \quad \forall x \in E, \lambda > 0.$$

It is shown in Reich and Sabach [22] that for any $x \in E$ and $\lambda > 0$, we have

(i) $\left(\mathrm{Res}^f_{\lambda A}(x), A_\lambda(x)\right) \in G(A)$;
(ii) $0^* \in Ax$ if and only if $0^* \in A_\lambda(x)$.

In 2011, Reich and Sabach [24] proved that if f is the Legendre function which is bounded uniformly Frêchet differentiable on bounded subsets of E, then $\hat{F}(\text{Res}^f_{\lambda A}) = F(\text{Res}^f_{\lambda A})$. We also know that if $\hat{F}(\text{Res}^f_{\lambda A}) = F(\text{Res}^f_{\lambda A})$, then a Bregman firmly nonexpansive mapping is a Bregman relatively nonexpansive mapping. Furthermore, if we take $MEP(G,\psi) = C$ and $T_i = \text{Res}^f_{\lambda A_i}$ for all $i = 1, 2, \ldots, N$ in Theorem 3.1, then we obtain the following consequence.

Theorem 4.1. *Let E be a reflexive Banch space with dual E^* and C be a nonempty closed convex subset of E such that $C \subset \text{int}(\text{dom} f)$. Let $f : E \longrightarrow (-\infty, +\infty]$ be a strong coercive Legendre function which is bounded uniformly Frêchet differentiable and totally convex on bounded subsets of E. Let $\{A_i : E \longrightarrow 2^{E^*}\}_{i=1}^N$ be a countable family of maximal monotone operators. Assume that $\bigcap_{i=1}^N A_i^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by the iterative scheme:*

$$\begin{cases} x_1 \in C, \text{Res}^f_{\lambda A_i}(x_1) = z_1^i \in C; \\ z^i_{n+1} = \nabla f^*(\alpha_n \nabla f(z^i_n) + (1-\alpha_n)\nabla f(\text{Res}^f_{\lambda A_i}(x_n))); \\ C^i_n = \{z \in C : D_f(z, z^i_{n+1}) \leqslant \alpha_n D_f(z, z^i_n) + (1-\alpha_n) D_f(z, x_n)\}; \\ C_n = \bigcap_{i=1}^N C^i_n; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n \rangle \leqslant 0\}; \\ x_{n+1} = \text{proj}^f_{C_n \cap Q_n} x_1, \quad \forall n \geqslant 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $\text{proj}^f_{\bigcap_{i=1}^N A_i^{-1}(0)} x_1$.

5. Numerical example

In this section, we present some numerical examples for comparing the values of sequences generated by iteration (1.3) and (3.1) and supporting Theorem 3.1.

Example 5.1. Let $E = \mathbb{R}$, $C = (-\infty, 0]$, let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = \frac{2}{3}x^2$ (f is a strong coercive Legendre function which is bounded uniformly Frêchet differentiable and totally convex on bounded subsets of E, see in the numerical example of [28]). Let $T : C \longrightarrow C$ be defined by $Tx = \frac{1}{3}x$, $G : C \times C \longrightarrow \mathbb{R}$ be defined by $G(x,y) = x - y$ for all $x, y \in C$, $\psi : C \longrightarrow \mathbb{R}$ be defined by $\psi(x) = x^2$ for all $x \in C$. Let $\phi : C \times C \longrightarrow \mathbb{R}$ in the iteration (1.3) be defined by $\phi(x,y) = y - x$ for all $x, y \in C$. By the numerical example section of [18], we obtain that T is a Bregman relatively nonexpansive mapping. It is easy to show that G and ψ satisfy the Assumption 2.19 and the Assumption 2.20, respectively, and ϕ is skew-symmetric, i.e., $\phi(x,x) - \phi(x,y) - \phi(y,x) + \phi(y,y) \geqslant 0$ for all $x, y \in C$, convex in the second argument and continuous. Let $\{x_n\}$ be generated by iteration (1.3) and (3.1). Given initial values $x_1 = -1 = z_1$ and $\alpha_n = \frac{1}{n^3}$ for all $n \geqslant 1$. Then the sequence $\{x_n\}$ converges strongly to 0, where $\text{proj}^f_{GEP(G,\phi)\cap F(T)}(x_1) = 0 = \text{proj}^f_{MEP(G,\psi)\cap F(T)}(x_1)$.

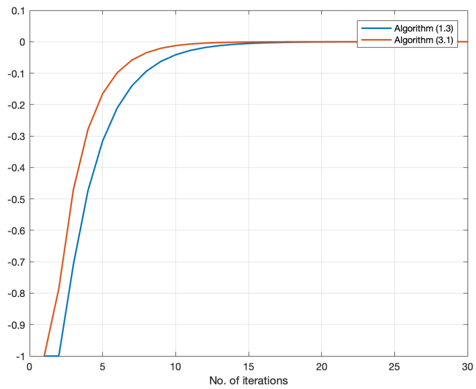

Figure 1: The numerical results for comparing Algorithm (3.1) and Algorithm (1.3).

We now illustrate the example supporting our main result.

