Existence and stability results for Hilfer-Katugampola-type fractional implicit differential equations with nonlocal conditions

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Abstract

This article contains a new discussion for Hilfer-Katugampola-type fractional derivative. We establish an existence and uniqueness results of Hilfer-Katugampola-type fractional derivative for implicit differential equations with the help of Schaefer's fixed point theorem and Banach contraction principle. Also, we use the Gronwall's lemma for singular kernels to prove the Ulam-Hyers-Rassias stability results. Further, the examples are given to illustrate our main results.

Keywords: Hilfer-Katugampola-type fractional derivative, implicit differential equation, Schaefer's fixed point theorem, existence, uniqueness, Ulam stability.

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1. Introduction

Fractional differential equations come to light in many scientific disciplines, engineering as the mathematical modeling of systems processes in the fields of chemistry, physics, electrodynamics of complex medium, aerodynamics, polymer rheology. In the last few years there has been considerable development on the ordinary and partial fractional differintegrals equations, see [2, 3, 5, 11–13, 15, 25, 32, 33] and references therein. There are many general definitions which interpolates to other differintegrals for particular values of parameters reported in the literature, but the most familiar definitions are the famous definitions of Riemann-Liouville, Caputo, Hadamard and Hilfer [15, 24, 28]. Recently, Katugampola has introduced a generalized form of the Riemann-Liouville fractional differential derivatives and integrals [22, 23]. Thereafter, OLiveira et al. [27], have established a more generalized form of Katugampola fractional derivative which is Hilfer-Katugampola. This formulation is an interpolation for the popular definitions of the Hilfer, Hilfer-Hadamard, Caputo-Hadamard and Riemann Liouville fractional derivatives, etc, we refer the reader to [27].

In 1940, Ulam gave a wide ranging talk before the mathematics club of the Wisconsin university in which he discussed several important unsolved problems [35, 36] and he raised the question concerning

the homomorphism stability. In the next years, Hyers positively answered the question of Ulam's partially [16]. Moreover, a generalized was obtained by Rassias for the Hyers results [29, 30]. And since 1940, the mathematics authors have been interested to studied the Ulam-type stability problems [4, 26, 31]. In last few decades, Jung extended the previously mentioned stability results, see [19–21]. The Ulam stability analysis is very useful in lot of applications, such as optimization, numerical analysis, etc, for more detailed study of Ulam-type stability with different approaches see [17, 38, 39]. Some recent work on stability of differential equations (or their solutions) with variety of initial or boundary conditions are reported in [1, 6–10, 18, 37].

Vivek et al. [37] in 2018, have obtained some existence and stability results for Hilfer fractional implicit differential equations with nonlocal conditions of the form

$$\begin{cases} D_{0^+}^{\alpha,\beta}x(t) = \varphi(t,x(t),D_{0^+}^{\alpha,\beta}x(t)), & t \in [0,T], \ 0 < \alpha < 1, 0 \leqslant \beta \leqslant 1, \\ I_{0^+}^{1-\gamma}x(0) = \sum_{i=1}^{n} \eta_i x(\xi_i), & \alpha \leqslant \gamma = \alpha + \beta(1-\alpha) < 1, \quad \xi_i \in (0,T), \end{cases}$$

where $D_{0^+}^{\alpha,\beta}$ is the Hilfer fractional derivative of order α and type β.

Bhairat [9], studied the existence and stability of fractional differential equations involving generalized Katugampola derivative

$$\begin{cases} {}_\rho D_{0^+}^{\alpha,\beta}x(t) = \varphi(t,x(t)), & t \in \Omega, \ 0 < \alpha < 1, 0 \leqslant \beta \leqslant 1, \\ {}_\rho I_{0^+}^{1-\gamma}x(0) = x_0, & \gamma = \alpha + \beta(1-\alpha), \quad x_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\rho > 0$ and ${}_\rho D_{0^+}^{\alpha,\beta}$ is the Hilfer-Katugampola-type fractional derivative of order α and type β.

Ibrahim et al. [18], have discuss the existence, uniqueness and stability results for a general class of Langevin equations with two Hilfer-Katugampola fractional derivatives as follows of the form

$$\begin{cases} {}_\rho D^{\alpha_1,\beta} \left({}_\rho D^{\alpha_2,\beta} + c \right) x(t) = \varphi(t,x(t)), & t \in (a,b], \\ {}_\rho I^{1-\gamma}x(a) = x_a, & \gamma = (\alpha_1 + \alpha_2)(1-\beta) + \beta, \end{cases}$$

where $\rho > 0$ and ${}_\rho D^{\alpha_1,\beta}$, ${}_\rho D^{\alpha_2,\beta}$ are Hilfer-Katugampola fractional differential operator of orders α_1, α_2, respectively and type β.

The aim of this article is to study the implicit differential equation with nonlocal condition involving Hilfer-Katugampola-type fractional derivative of the following form

$$\begin{cases} {}_\rho D_{0^+}^{\alpha,\beta}x(t) = \varphi(t,x(t), {}_\rho D_{0^+}^{\alpha,\beta}x(t)), & t \in \mathbb{I} := [0,T], \ 0 < \alpha < 1, 0 \leqslant \beta \leqslant 1, \\ {}_\rho I_{0^+}^{1-\gamma}x(0) = \sum_{i=1}^{n} \xi_i x(\eta_i), & \alpha \leqslant \gamma = \alpha + \beta(1-\alpha) < 1, \end{cases} \tag{1.2}$$

where $\rho > 0$ and ${}_\rho D_{0^+}^{\alpha,\beta}$ is the Hilfer-Katugampola-type fractional derivative of order α and type β, ${}_\rho I_{0^+}^{1-\gamma}$ is the Katugampola fractional integral of order $1-\gamma$, $\eta_i \in (0,T)$, $\xi_i \in \mathbb{R}$, for all $i = 1, 2, \ldots, n$, $\eta_1 \leqslant \eta_2 \leqslant \cdots \leqslant \eta_n$ and here we let $\mathbb{R}$ be a Banach space, $\varphi : \mathbb{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The rest of this paper is structured as follows: Section 2 contains some fundamental concepts on Hilfer-Katugampola-type fractional derivative. In Section 3, we present our main result of existence and uniqueness by using Schaefer's fixed point theorem and Banach contraction principle. Further, we discuss stability analysis of Hilfer-Katugampola-type fractional derivative. The last section contains some examples to illustrate our main results.

