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Some new approach of spaces of non-integral order

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Abstract

The aim of this work is to develop the new techniques of sequences by employing the gamma function by introducing the space $r^q(\triangle_g^p, \kappa)$ of non-integral order. The completeness property concerning to this non-integral order space will be developed. Many interesting properties will be illustrated.

Keywords: Sequence space, non-absolute property, basis.

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1. Introduction

It is well known fact that gamma functions plays an explicit series and integral functional representations, and thus provide basic building for developing the useful products and transformation formulae. Moreover, many applied problems often need solutions of a function in terms of parameters, rather than merely in terms of a variable, and such a solution is often given by the parametric character of the Gamma function. As a consequence, this function can be operated to establish the physical problems in many areas of science, engineering and technology. Its origin is almost as often as the well-known factorial symbol $n!$ and were given by famous mathematician L. Euler (1729) as a natural extension of the factorial operation $n!$ from natural numbers n to real and even complex values of this argument [8].

Sequence space is referred to be a function space with entries as functions from positive numbers $\mathbb{N}$ to the field $\mathbb{R}$ of real numbers or $\mathbb{C}$ the complex numbers. The set of every sequences (real or complex) will be abbreviated by Ω. The bounded sequences, convergent sequences and null sequences will be abbreviated by ℓ_∞, c and c_0 respectively.

For an infinite matrix $\mathcal{C} = (c_{i,j})$ and $v = (v_k) \in \Omega$, the $\mathcal{C}$-transform of v is $\mathcal{C}v = \{(\mathcal{C}v)_i\}$ provided it exists $\forall\, i \in \mathbb{N}$, where $(\mathcal{C}v)_i = \sum_{j=0}^{\infty} c_{i,j} v_j$.

For an infinite matrix $\mathcal{C} = (c_{ij})$, the set $G_{\mathcal{C}}$, where

$$G_{\mathcal{C}} = \{u = (u_i) \in \Omega \ : \ \mathcal{C}u \in G\}, \tag{1.1}$$

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is said to be as the matrix domain of $\mathcal{C}$ in G as can be found in [13, 15]. Also the set of all such maps will be symbolized by (G, L) with $G \subseteq L_{\mathcal{C}}$ as can be seen in [11, 16, 18, 21] and many others.

In [20], the author has introduced the following spaces $\mathcal{V}(\triangle)$ viz.,

$$\mathcal{V}(\triangle) = \{\rho = (\rho_j) \in \Omega : (\triangle \rho_j) \in \mathcal{V}\},$$

where $\mathcal{V} \in \{\ell_\infty, c, c_0\}$ and $\Delta\rho_j = \rho_j - \rho_{j+1}$, $\forall\, j \in \mathbf{N}$. Also naught will be taken for a term with negative subscript. It has been further modified and generalized by authors as can be seen in [3, 5, 7, 10, 12, 17, 34] and many others.

The space bv_p in [4] has been defined as follows

$$bv_p = \left\{\rho = (\rho_k) \in \Omega : \sum_k |\rho_k - \rho_{k-1}|^p < \infty\right\},$$

where $1 \leqslant p < \infty$. As in (1.1), the space bv_p can be written as

$$bv_p = (\ell_p)_\triangle, \quad 1 \leqslant p < \infty,$$

where, $\triangle$ denotes the matrix $\triangle = (\triangle_{nk})$ defined as

$$\triangle_{nk} = \begin{cases} (-1)^{n-k}, & \text{if } n-1 \leqslant k \leqslant n, \\ 0, & \text{if } k < n-1 \text{ or } k > n. \end{cases}$$

As in [5], the authors have generalized spaces given in [20] and have given the following

$$\triangle^l(\mathcal{V}) = \{\rho = (\rho_j) \in \Omega : (\triangle^l \rho_j) \in \mathcal{V}\},$$

where l is non-negative integer and $\triangle^l\rho_j = \triangle^{l-1}\rho_j - \triangle^l\rho_{l+1}$, so that

$$\triangle^l \rho_z = \sum_{t=0}^{l} (-1)^t \binom{l}{t} \rho_{z+t}.$$

These are Banach spaces with the following norm

$$\|\rho\| = \sum_{t=0}^{l} |\rho_t| + \|\triangle^l \rho\|_\infty.$$

Choose the sequence of positive numbers as (q_k) and for j$\in$ $\mathbf{N}$ set $\mathfrak{A}_n = \sum\limits_{j=0}^{n} q_j$. So the matrix $\mathcal{R}^q = (r_{ij})$ as defined in [25] is defined as follows

$$r_{ij} = \begin{cases} \frac{q_j}{\mathfrak{A}_i}, & \text{if } 0 \leqslant j \leqslant i, \\ 0, & \text{if } j > i\,. \end{cases}$$