Example 5.2. Let $E = \mathbb{R}$, $C = (-\infty, 0]$, and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = \frac{2}{3}x^2$. Let $\{T_i : C \longrightarrow C\}_{i=1}^{5}$ be defined by $T_i x = \frac{1}{i+1}x$, and let $G : C \times C \longrightarrow \mathbb{R}$ be defined by $G(x, y) = x - y$ for all $x, y \in C$, $\psi : C \longrightarrow \mathbb{R}$ be defined by $\psi(x) = x^2$ for all $x \in C$. Setting $\alpha_n = \frac{1}{n}$ for all $n \geqslant 1$. Let $\{x_n\}$ be the sequence generated by the iterative scheme. Given initial values $x_1, T_i x_1 = z_1 \in C$ for $i = 1, 2, \ldots, 5$,

$$\begin{cases} u_n^i = \nabla f^*(\alpha_n \nabla f(z_n^i) + (1-\alpha_n)\nabla f(T_i x_n)); \\ z_{n+1}^i = \operatorname{Res}_{G,\psi}^f(u_n^i); \\ C_n^i = \{z \in C : D_f(z, z_{n+1}^i) \leqslant \alpha_n D_f(z, z_n^i) + (1-\alpha_n) D_f(z, x_n)\}; \\ C_n = \bigcap_{i=1}^{5} C_n^i; \\ Q_n = \{z \in C : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n\rangle \leqslant 0\}; \\ x_{n+1} = \operatorname{proj}_{C_n \cap Q_n}^f x_1, \quad \forall n \geqslant 1. \end{cases} \tag{5.1}$$

It follows from Example 5.1, we know that f is a strong coercive Legendre function which is bounded uniformly Frêchet differentiable and totally convex on bounded subsets of $\mathbb{R}$ such that $\nabla f(x) = \frac{4}{3}x$ and G, ψ satisfy the Assumption 2.19 and the Assumption 2.20, respectively. Since $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}$, $f^*(z) = \frac{3}{8}z^2$ such that $\nabla f^* = \frac{3}{4}z$. Next, we show that T_i is a Bregman relatively nonexpansive mapping for all $i = 1, 2, \ldots, 5$. Clearly $F(T_i) = 0 = \hat{F}(T_i)$ for all $i = 1, 2, \ldots, 5$. Furthermore, we obtain that

$$\begin{aligned} D_f(0, T_i x) &= f(0) - f(T_i x) - \langle 0 - T_i x, \nabla f(T_i x)\rangle \\ &= 0 - \frac{2}{3(i+1)}x^2 - \langle -\frac{1}{i+1}x, \frac{4}{3(i+1)}x\rangle = \frac{4}{3(i+1)^2}x^2 - \frac{2}{3(i+1)}x^2 = \frac{2}{3}\left(\frac{1-i}{(i+1)^2}\right), \end{aligned}$$

and

$$D_f(0, x) = f(0) - f(x) - \langle 0 - x, \nabla f(x)\rangle = 0 - \frac{2}{3}x^2 - \langle -x, \frac{4}{3}x\rangle = \frac{4}{3}x^2 - \frac{2}{3}x^2 = \frac{2}{3}x^2.$$

Since $\frac{1-i}{(i+1)^2} \leqslant 0$ for all $i = 1, 2, \ldots, 5$, $D_f(0, T_i x) \leqslant D_f(0, x)$ for all $i = 1, 2, \ldots, 5$. It follows that $\{T_i\}_{i=1}^{5}$ is a countable family of Bregman relatively nonexpansive mappings. We also know that

$$G(0, y) + \psi(y) - \psi(0) = (0 - y) + y^2 - 0 = y(y-1) \geqslant 0, \quad \forall y \in C,$$

this implies that $0 \in MEP(G, \psi)$ and $\Omega = \bigcap_{i=1}^{5} F(T_i) \cap MEP(G, \psi) = \{0\}$. It follows from iteration (5.1), we have

$$\begin{aligned} u_n^i &= \alpha_n z_n^i + (1-\alpha_n)\left(\frac{1}{i+1}\right)x_n; \\ z_{n+1}^i &= \frac{4}{7}u_n^i; \\ C_n^i &= [e_n^i, \infty), \quad \text{where } e_n^i = \frac{(z_{n+1}^i)^2 + (\alpha_n - 1)x_n^2 - \alpha_n (z_n^i)^2}{2\left(z_{n+1}^i - \alpha_n z_n^i + (\alpha_n - 1)x_n\right)}; \\ C_n &= \bigcap_{i=1}^{5} C_n^i; \\ Q_n &= [x_n, \infty); \\ x_{n+1} &= \operatorname{proj}_{C_n \cap Q_n}^f x_1, \quad \forall n \geqslant 1, i = 1, 2, \ldots, 5. \end{aligned}$$

Then the sequence $\{x_n\}$ generated by (5.1) converges strongly to $x^* = 0 \in \Omega$ as $n \longrightarrow \infty$. The Figure 2 shows the comparision of the values of the sequence $\{x_n\}$. Given initial values $x_1 = -5$, let $x_n(i)$ denote by the values of the sequence $\{x_n\}$ for $i = 1, 2, \ldots, 5$.

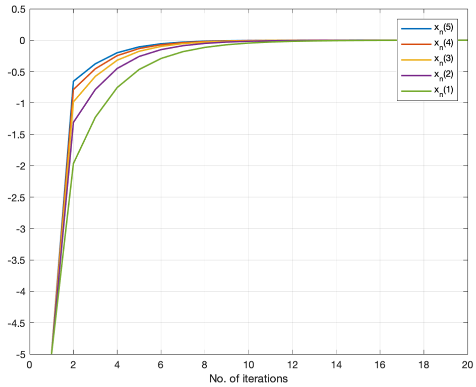

Figure 2: The numerical results for different $i = 1, 2, \ldots, 5$.

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