2. Preliminaries

In this section, we introduce some basic notations, definitions and lemmas regarding the theory of fractional differential equations, which will be used later in this paper, to obtain our main results.

In the light of the [27, Definition 3] we have the following definition.

Definition 2.1. Let $C[\mathbb{I}, \mathbb{R}]$ is a Banach space of continuous functions φ from $\mathbb{I}$ into $\mathbb{R}$ with the norm $\|\varphi\|_C = \max\{|\varphi(t)| : t \in \mathbb{I}\}$ and the Lebesgue space $L^1(\mathbb{I})$ of integrable functions $\varphi : \mathbb{I} \to \mathbb{R}$ with the norm $\|\varphi\|_1 = \int_0^T |\varphi(\tau)| d\tau$. For $\rho > 0$ and $0 \leqslant \gamma < 1$, the weighted space $C_{\gamma, \rho}[\mathbb{I}, \mathbb{R}]$ of continuous functions φ on $(0, T]$ is defined by

$$C_{\gamma, \rho}[\mathbb{I}, \mathbb{R}] = \left\{ \varphi : (0, T] \to \mathbb{R} : [(t^\rho/\rho)]^\gamma \varphi(t) \in C[\mathbb{I}, \mathbb{R}] \right\}$$

with the norm

$$\|\varphi\|_{C_{\gamma, \rho}} = \left\| [(t^\rho/\rho)]^\gamma \varphi(t) \right\|_C = \max_{t \in \mathbb{I}} \left| [(t^\rho/\rho)]^\gamma \varphi(t) \right|,$$

where $C_{0, \rho}[\mathbb{I}, \mathbb{R}] = C[\mathbb{I}, \mathbb{R}]$. Obviously,

$$C_{\gamma, \delta_\rho}^n[\mathbb{I}, \mathbb{R}] = \left\{ \varphi \in C_{\delta_\rho}^{n-1}[\mathbb{I}, \mathbb{R}] : \delta_\rho^n \varphi \in C_{\gamma, \rho}[\mathbb{I}, \mathbb{R}] \right\}$$

is the Banach space with the norm

$$\|\varphi\|_{C_{\gamma, \delta_\rho}^n} = \sum_{k=0}^{n-1} \left\| \delta_\rho^k \varphi \right\|_C + \left\| \delta_\rho^n \varphi \right\|_{C_{\gamma, \rho}}, \quad \|\varphi\|_{C_{\gamma, \delta_\rho}^n} = \sum_{k=0}^n \max_{t \in \mathbb{I}} \left| \delta_\rho^k \varphi(t) \right|, \quad n \in \mathbb{N}.$$

For $n = 0$, we have

$$C_{\gamma, \delta_\rho}^0 = C_{\gamma, \rho}[\mathbb{I}, \mathbb{R}].$$

Also, for $0 < \alpha < 1, 0 \leqslant \beta \leqslant 1$, we define the following spaces

$$C_{1-\gamma, \rho}^{\alpha, \beta}[\mathbb{I}, \mathbb{R}] = \left\{ \varphi \in C_{1-\gamma, \rho}[\mathbb{I}, \mathbb{R}], \ _\rho D_{0+}^{\alpha, \beta} \varphi \in C_{1-\gamma, \rho}[\mathbb{I}, \mathbb{R}] \right\},$$

and

$$C_{1-\gamma, \rho}^{\gamma}[\mathbb{I}, \mathbb{R}] = \left\{ \varphi \in C_{1-\gamma, \rho}[\mathbb{I}, \mathbb{R}], \ _\rho D_{0+}^{\gamma} \varphi \in C_{1-\gamma, \rho}[\mathbb{I}, \mathbb{R}] \right\}.$$

Clearly,

$$C_{1-\gamma, \rho}^{\gamma}[\mathbb{I}, \mathbb{R}] \subset C_{1-\gamma, \rho}^{\alpha, \beta}[\mathbb{I}, \mathbb{R}].$$

Definition 2.2 ([27]). Let $\Omega = (0, T]$ and $\varphi : (0, \infty) \to \mathbb{R}$, the Katugampola fractional integral $_\rho I_{0+}^\alpha \varphi$ of order $\gamma \in \mathbb{C}(\mathfrak{R}(\alpha) > 0)$ is defined for $\rho > 0$ as

$$(_\rho I_{0+}^\alpha \varphi)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1} \varphi(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau, \quad (t > 0),$$

and the corresponding Katugampola fractional derivative $_\rho D_{0+}^\alpha \varphi$ is defined as

$$(_\rho D_{0+}^\alpha \varphi)(t) := \left(t^{1-\rho} \frac{d}{dt}\right)^n (_\rho I_{0+}^{n-\alpha} \varphi)(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right)^n \int_0^t \frac{\tau^{\rho-1} \varphi(\tau)}{(t^\rho - \tau^\rho)^{\alpha-n+1}} d\tau, \quad (t > 0),$$

Definition 2.3 ([27]). Let $0 < \alpha < 1, \ 0 \leqslant \beta \leqslant 1, \ \varphi \in C_{1-\gamma, \rho}[0, T]$. The Hilfer-Katugampola fractional derivative $_\rho D^{\alpha, \beta}$ of order α and type β of φ is defined as

$$(_\rho D^{\alpha, \beta} \varphi)(t) = \left(_\rho I^{\beta(1-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right) {}_\rho I^{(1-\alpha)(1-\beta)} \varphi \right)(t)$$

$$= \left(_\rho I^{\beta(1-\alpha)} (\delta_\rho) \ _\rho I^{(1-\alpha)(1-\beta)} \varphi \right)(t); \quad \gamma = \alpha + \beta(1-\alpha),$$

where $_\rho I^{(.)}$ is the Katugampola fractional integral given in Definition 2.2.