This as in [29], we have following space

$$\mathcal{R}^q(\triangle_g^p) = \left\{\rho = (\rho_j) \in \Omega : \sum_j \left|\frac{1}{\mathfrak{A}_j} \sum_{m=0}^{j} g_j q_m \Delta\rho_m\right|^{p_j}\right\}.$$

Choose the sequence of positive numbers as (q_k) and for j$\in$ $\mathbf{N}$ set $\mathfrak{A}_n = \sum\limits_{j=0}^{n} q_j$. So the matrix $\mathcal{R}^q = (r_{ij})$

as defined in [25] is defined as follows

$$r_{ij} = \begin{cases} \frac{q_j}{\mathfrak{A}_i}, & \text{if } 0 \leqslant j \leqslant i, \\ 0, & \text{if } j > i\,. \end{cases}$$

For a positive proper fraction τ, the author in [3] has defined a new pattern of this kind as follows

$$\Delta^{\tau}\rho_r = \sum_{r=0}^{\infty} (-1)^r \frac{\Gamma(\tau+1)}{r!\Gamma(\tau-r+1)}\rho_{i+r},$$

for $i \in \mathbb{N}$, where the function $\Gamma(\tau)$ (or the Euler gamma function) of a real number τ with $\tau \notin \{0, -1, -2, \cdots\}$ has be represented as follows:

$$\Gamma(\tau) = \int_0^{\infty} e^{-t} t^{\tau-1}\, dt.$$

It is important to note that

(i) $\Gamma(\tau+1) = \tau!, \text{ for } \tau \in \mathbb{N};$

(ii) $\Gamma(\tau+1) = \tau\Gamma(\tau), \text{ for } \tau \in \mathbb{R}\setminus\{0,-1,-2,\ldots\}.$

Some definitions of non-integral derivatives have been generalized by using techniques of new difference sequence spaces of non-integral order as can be seen in [3, 19] etc. To establish a new space with the help of matrix methods were studied by several authors as can be found in [1, 11–17], [23–31, 33] and many more. Following the references cited, the scenario here is to put forward and synthesis the spaces $\mathcal{R}^q(\triangle_g^p, \kappa)$ of order κ (non-integral) for which $\Delta_g^{(\kappa)}$-transform is in space $\ell(p)$, where $g = (g_i)$ is a sequence with $g_i \neq 0$, $\forall i \in \mathbf{N}$.

2. The space $\mathcal{R}^q(\triangle_g^p, \kappa)$

In this section, we introduce the space $\mathcal{R}^q(\triangle_g^p, \kappa)$ of non-integral order κ and discuss some of its basic properties.

A linear topological space $\mathfrak{K}$ is said to be paranormed space over $\mathbf{R}$ if for a function $\mathfrak{G} : \mathfrak{K} \to \mathbf{R}$ which is subadditive satisfies $\mathfrak{G}(\theta) = 0$, $\mathfrak{G}(-\rho) = \mathfrak{G}(\rho)$ and continuity of scalar multiplication holds, which means for $|a_n - a| \to 0$ and $\mathfrak{G}(\rho_n - \rho) \to 0$ imply $\mathfrak{K}(a_n\rho_n - a\rho) \to 0$, $\forall\ a's \in \mathbf{R}$ and $\rho's \in \mathfrak{K}$ with zero vector as θ and is in space $\mathfrak{K}$. From here on words, (p_k) will represent a bounded sequence of strictly positive real numbers with $\sup_k p_k = \mathcal{H}$ and $M = \max\{1, \mathcal{H}\}$. Then, as in [22, 32], we write

$$\ell(p) = \{\rho = (\rho_k) : \sum_k |\rho_k|^{p_k} < \infty\}.$$

Under the following paranorm, this space is complete

$$\mathfrak{G}(\rho) = \left[\sum_k |\rho_k|^{p_k}\right]^{\frac{1}{\mathcal{M}}}.$$

Throughout the text, we employ the fact that $p_i^{-1} + \{p_i'\}^{-1} = 1$ only if $1 < \inf p_i \leqslant \mathcal{H} < \infty$.