Lemma 2.4 (Schaefer fixed point Theorem, [14]). *Let* E *be a Banach space and* $K : E \to E$ *is a continuous and compact mapping. If,*

$$X = \left\{ x \in E : x = \omega K(x) \text{ for some } \omega \in [0,1] \right\}$$

is a bounded set, then K *has a fixed point.*

Lemma 2.5 ([27]). *Let* $0 < \alpha < 1, 0 \leqslant \beta \leqslant 1, \gamma = \alpha + \beta(1 - \alpha)$, *and assume that* $\varphi(t, x(t)) \in C_{1-\gamma,\rho}[0, T]$ *where* $\varphi : (0, T] \times \mathbb{R} \to \mathbb{R}$ *be a function for any* $x \in C_{1-\gamma,\rho}[0, T]$. *If* $x \in C_{1-\gamma,\rho}^{\gamma}[0, T]$, *then* x *satisfies* (1.1), *if and only if* x *satisfies the second kind Volterra fractional integral equation*

$$x(t) = \frac{x_0}{\Gamma(\gamma)} \left(t^\rho / \rho \right)^{\gamma - 1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1} \varphi(\tau, x(\tau))}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau, \quad (t > 0).$$

Lemma 2.6 ([27]). *Let* $t > 0$, $_\rho I_{0+}^\alpha$ *and* $_\rho D_{0+}^\alpha$, *as defined in Definition 2.2. Then for* $\alpha \geqslant 0$, $\beta > 0$, *we have*

$$\left({_\rho I_{0+}^\alpha} \left(\tau^\rho / \rho \right)^{\beta - 1} \right)(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)} \left(t^\rho / \rho \right)^{\alpha + \beta - 1},$$

and for $0 < \alpha < 1$

$$\left({_\rho D_{0+}^\alpha} \left(\tau^\rho / \rho \right)^{\alpha - 1} \right)(t) = 0.$$

Lemma 2.7 ([22]). *Let* $0 < a < b < \infty$, $1 \leqslant p \leqslant \infty$, $\alpha, \beta > 0$ *and let* $(c \leqslant \rho) \rho, c \in \mathbb{R}$. *Then, for* $\varphi \in X_c^p(a, b)$ *the following semigroup property holds*

$$\left({_\rho I_{0+}^\alpha} \, {_\rho I_{0+}^\beta} \varphi \right)(t) = \left({_\rho I_{0+}^{\alpha+\beta}} \varphi \right)(t),$$

and (see [14]*)*

$$\left({_\rho D_{0+}^\beta} \, {_\rho I_{0+}^\beta} \varphi \right)(t) = \varphi(t).$$

In the light of the lemmas (Lemma 7, Lemma 9 , and Lemma 12, see [27]), we have the following lemmas.

Lemma 2.8. *Assume that* $0 < \alpha < 1$, $0 \leqslant \gamma < 1$, $\rho > 0$. *If* $\varphi \in C_{\gamma,\rho}[\mathbb{I}, \mathbb{R}]$ *and* $_\rho I_{0+}^{1-\beta} \varphi \in C_{\gamma,\rho}^1[\mathbb{I}, \mathbb{R}]$, *then*

$$\left({_\rho I_{0+}^\alpha} \, {_\rho D_{0+}^\alpha} \varphi \right)(t) = \varphi(t) - \frac{\left({_\rho I_{0+}^{1-\alpha}} \varphi \right)(0)}{\Gamma(\alpha)} \left(t^\rho / \rho \right)^{\alpha - 1}, \quad \text{for all } t \in \mathbb{I}.$$

Lemma 2.9. *Assume that* $\alpha > 0$, $0 \leqslant \gamma < 1$, $\rho > 0$ *and* $\varphi \in C_{\gamma,\rho}[\mathbb{I}, \mathbb{R}]$. *If* $\gamma < \alpha$, *then*

$$\left({_\rho I_{0+}^\alpha} \varphi \right)(0) = \lim_{t \to 0^+} \left({_\rho I_{0+}^\alpha} \varphi \right)(t) = 0.$$

Lemma 2.10. *Assume that* $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$, $\rho > 0$ *and* $\gamma = \alpha + \beta(1 - \alpha)$. *If* $\varphi \in C_{1-\gamma,\rho}^{\gamma}[\mathbb{I}, \mathbb{R}]$, *then*

$${_\rho I_{0+}^\gamma} \, {_\rho D_{0+}^\gamma} \varphi = {_\rho I_{0+}^\alpha} \, {_\rho D_{0+}^{\alpha,\beta}} \varphi, \quad {_\rho D_{0+}^\gamma} \, {_\rho I_{0+}^\alpha} \varphi = {_\rho D_{0+}^{\beta(1-\alpha)}} \varphi.$$

Now, in the following lemma we establish an important equivalent mixed-type Volterra fractional integral equation for our problem (1.2), *which we will utilize later in our main results.*

Lemma 2.11. *Let* $0 < \alpha < 1, 0 \leqslant \beta \leqslant 1, \gamma = \alpha + \beta(1 - \alpha)$, *and assume that* $\varphi(t, x(t), {_\rho D_{0+}^{\alpha,\beta}} x(t)) \in C_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}]$ *where* $\varphi : \mathbb{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a function for any* $x \in C_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}]$. *If* $x \in C_{1-\gamma,\rho}^{\gamma}[\mathbb{I}, \mathbb{R}]$, *then* x *satisfies* (1.2), *if and only if* x *satisfies the mixed-type Volterra fractional integral equation*

$$x(t) = \frac{\Psi \, \rho^{1-\alpha}}{\Gamma(\alpha)} \left(t^\rho / \rho \right)^{\gamma - 1} \sum_{i=1}^{n} \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} d\tau + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau, \tag{2.1}$$

where

$$\Psi := \frac{1}{\Gamma(\gamma) - \sum_{i=1}^{n} \xi_i \left(\eta_i^{\rho}/\rho\right)^{\gamma-1}}, \quad \text{if} \quad \Gamma(\gamma) \neq \sum_{i=1}^{n} \xi_i \left(\eta_i^{\rho}/\rho\right)^{\gamma-1} \tag{2.2}$$

and

$$\Lambda_x(t) :=_{\rho} D_{0+}^{\alpha,\beta} x(t) = \varphi(t, x(t), \Lambda_x(t)).$$

Proof. Firstly, we will prove the necessary condition. According to Lemma 2.5, we can express the solution of the problem (1.2) as follows

$$x(t) = \frac{\rho I_{0+}^{1-\gamma} x(0)}{\Gamma(\gamma)} \left(t^{\rho}/\rho\right)^{\gamma-1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(t^{\rho} - \tau^{\rho})^{1-\alpha}} d\tau. \tag{2.3}$$

Now, by using the substitution $t = \eta_i$ in (2.3), we have

$$x(\eta_i) = \frac{\rho I_{0+}^{1-\gamma} x(0)}{\Gamma(\gamma)} \left(\eta_i^{\rho}/\rho\right)^{\gamma-1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(\eta_i^{\rho} - \tau^{\rho})^{1-\alpha}} d\tau, \tag{2.4}$$

by multiply ξ_i on both sides of the equation (2.4), we get

$$\xi_i x(\eta_i) = \frac{\rho I_{0+}^{1-\gamma} x(0)}{\Gamma(\gamma)} \xi_i \left(\eta_i^{\rho}/\rho\right)^{\gamma-1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(\eta_i^{\rho} - \tau^{\rho})^{1-\alpha}} d\tau.$$

Next, by using the initial value given in problem (1.2), we have

$$\rho I_{0+}^{1-\gamma} x(0) = \sum_{i=1}^{n} \xi_i x(\eta_i) = \frac{\rho I_{0+}^{1-\gamma} x(0)}{\Gamma(\gamma)} \sum_{i=1}^{n} \xi_i \left(\eta_i^{\rho}/\rho\right)^{\gamma-1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{n} \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(\eta_i^{\rho} - \tau^{\rho})^{1-\alpha}} d\tau,$$

which implies that

$$\rho I_{0+}^{1-\gamma} x(0) \left(\Gamma(\gamma) - \sum_{i=1}^{n} \xi_i \left(\eta_i^{\rho}/\rho\right)^{\gamma-1}\right) = \frac{\Gamma(\gamma)\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{n} \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(\eta_i^{\rho} - \tau^{\rho})^{1-\alpha}} d\tau.$$

Thus,

$$\rho I_{0+}^{1-\gamma} x(0) = \frac{\Gamma(\gamma)\rho^{1-\alpha}}{\Gamma(\alpha)} \Psi \sum_{i=1}^{n} \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(\eta_i^{\rho} - \tau^{\rho})^{1-\alpha}} d\tau. \tag{2.5}$$

Therefore, by substitution of (2.5) in (2.3), we obtain that (2.1).

Secondly, we will prove the sufficient condition. By applying $\rho I_{0+}^{1-\gamma}$ on both sides of the equation (2.1), we obtain

$$\rho I_{0+}^{1-\gamma} x(t) =_{\rho} I_{0+}^{1-\gamma} \left(t^{\rho}/\rho\right)^{\gamma-1} \frac{\Psi \rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{n} \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(\eta_i^{\rho} - \tau^{\rho})^{1-\alpha}} d\tau +_{\rho} I_{0+}^{1-\gamma} {}_{\rho} I_{0+}^{\alpha} \Lambda_x(t),$$

Now, by using Lemmas 2.6 and 2.7, we have

$$\rho I_{0+}^{1-\gamma} x(t) = \frac{\Gamma(\alpha) \rho^{1-\alpha}}{\Gamma(\alpha)} \Psi \sum_{i=1}^{n} \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(\eta_i^{\rho} - \tau^{\rho})^{1-\alpha}} d\tau +_{\rho} I_{0+}^{1-\beta(1-\alpha)} \Lambda_x(t).$$

Since $1 - \beta(1-\alpha) > 1 - \gamma$, then by Lemma 2.9 and taking the limit as $t \to 0^+$ yields

$$\rho I_{0+}^{1-\gamma} x(0) = \frac{\Gamma(\gamma) \rho^{1-\alpha}}{\Gamma(\alpha)} \Psi \sum_{i=1}^{n} \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(\eta_i^{\rho} - \tau^{\rho})^{1-\alpha}} d\tau. \tag{2.6}$$

Using the substitution $t = \eta_i$ in (2.1), we have

$$x(\eta_i) = \frac{\Psi \, \rho^{1-\alpha}}{\Gamma(\alpha)} \left(\eta_i^\rho/\rho\right)^{\gamma-1} \sum_{i=1}^{n} \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} \, d\tau + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} \, d\tau.$$

Furthermore, we derive

$$
\begin{aligned}
\sum_{i=1}^{n} \xi_i x(\eta_i) &= \frac{\Psi \, \rho^{1-\alpha}}{\Gamma(\alpha)} \left(\sum_{i=1}^{n} \xi_i \left(\eta_i^\rho/\rho\right)^{\gamma-1} \right) \sum_{i=1}^{n} \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} \, d\tau \\
&\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{n} \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} \, d\tau \\
&= \left[1 + \Psi \sum_{i=1}^{n} \xi_i \left(\eta_i^\rho/\rho\right)^{\gamma-1} \right] \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{n} \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} \, d\tau \\
&= \frac{\Gamma(\gamma) \, \rho^{1-\alpha}}{\Gamma(\alpha)} \Psi \sum_{i=1}^{n} \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} \, d\tau, \qquad \text{(by (2.2))}.
\end{aligned}
\tag{2.7}
$$

From equations (2.6) and (2.7), we obtain

$$\,_\rho I_{0+}^{1-\gamma} x(0) = \sum_{i=1}^{n} \xi_i x(\eta_i).$$

Next, applying $\,_\rho D_{0+}^{\gamma}$ on both sides of the equation (2.1) and by using Lemmas 2.6 and 2.10, we get

$$\,_\rho D_{0+}^{\gamma} x(t) = \,_\rho D_{0+}^{\beta(1-\alpha)} \Lambda_x(t) = \,_\rho D_{0+}^{\beta(1-\alpha)} \varphi(t, x(t), \,_\rho D_{0+}^{\alpha,\beta} x(t)). \tag{2.8}$$