Following the authors as cited in the references [2, 5, 6, 9, 29], the space $\mathcal{R}^q(\triangle_g^p, \kappa)$ is defined as the set of those sequences whose $\mathcal{R}^q(\Delta_g^{\kappa})$ transform is in the space $\ell(p)$, this shows that

$$\mathcal{R}^q(\triangle_g^p, \kappa) = \left\{\rho = (\rho_j) \in \Omega : \mathcal{R}_g^q(\Delta^{\kappa})\rho \in \ell(p)\right\},$$

where, $0 < p_k \leqslant \mathcal{H} < \infty$.

Using (1.1), the space $\mathcal{R}^q(\triangle_g^p,\kappa)$ can be redefined as

$$\mathcal{R}^q(\triangle_g^p,\kappa)=\{\ell(p)\}_{R^q(\Delta_g^\kappa)}.$$

We define the sequence $\sigma=(\sigma_k)$ as the $R^q(\Delta_g^\kappa)$-transform of a sequence $\rho=(\rho_n)$ with $n\in\mathbf{N}$, via,

$$\sigma_n=\sum_{k=0}^{n-1}\left[\sum_{i=k}^{n}(-1)^{i-k}\frac{\Gamma(\tau+1)}{(i-k)!\Gamma(\tau-i+k+1)}\frac{g_k q_i}{\mathfrak{A}_n}\right]\rho_k+\frac{g_k q_n}{\mathfrak{A}_n}\rho_n. \tag{2.1}$$

Definition 2.1. For the space $\mathcal{R}^q(\Delta_g^\kappa)$ we have

$$\left(\mathcal{R}^q(\Delta_g^\kappa)\right)_{nk}^{-1}=\begin{cases}(-1)^{n-k}\sum\limits_{j=k}^{k+1}\frac{\Gamma(-\tau+1)}{(n-j)!\Gamma(-\tau-n+j+1)}\frac{\mathfrak{A}_k}{g_k q_j}, & \text{if } 0\leqslant k<n,\\ \frac{\mathfrak{A}_n}{g_k q_n}, & \text{if } k=n,\\ 0, & \text{if } k>n,\end{cases}$$

which is known as the inverse of $\mathcal{R}^q(\Delta_g^\kappa)$.

Definition 2.2. By choosing different values of κ and g, we have following deductions:

(i) For $\kappa=0$, this space is reduced to $\mathcal{R}^t(g,p)$ introduced and studied in [29].

(ii) For $\kappa=1$, this space is reduced to $\mathcal{R}^t(\Delta,g,p)$ introduced and studied in [24].

(iii) For $\kappa=0$ and $g=1$, this space is reduced to $\mathcal{R}^t(p)$ introduced and studied in [1].

Theorem 2.3. *For $0<p_k\leqslant\mathcal{H}<\infty$, the space $\mathcal{R}^q(\triangle_g^p,\kappa)$ is a complete linear metric space paranormed by $\mathfrak{H}_\triangle$ given by*

$$\mathfrak{H}_\triangle(\rho)=\left[\sum_m\left|\left[\mathcal{R}^q(\Delta_g^\kappa)\rho\right]_m\right|^{p_m}\right]^{\frac{1}{\mathcal{M}}}.$$

Proof. To prove $\mathcal{R}^q(\triangle_g^p,\kappa)$ is linear with respect to the coordinate wise addition and scalar multiplication, we first let $\tau,\rho\in\mathcal{R}^q(\triangle_g^p,\kappa)$ and have