Since $x \in C_{1-\gamma,\rho}^{\gamma}[\mathbb{I}, \mathbb{R}]$ and by definition of $C_{1-\gamma,\rho}^{\gamma}[\mathbb{I}, \mathbb{R}]$ given in Definition 2.1, we have $\,_\rho D_{0+}^{\gamma} x(t) \in C_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}]$ which implies that $\,_\rho D_{0+}^{\beta(1-\alpha)} \varphi = \delta_\rho \,_\rho I_{0+}^{1-\beta(1-\alpha)} \varphi \in C_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}]$. For $\varphi \in C_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}]$ it is clear that $\,_\rho I_{0+}^{1-\beta(1-\alpha)} \varphi \in C_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}]$. So, we have $\,_\rho I_{0+}^{1-\beta(1-\alpha)} \varphi \in C_{1-\gamma,\rho}^{1}[\mathbb{I}, \mathbb{R}]$. Therefore, φ and $\,_\rho I_{0+}^{1-\beta(1-\alpha)} \varphi$ satisfy conditions of Lemma 2.8. Applying $\,_\rho I_{0+}^{\beta(1-\alpha)}$ on both sides of equation (2.8) and by using Lemma 2.8, we can get

$$\,_\rho D_{0+}^{\alpha,\beta} x(t) = \Lambda_x(t) - \frac{\left(\,_\rho I_{0+}^{1-\beta(1-\alpha)} \Lambda_x \right)(0)}{\Gamma(\beta(1-\alpha))} \left(t^\rho/\rho\right)^{\beta(1-\alpha)-1}.$$

From Lemma 2.9, easily we can obtain that $\,_\rho I_{0+}^{1-\beta(1-\alpha)} \Lambda_x)(0) = 0$. Thus, it reduces to $\,_\rho D_{0+}^{\alpha,\beta} x(t) = \Lambda_x(t) = \varphi(t, x(t), \,_\rho D_{0+}^{\alpha,\beta} x(t))$. □

3. Main results

In this section, we present the existence and uniqueness results of solutions for Hilfer-Katugampola-type fractional implicit differential equations given in the problem (1.2), then we discuss Ulam stability results for the Problem (1.2). For our main results, we need the following suppositions.

$(\mathcal{H}_1)$ Suppose that there exist $\phi, \psi, \chi \in C_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}]$ with $\phi^* = \sup_{t\in\mathbb{I}} \phi(t)$, $\psi^* = \sup_{t\in\mathbb{I}} \psi(t)$ and $\chi^* = \sup_{t\in\mathbb{I}} \chi(t) < 1$ such that

$$\left| \varphi(t, p, q) \right| \leqslant \phi(t) + \psi(t) \, |p| + \chi(t) \, |q| \quad \text{for all } t \in \mathbb{I}; \ p, q \in \mathbb{R}.$$

($\mathcal{H}_2$) Suppose that $\varphi \in C_{1-\gamma,\rho}^{\beta(1-\alpha)}[\mathbb{I},\mathbb{R}]$ where $\varphi : \mathbb{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function for any $x \in C_{1-\gamma,\rho}^{\gamma}[\mathbb{I},\mathbb{R}]$ and there exist $\mu > 0$ and $\nu > 0$ where μ, ν are positive constants such that

$$\left| \varphi(t,p,q) - \varphi(t,\tilde{p},\tilde{q}) \right| \leqslant \mu \left| p - \tilde{p} \right| + \nu \left| q - \tilde{q} \right| \quad \text{for any } p,q,\tilde{p},\tilde{q} \in \mathbb{R}; \ t \in \mathbb{I}.$$

($\mathcal{H}_3$) Suppose that the constant Θ satisfies the following estimate

$$\Theta := \left(\frac{\mu}{1-\nu}\right) \frac{B(\gamma,\alpha)}{\Gamma(\alpha)} \left[|\Psi| \sum_{i=1}^{n} \xi_i \left(\eta_i^\rho/\rho\right)^{\alpha+\gamma-1} + \left(T^\rho/\rho\right)^\alpha \right] < 1.$$

($\mathcal{H}_4$) Suppose that there exist an increasing function $h \in C_{1-\gamma,\rho}[\mathbb{I},\mathbb{R}]$ and $\zeta_h > 0$ such that

$$_\rho I_{0^+}^\alpha h(t) \leqslant \zeta_h h(t), \quad \text{for any } t \in \mathbb{I}.$$

Theorem 3.1 (Existence). *Assume that* ($\mathcal{H}_1$) *holds. Then, the problem* (1.2) *has at least one solution in* $C_{1-\gamma,\rho}^{\gamma}[\mathbb{I},\mathbb{R}]$ $\subset C_{1-\gamma,\rho}^{\alpha,\beta}[\mathbb{I},\mathbb{R}]$.

Proof. We will present the proof into several steps.

We consider the well defined operator $\mathcal{K} : C_{1-\gamma,\rho}[\mathbb{I},\mathbb{R}] \to C_{1-\gamma,\rho}[\mathbb{I},\mathbb{R}]$, which is given as follows

$$(\mathcal{K}x)(t) = \frac{\Psi \rho^{1-\alpha}}{\Gamma(\alpha)} \left(t^\rho/\rho\right)^{\gamma-1} \sum_{i=1}^{n} \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} d\tau \ + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau. \tag{3.1}$$

Step 1. $\mathcal{K}$ is continuous.

Assume that x_m is a sequence such that $x_m \to x$ in $C_{1-\gamma,\rho}[\mathbb{I},\mathbb{R}]$. Then, for each $t \in \mathbb{I}$, we have

$$\left| \left((\mathcal{K}x_m)(t) - (\mathcal{K}x)(t) \right) \left(t^\rho/\rho\right)^{1-\gamma} \right|$$

$$\leqslant \frac{|\Psi| \rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{n} \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1}}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} \left| \Lambda_{x_m}(\tau) - \Lambda_x(\tau) \right| d\tau$$

$$+ \frac{\left(t^\rho/\rho\right)^{1-\gamma} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} \left| \Lambda_{x_m}(\tau) - \Lambda_x(\tau) \right| d\tau$$

$$\leqslant \frac{|\Psi| \rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{n} \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \left(\tau^\rho/\rho\right)^{\gamma-1}}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} d\tau \left\| \Lambda_{x_m}(\cdot) - \Lambda_x(\cdot) \right\|_{C_{1-\gamma,\rho}}$$

$$+ \frac{\left(t^\rho/\rho\right)^{1-\gamma} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1} \left(\tau^\rho/\rho\right)^{\gamma-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau \left\| \Lambda_{x_m}(\cdot) - \Lambda_x(\cdot) \right\|_{C_{1-\gamma,\rho}}$$

$$\leqslant \frac{B(\gamma,\alpha)}{\Gamma(\alpha)} |\Psi| \sum_{i=1}^{n} \xi_i \left(\eta_i^\rho/\rho\right)^{\alpha+\gamma-1} \left\| \Lambda_{x_m}(\cdot) - \Lambda_x(\cdot) \right\|_{C_{1-\gamma,\rho}}$$

$$+ \frac{B(\gamma,\alpha)}{\Gamma(\alpha)} \left(T^\rho/\rho\right)^\alpha \left\| \Lambda_{x_m}(\cdot) - \Lambda_x(\cdot) \right\|_{C_{1-\gamma,\rho}}$$

$$\leqslant C \left\| \Lambda_{x_m}(\cdot) - \Lambda_x(\cdot) \right\|_{C_{1-\gamma,\rho}}, \tag{3.2}$$

where $C = \frac{B(\gamma,\alpha)}{\Gamma(\alpha)} \left[|\Psi| \sum_{i=1}^{n} \xi_i \left(\eta_i^\rho/\rho\right)^{\alpha+\gamma-1} + \left(T^\rho/\rho\right)^\alpha \right]$. Since Λ_x is continuous (i.e. φ is continuous), then (3.2) implies that

$$\left\| \mathcal{K}x_m - \mathcal{K}x \right\|_{C_{1-\gamma,\rho}} \to 0 \quad \text{as} \quad n \to \infty.$$

Step 2. $\mathcal{K}$ maps bounded sets into bounded sets in $C_{1-\gamma,\rho}[\mathbb{I},\mathbb{R}]$.

Actually, it is enough to show that for $\lambda > 0$ there exists $\theta > 0$ (positive constant) such that $x \in B_\lambda = \{x \in C_{1-\gamma,\rho}[\mathbb{I},\mathbb{R}] : \|x\|_{C_{1-\gamma,\rho}} \leqslant \lambda\}$, we have $\|\mathcal{K}x\|_{C_{1-\gamma,\rho}} \leqslant \theta$.

$$\left|\left((\mathcal{K}x)(t)\right)\left(t^\rho/\rho\right)^{1-\gamma}\right| \leqslant \frac{|\Psi|\,\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^n \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1}}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} |\Lambda_x(\tau)| d\tau$$
$$+ \frac{\left(t^\rho/\rho\right)^{1-\gamma} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} |\Lambda_x(\tau)| d\tau. \tag{3.3}$$

Here, we set

$$\mathscr{A} = \frac{|\Psi|\,\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^n \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1}}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} |\Lambda_x(\tau)| d\tau, \tag{3.4}$$

$$\mathscr{B} = \frac{\left(t^\rho/\rho\right)^{1-\gamma} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} |\Lambda_x(\tau)| d\tau. \tag{3.5}$$

Now, by supposition $(\mathcal{H}_1)$, we have

$$\left|\Lambda_x(t)\right| = \left|\varphi(t, x(t), \Lambda_x(t))\right| \leqslant \phi(t) + \psi(t)\,|x(t)| + \chi(t)\,|\Lambda_x(t)|$$
$$\leqslant \phi^* + \psi^*\,|x(t)| + \chi^*\,|\Lambda_x(t)| \leqslant \frac{\phi^* + \psi^*\,|x(t)|}{1 - \chi^*}.$$

Thus, equations (3.4) and (3.5) become as follows:

$$\mathscr{A} = \frac{|\Psi|}{1-\chi^*} \sum_{i=1}^n \xi_i \left[\frac{\phi^*\left(\eta_i^\rho/\rho\right)^\alpha}{\Gamma(1+\alpha)} + B(\gamma,\alpha) \frac{\psi^*\left(\eta_i^\rho/\rho\right)^{\alpha+\gamma-1}}{\Gamma(\alpha)} \|x\|_{C_{1-\gamma,\rho}} \right], \tag{3.6}$$

$$\mathscr{B} = \frac{1}{1-\chi^*} \left[\frac{\phi^*\left(T^\rho/\rho\right)^{\alpha-\gamma+1}}{\Gamma(1+\alpha)} + B(\gamma,\alpha) \frac{\psi^*\left(T^\rho/\rho\right)^\alpha}{\Gamma(\alpha)} \|x\|_{C_{1-\gamma,\rho}} \right]. \tag{3.7}$$

By putting equations (3.6) and (3.7) into (3.3), we obtain

$$\left|\left((\mathcal{K}x)(t)\right)\left(t^\rho/\rho\right)^{1-\gamma}\right| \leqslant \frac{\phi^*}{(1-\chi^*)\Gamma(1+\alpha)} \left[|\Psi| \sum_{i=1}^n \xi_i \left(\eta_i^\rho/\rho\right)^\alpha + \left(T^\rho/\rho\right)^{\alpha-\gamma+1} \right]$$
$$+ \frac{\psi^* B(\gamma,\alpha)}{(1-\chi^*)\Gamma(\alpha)} \left[|\Psi| \sum_{i=1}^n \xi_i \left(\eta_i^\rho/\rho\right)^{\alpha+\gamma-1} + \left(T^\rho/\rho\right)^\alpha \right] \|x\|_{C_{1-\gamma,\rho}} := \theta.$$

Step: 3. $\mathcal{K}$ maps bounded sets into equicontinuous set of $C_{1-\gamma,\rho}[\mathbb{I},\mathbb{R}]$.