$$\begin{aligned}\mathfrak{H}_\triangle(\rho+\tau)&=\left[\sum_m\left|\sum_{j=0}^{m-1}\left[\sum_{i=j}^{m}(-1)^{i-j}\frac{\Gamma(\tau+1)}{(i-j)!\Gamma(\tau-i+j+1)}\frac{g_m q_i}{\mathfrak{A}_m}\right](\rho_j+\tau_j)\right.\right.\\&\qquad\left.\left.+\frac{g_m q_m}{\mathfrak{A}_m}(\rho_m+\zeta_m)\right|^{p_m}\right]^{\frac{1}{\mathcal{M}}}\\&\leqslant\left[\sum_m\left|\sum_{j=0}^{m-1}\left[\sum_{i=j}^{m}(-1)^{i-j}\frac{\Gamma(\tau+1)}{(i-j)!\Gamma(\tau-i+j+1)}\frac{g_m q_i}{\mathfrak{A}_m}\right]\rho_j+\frac{g_m q_m}{\mathfrak{A}_m}\rho_m\right|^{p_m}\right]^{\frac{1}{\mathcal{M}}}\\&\qquad+\left[\sum_m\left|\sum_{j=0}^{m-1}\left[\sum_{i=j}^{m}(-1)^{i-j}\frac{\Gamma(\tau+1)}{(i-j)!\Gamma(\tau-i+j+1)}\frac{g_m q_i}{\mathfrak{A}_m}\right]\zeta_j+\frac{g_m q_m}{\mathfrak{A}_m}\zeta_m\right|^{p_m}\right]^{\frac{1}{\mathcal{M}}},\end{aligned} \tag{2.2}$$

and for any $\beta\in\mathbf{R}$ (see, [22])

$$|\beta|^{p_m}\leqslant\max(1,|\beta|^{\mathcal{M}}). \tag{2.3}$$

It is clear that, $\mathfrak{H}_\triangle(\theta)$=0 and $\mathfrak{H}_\triangle(\rho) = \mathfrak{H}_\triangle(-\rho)$, for all $\rho \in \mathcal{R}^q(\triangle_g^p, \kappa)$. Again the inequality (2.2) and (2.3), yield the subadditivity of $\mathfrak{H}_\triangle$ and

$$\mathfrak{H}_\triangle(\beta\rho) \leqslant \max(1, |\beta|)\mathfrak{H}_\triangle(\rho).$$

Let $\{\rho^n\}$ be any sequence of points of the space $\mathcal{R}^q(\triangle_g^p, \kappa)$ such that $\mathfrak{H}_\triangle(\rho^n - \rho) \to 0$ and (β_n) is a sequence of scalars such that $\beta_n \to \beta$. Then, since the inequality,

$$\mathfrak{H}_\triangle(\rho^n) \leqslant \mathfrak{H}_\triangle(\rho) + \mathfrak{H}_\triangle(\rho^n - \rho),$$

holds by subadditivity of $\mathfrak{H}_\triangle$,$\{\mathfrak{H}_\triangle(\rho^n)\}$ is bounded and we thus have

$$\begin{aligned}\mathfrak{H}_\triangle(\beta_n\rho^n - \beta\rho) = &\Bigg[\sum_m \Bigg| \sum_{j=0}^{m-1} \Bigg[\sum_{i=j}^{m} (-1)^{i-j} \frac{\Gamma(\tau+1)}{(i-j)!\Gamma(\tau-i+j+1)} \frac{g_m q_i}{\mathfrak{A}_m}\Bigg] \\ &\times (\beta_n \rho_j^n - \beta\rho_j) + \frac{g_m q_m}{\mathfrak{A}_m}(\beta_n\rho_j^n - \beta\rho_j)\Bigg|^{p_m}\Bigg]^{\frac{1}{\mathcal{M}}} \\ &\leqslant |\beta_n - \beta|^{\frac{1}{\mathcal{M}}} \mathfrak{H}_\triangle(\rho^n) + |\beta|^{\frac{1}{\mathcal{M}}} \mathfrak{H}_\triangle(\rho^n - \rho),\end{aligned}$$

and approaches to zero as $n \to \infty$. This shows that the continuity of scalar multiplication. Hence, $\mathfrak{H}_\triangle$ is paranorm on the space $\mathcal{R}^q(\triangle_g^p, \kappa)$.