Assume that $t_2 \leqslant t_1$ for $t_2, t_1 \in \mathbb{I}$, B_λ is the bounded set of $C_{1-\gamma,\rho}[\mathbb{I},\mathbb{R}]$ as in the Step 2, and $x \in B_\lambda$. Then, we have

$$\left|\left((\mathcal{K}x)(t_1)\right)\left(t_1^\rho/\rho\right)^{1-\gamma} - \left((\mathcal{K}x)(t_2)\right)\left(t_2^\rho/\rho\right)^{1-\gamma}\right|$$
$$\leqslant \left| \frac{\left(t_1^\rho/\rho\right)^{1-\gamma} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} \frac{\tau^{\rho-1}\Lambda_x(\tau)}{(t_1^\rho - \tau^\rho)^{1-\alpha}} d\tau - \frac{\left(t_2^\rho/\rho\right)^{1-\gamma} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} \frac{\tau^{\rho-1}\Lambda_x(\tau)}{(t_2^\rho - \tau^\rho)^{1-\alpha}} d\tau \right|$$
$$\leqslant \left| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} \left[\frac{\left(t_1^\rho/\rho\right)^{1-\gamma}}{(t_1^\rho - \tau^\rho)^{1-\alpha}} - \frac{\left(t_2^\rho/\rho\right)^{1-\gamma}}{(t_2^\rho - \tau^\rho)^{1-\alpha}} \right] \tau^{\rho-1}\Lambda_x(\tau) d\tau \right|$$
$$+ \left| \frac{\left(t_2^\rho/\rho\right)^{1-\gamma} \rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{\tau^{\rho-1}\Lambda_x(\tau)}{(t_2^\rho - \tau^\rho)^{1-\alpha}} d\tau \right|. \tag{3.8}$$

The right-hand side of the inequality (3.8) tends to zero, as $t_1 \to t_2$.

Consequently, from Steps 1-3, together with Arzela-Ascoli theorem, we can deduce that

$$\mathcal{K} : C_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}] \to C_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}]$$

is a completely continuous.

Final Step. We will show that the set

$$G = \left\{ x \in C_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}] : x = \varepsilon(\mathcal{K}x), \ 0 \leqslant \varepsilon \leqslant 1 \right\}$$

is a bounded set.

Assume that $x \in G$, $x = \varepsilon(\mathcal{K}x)$ for some $0 \leqslant \varepsilon \leqslant 1$. Then, for each $t \in \mathbb{I}$, we have

$$x(t) = \varepsilon\left(\frac{|\Psi| \, \rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{n} \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} d\tau \ + \frac{(t^\rho/\rho)^{1-\gamma} \, \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau \right). \qquad (3.9)$$

Hence, by supposition $(\mathcal{H}_1)$, the equation (3.9) implies that

$$\left| \left(x(t) \right) \left(t^\rho/\rho \right)^{1-\gamma} \right| \leqslant \left| \left((\mathcal{K}x)(t) \right) \left(t^\rho/\rho \right)^{1-\gamma} \right|$$

$$\leqslant \frac{\phi^*}{(1-\chi^*)\Gamma(1+\alpha)} \left[|\Psi| \sum_{i=1}^{n} \xi_i \left(\eta_i^\rho/\rho \right)^\alpha + \left(T^\rho/\rho \right)^{\alpha-\gamma+1} \right]$$

$$+ \frac{\psi^* B(\gamma, \alpha)}{(1-\chi^*)\Gamma(\alpha)} \left[|\Psi| \sum_{i=1}^{n} \xi_i \left(\eta_i^\rho/\rho \right)^{\alpha+\gamma-1} + \left(T^\rho/\rho \right)^\alpha \right] \|x\|_{C_{1-\gamma,\rho}} := \theta_1.$$

Therefore, the set G is a bounded. Consequently, by Lemma 2.4 (Schaefer's fixed point theorem), we conclude that $\mathcal{K}$ has a fixed point which is a solution of problem (1.2). $\qquad \square$

Theorem 3.2 (Uniqueness). *Assume that* $(\mathcal{H}_2)$ *and* $(\mathcal{H}_3)$ *are satisfied. Then, the problem* (1.2) *has a unique solution.*

Proof. Assume that the operator $\mathcal{K} : C_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}] \to C_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}]$, defined in equation (3.1). Clearly, by Lemma 2.11, the fixed points of the operator $\mathcal{K}$ are solutions of the problem (1.2). Now, it remains to show that the solution is unique. Suppose that $x_1, x_2 \in C_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}]$ and $t \in \mathbb{I}$, then we have

$$\left| \left((\mathcal{K}x_1)(t) - (\mathcal{K}x_2)(t) \right) \left(t^\rho/\rho \right)^{1-\gamma} \right| \leqslant \frac{|\Psi| \, \rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{n} \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1}}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} \left| \Lambda_{x_1}(\tau) - \Lambda_{x_2}(\tau) \right| d\tau$$

$$+ \frac{\left(t^\rho/\rho \right)^{1-\gamma} \, \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} \left| \Lambda_{x_1}(\tau) - \Lambda_{x_2}(\tau) \right| d\tau. \qquad (3.10)$$

By supposition $(\mathcal{H}_2)$, we have

$$\left| \Lambda_{x_1}(t) - \Lambda_{x_2}(t) \right| = \left| \varphi(t, x_1(t), \Lambda_{x_1}(t)) - \varphi(t, x_2(t), \Lambda_{x_2}(t)) \right|$$

$$\leqslant \mu \left| x_1(t) - x_2(t) \right| + \nu \left| \Lambda_{x_1}(t) - \Lambda_{x_2}(t) \right| \leqslant \frac{\mu}{1-\nu} \left| x_1(t) - x_2(t) \right|. \qquad (3.11)$$