We now show the completeness property of $\mathcal{R}^q(\triangle_g^p, \kappa)$. For that, let $\{\rho^j\}$ be any Cauchy sequence in $\mathcal{R}^q(\triangle_g^p, \kappa)$, where $\rho^i = \{\rho_0^i, \rho_1^i, \cdots\}$. Then, for a given $\epsilon > 0$. we can find a positive integer $n_0(\epsilon)$ such that

$$\mathfrak{H}_\triangle(\rho^i - \rho^j) < \epsilon, \tag{2.4}$$

for all $i, j \geqslant n_0(\epsilon)$. Using definition of $\mathfrak{H}_\triangle$ and for each fixed $m \in \mathbf{N}$, we have

$$\left|\left(\mathcal{R}^q(\Delta_g^\kappa)\rho^i\right)_m - \left(\mathcal{R}^q(\Delta_g^\kappa)\rho^j\right)_m\right| \leqslant \left[\sum_m \left|\left(\mathcal{R}^q(\Delta_g^\kappa)\rho^i\right)_m - \left(\mathcal{R}^q(\Delta_g^\kappa)\rho^j\right)_m\right|^{p_m}\right]^{\frac{1}{\mathcal{M}}} < \epsilon,$$

for $i, j \geqslant n_0(\epsilon)$. This shows that $\{(\mathcal{R}^q(\Delta_g^\kappa)\rho^0)_k, (\mathcal{R}^q(\Delta_g^\kappa)\rho^1)_k, \cdots\}$ is a Cauchy sequence of real numbers for every fixed $m \in \mathbf{N}$. Since $\mathbf{R}$ is complete and hence converges, say, $(\mathcal{R}^q(\Delta_g^\kappa)\rho^i)_m \to (\mathcal{R}^q(\Delta_g^\kappa)\rho)_m$ for $i \to \infty$. Utilizing these infinitely many limits $(\mathcal{R}^q(\Delta_g^\kappa)\rho)_0, (\mathcal{R}^q(\Delta_g^\kappa)\rho)_1, \cdots$, we consider the sequence $\{(\mathcal{R}^q(\Delta_g^\kappa)\rho)_0, (\mathcal{R}^q(\Delta_g^\kappa)\rho)_1, \ldots\}$. Now for each $m \in \mathbf{N}$ and $i, j \geqslant n_0(\epsilon)$, we see from (2.4) that

$$\sum_{k=0}^{r} \left|(\mathcal{R}^q(\Delta_g^\kappa)\rho^i)_m - (\mathcal{R}^q(\Delta_g^\kappa)\rho^j)_m\right|^{p_m} \leqslant \mathfrak{H}_\triangle(\rho^i - \rho^j)^{\mathcal{M}} < \epsilon^{\mathcal{M}}. \tag{2.5}$$

Take any $i, j \geqslant n_0(\epsilon)$, letting first $j \to \infty$ in (2.5) and then $r \to \infty$, we obtain

$$\mathfrak{H}_\triangle(\rho^i - \rho) \leqslant \epsilon.$$

Finally, taking $\epsilon = 1$ in (2.5) and letting $i \geqslant n_0(1)$ we have by Minkowski's inequality for each $r \in \mathbb{N}$ that

$$\left[\sum_{k=0}^{r} \left|(\mathcal{R}^q(\Delta_g^\kappa)\rho)_m\right|^{p_m}\right]^{\frac{1}{\mathcal{M}}} \leqslant \mathfrak{H}_\triangle(\upsilon^i - \upsilon) + \mathfrak{H}_\triangle(\upsilon^i) \leqslant 1 + \mathfrak{H}_\triangle(\upsilon^i),$$

which shows that $\upsilon \in \mathcal{R}^q(\triangle_g^p, \kappa)$. Since $\mathfrak{H}_\triangle(\rho - \rho^i) \leqslant \epsilon$, for all $i \geqslant n_0(\epsilon)$, it follows that $\rho^i \to \rho$ as $i \to \infty$, hence we have shown that $\mathcal{R}^q(\triangle_g^p, \kappa)$ is complete. $\square$

Remark 2.4. It is easy to see that the property of absoluteness is not satisfied for $\mathcal{R}^q(\triangle_g^p, \kappa)$, this implies that $\mathfrak{H}_\triangle(\rho) \neq \mathfrak{H}_\triangle(|\rho|)$ for atleast one sequence in the given space and thus $\mathcal{R}^q(\triangle_g^p, \kappa)$ is a sequence space with non-absolute nature.

We will now study for computing the linear isomorphism property of $\mathcal{R}^q(\triangle_g^p, \kappa)$.

Theorem 2.5. *For $0 < p_k \leqslant \mathcal{H} < \infty$, the introduced space is linearly isomorphic to space $\ell(p)$.*

Proof. In order to establish the result, we should determine the presence of a linear bijection to the spaces $\mathcal{R}^q(\triangle_g^p, \kappa)$ and $\ell(p)$. Employing (2.1), consider the mapping $\mathcal{G} : \mathcal{R}^q(\triangle_g^p, \kappa) \to \ell(p)$ given by $\rho \to \sigma = \mathcal{G}\rho$. Its linearity is trivial and $\rho = \theta$ for $\mathcal{G}\rho = \theta$ and consequently the injective property of $\mathcal{G}$ follows.

For $\sigma = (\sigma_m) \in \ell(p)$ and $k \in \mathbb{N}$, choose sequence $\nu = (\rho_m)$ given by

$$\rho_k = \sum_{j=0}^{k-1} \left[\sum_{i=j}^{j+1} (-1)^{k-j} \frac{\Gamma(-\tau+1)}{(k-i)!\Gamma(-\tau-k+i+1)} \frac{\mathfrak{A}_j}{g_j q_i} \sigma_j \right] + \frac{\mathfrak{A}_k}{g_j q_k} \sigma_k.$$

So that

$$\begin{aligned} \mathfrak{H}_\triangle(\rho) &= \left[\sum_k \left| \sum_{j=0}^{k-1} \left[\sum_{i=j}^{k} (-1)^{i-j} \frac{\Gamma(\tau+1)}{(i-j)!\Gamma(\tau-i+j+1)} \frac{g_k q_i}{\mathfrak{A}_k} \right] \rho_j + \frac{g_k q_k}{\mathfrak{A}_k} \rho_k \right|^{p_k} \right]^{\frac{1}{M}} \\ &= \left[\sum_k \left| \delta_{kj} \sigma_k \right|^{p_k} \right]^{\frac{1}{M}} \\ &= \left[\sum_k |\sigma_k|^{p_k} \right]^{\frac{1}{M}} = \mathfrak{S}(\sigma) < \infty, \end{aligned}$$

where, Kronecker delta δ_{kj} is given by

$$\delta_{kj} = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \neq j. \end{cases}$$

Hence, it follows that $\rho \in \mathcal{R}^q(\triangle_g^p, \kappa)$, which implies that $\mathcal{G}$ is surjective and hence preserves the property of paranorm. Thus, it follows that $\mathcal{R}^q(\triangle_g^p, \kappa)$ and $\ell(p)$ are linearly isomorphic. □

3. The Schauder basis of the given space

This section deals with the calculation of Schauder basis of the given space.

Definition 3.1. Let $\mathcal{X}$ be a Banach space. A sequence $(w_n) \subset \mathcal{X}$ is a Schauder basis if for every $\nu \in \mathcal{X}$ there exists a unique convergent series of the form $\nu = \sum_{j=0}^{\infty} a_j w_j$, where (a_j) is a sequence of scalars and is known as expansion of ν.

Theorem 3.2. *Consider the sequence $\vartheta^{(r)}(q) = \{\vartheta_n^{(r)}(q)\}$ of objects of space $\mathcal{R}^q(\triangle_g^p, \kappa)$ for all fixed $r \in \mathbf{N}$ given by*

$$\vartheta_n^{(r)}(q) = \begin{cases} \sum_{i=j}^{j+1} (-1)^{r-j} \frac{\Gamma(-\tau+1)}{(r-i)!\Gamma(-\tau-r+i+1)} \frac{\mathfrak{A}_j}{g_r q_i}, & \text{if } 0 \leqslant r < n, \\ \frac{\mathfrak{A}_n}{g_r q_n}, & \text{if } r = n, \\ 0, & \text{if } r > n. \end{cases}$$

Then, the basis for $\mathcal{R}^q(\triangle_g^p, \kappa)$ is $\{\vartheta^{(r)}(q)\}$ and any $\rho \in \mathcal{R}^q(\triangle_g^p, \kappa)$ can be expressed in one and only one way as

$$\rho = \sum_r \Lambda_r(q) \vartheta^{(r)}(q), \tag{3.1}$$

where, $\Lambda_r(q) = \left(\mathcal{R}^q(\Delta_g^\kappa)\rho\right)_r$, for all $r \in \mathbf{N}$ and $0 < p_r \leqslant \mathcal{H} < \infty$.

Proof. Trivially, we have $\vartheta^{(m)}(q) \subset \mathcal{R}^q(\triangle_g^p, \kappa)$, it is due to the fact that

$$\mathcal{R}^q(\Delta_g^\kappa)\vartheta^{(m)}(q) = e^{(m)} \in \ell(p), \quad \text{for } m \in \mathbf{N}, \tag{3.2}$$

and $0 < p_m \leqslant \mathcal{H} < \infty$, where $e^{(m)}$ is that sequence having non-zero entry as unity in m^{th} place with $m \in \mathbf{N}$.

We now suppose that $\rho \in \mathcal{R}^q(\triangle_g^p, \kappa)$ and write

$$\rho^{[l]} = \sum_{r=0}^{l} \Lambda_r(q)\vartheta^{(r)}(q), \tag{3.3}$$

for all non-negative integer l. Then, clearly by using $\mathcal{R}^q(\Delta_g^\kappa)$ to (3.3) with (3.2), we see that

$$\mathcal{R}^q(\Delta_g^\kappa)\rho^{[l]} = \sum_{r=0}^{l} \Lambda_r(q)\mathcal{R}^q(\Delta_g^\kappa)\vartheta^{(r)}(q) = \sum_{r=0}^{l} \left(\mathcal{R}^q(\Delta_g^\kappa)\right)_r e^{(r)},$$

and

$$\left(\mathcal{R}^q(\Delta_g^\kappa)\left(\rho - \rho^{[l]}\right)\right)_i = \begin{cases} 0, & \text{if } 0 \leqslant i \leqslant l, \\ \left(\mathcal{R}^q(\Delta_g^\kappa)\rho\right)_i, & \text{if } i > l, \end{cases}$$

with $i, l \in \mathbf{N}$. Since $\varepsilon > 0$, there exists an integer l_0 in such a way that

$$\left(\sum_{i=l}^{\infty} |\left(\mathcal{R}^q(\Delta_g^\kappa)\rho\right)_i|^{p_r}\right)^{\frac{1}{\mathcal{M}}} < \frac{\varepsilon}{2},$$

for all $l \geqslant l_0$. Hence,

$$\begin{aligned} \mathfrak{H}_\triangle\left(\rho - \rho^{[l]}\right) &= \left(\sum_{i=l}^{\infty} \left|\left(\mathcal{R}^q(\Delta_g^\kappa)\rho\right)_i\right|^{p_r}\right)^{\frac{1}{\mathcal{M}}} \\ &\leqslant \left(\sum_{i=l_0}^{\infty} \left|\left(\mathcal{R}^q(\Delta_g^\kappa)\rho\right)_i\right|^{p_r}\right)^{\frac{1}{\mathcal{M}}} \\ &< \frac{\varepsilon}{2} < \varepsilon, \end{aligned}$$

for every $l \geqslant l_0$, employing there by $\rho \in \mathcal{R}^q(\triangle_g^p, \kappa)$ is represented as (3.1).

We now need to prove this representation to be unique for $\rho \in \mathcal{R}^q(\triangle_g^p, \kappa)$ given by (3.1). On contrary, assume that we can find another form of the type $\rho = \sum_r (q)b^r(q)$. Since the mapping $\mathcal{G} : \mathcal{R}^q(\triangle_g^p, \kappa) \to \ell(p)$ employed is continuous, thus for $m \in \mathbf{N}$, we see

$$\begin{aligned} \left(\mathcal{R}^q(\Delta_g^\kappa)\rho\right)_m &= \sum_r \mathfrak{J}_r(q)\left(\mathcal{R}^q(\Delta_g^\kappa)\vartheta^r(q)\right)_m \\ &= \sum_r \mathfrak{J}_r(q)e_m^{(r)} = \mathfrak{J}_m(q). \end{aligned}$$

This is contradiction to the fact that $\left(\mathcal{R}^q(\Delta_g^\kappa)\rho\right)_m = \Lambda_m(q)$, $\forall m \in \mathbf{N}$. Consequently, the representation which is set by (3.1) is unique. □

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