By substituting (3.11) in the inequality (3.10), we obtain

$$\left| \left((\mathcal{K}x_1)(t) - (\mathcal{K}x_2)(t) \right) \left(t^\rho/\rho \right)^{1-\gamma} \right| \leqslant \frac{B(\gamma, \alpha)}{\Gamma(\alpha)} \left(\frac{\mu \, |\Psi|}{1-\nu} \right) \sum_{i=1}^{n} \xi_i \left(\eta_i^\rho/\rho \right)^{\alpha+\gamma-1} \|x_1 - x_1\|_{C_{1-\gamma,\rho}}$$

$$+ \frac{B(\gamma, \alpha)}{\Gamma(\alpha)} \left(\frac{\mu}{1-\nu} \right) \left(T^\rho/\rho \right)^\alpha \|x_1 - x_1\|_{C_{1-\gamma,\rho}} \leqslant \Theta \|x_1 - x_1\|_{C_{1-\gamma,\rho}}.$$

Thus, $\left\|(\mathcal{K}x_1)(t) - (\mathcal{K}x_2)(t)\right\|_{C_{1-\gamma,\rho}} \leqslant \Theta \|x_1 - x_1\|_{C_{1-\gamma,\rho}}$. It follows from supposition $(\mathcal{H}_3)$, that the operator $\mathcal{K}$ is contraction. By well known Banach contraction principle, we can deduce that $\mathcal{K}$ has a unique fixed point which is a solution of the problem (1.2). $\square$

Next, we will prove some different types of Ulam stability results for the fractional implicit differential equations (1.2). Now, we shall give definitions and criteria of Ulam-Hyers stable (U.H.S.) and Ulam-Hyers-Rassias stable (U.H.R.S.) for the Hilfer-Katugampola fractional implicit differential equations (1.2).

For $\delta > 0$ and $h : \mathbb{I} \to [0, \infty)$ is a continuous function, we theorize the following inequalities

$$\left| {}_\rho D_{0^+}^{\alpha,\beta} y(t) - \varphi(t, y(t), \, {}_\rho D_{0^+}^{\alpha,\beta} y(t)) \right| \leqslant \delta, \quad t \in \mathbb{I} \tag{3.12}$$

$$\left| {}_\rho D_{0^+}^{\alpha,\beta} y(t) - \varphi(t, y(t), \, {}_\rho D_{0^+}^{\alpha,\beta} y(t)) \right| \leqslant \delta \, h(t), \quad t \in \mathbb{I} \tag{3.13}$$

$$\left| {}_\rho D_{0^+}^{\alpha,\beta} y(t) - \varphi(t, y(t), \, {}_\rho D_{0^+}^{\alpha,\beta} y(t)) \right| \leqslant h(t), \quad t \in \mathbb{I}. \tag{3.14}$$

Definition 3.3. The problem (1.2) is U.H.S. if there exists the real number $\lambda_\varphi > 0$ such that for all $\delta > 0$ and for all solution $y \in C_{1-\gamma,\rho}^{\gamma}[\mathbb{I}, \mathbb{R}]$ of the inequality (3.12) there exists the solution $x \in C_{1-\gamma,\rho}^{\gamma}[\mathbb{I}, \mathbb{R}]$ of the problem (1.2) satisfying

$$\left| y(t) - x(t) \right| \leqslant \lambda_\varphi \delta, \quad t \in \mathbb{I}.$$

Definition 3.4. The problem (1.2) is generalized U.H.S. if there exists $z_\varphi \in C([0, \infty), [0, \infty))$, $z_\varphi(0) = 0$ such that for all solution $y \in C_{1-\gamma,\rho}^{\gamma}[\mathbb{I}, \mathbb{R}]$ of the inequality (3.12) there exists the solution $x \in C_{1-\gamma,\rho}^{\gamma}[\mathbb{I}, \mathbb{R}]$ of the problem (1.2) satisfying

$$\left| y(t) - x(t) \right| \leqslant z_\varphi \delta, \quad t \in \mathbb{I}.$$

Definition 3.5. The problem (1.2) is U.H.R.S. with respect to $h \in C_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}]$ if there exists the real number $\lambda_\varphi > 0$ such that for all $\delta > 0$ and for all solution $y \in C_{1-\gamma,\rho}^{\gamma}[\mathbb{I}, \mathbb{R}]$ of the inequality (3.13) there exists the solution $x \in C_{1-\gamma,\rho}^{\gamma}[\mathbb{I}, \mathbb{R}]$ of the problem (1.2) satisfying

$$\left| y(t) - x(t) \right| \leqslant \lambda_\varphi \delta \, h(t), \quad t \in \mathbb{I}.$$

Definition 3.6. The problem (1.2) is generalized U.H.R.S. with respect to $h \in C_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}]$ if there exists the real number $\lambda_{\varphi,h} > 0$ such that for all solution $y \in C_{1-\gamma,\rho}^{\gamma}[\mathbb{I}, \mathbb{R}]$ of the inequality (3.14) there exists the solution $x \in C_{1-\gamma,\rho}^{\gamma}[\mathbb{I}, \mathbb{R}]$ of the problem (1.2) satisfying

$$\left| y(t) - x(t) \right| \leqslant \lambda_{\varphi,h} \, h(t), \quad t \in \mathbb{I}.$$

Remark 3.7. From the above definitions it is clear that

1. Definition 3.3⇒Definition 3.4;
2. Definition 3.5⇒Definition 3.6;
3. Definition 3.5 for $h(\cdot) = 1 \Rightarrow$ Definition 3.3.

Remark 3.8. The function $y \in C_{1-\gamma,\rho}^{\gamma}[\mathbb{I}, \mathbb{R}]$ is the solution of the inequality (3.12), if and only if there exists the function $z \in C_{1-\gamma,\rho}^{\gamma}[\mathbb{I}, \mathbb{R}]$ such that

(a) $\left| z(t) \right| \leqslant \delta, \ t \in \mathbb{I}$;
(b) ${}_\rho D_{0^+}^{\alpha,\beta} y(t) = \varphi(t, y(t), \, {}_\rho D_{0^+}^{\alpha,\beta} y(t)) + z(t), \quad t \in \mathbb{I}.$

We have similar remarks for the inequalities (3.13) and (3.14).

Lemma 3.9. *Assume that $\rho > 0$, $0 < \alpha < 1$, and $\beta \in [0,1]$. If the function $y \in C^{\gamma}_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}]$ is the solution of the inequality* (3.12), *then y is the solution of the following integral inequality*

$$\left| y(t) - \Xi_y - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1} \Lambda_y(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau \right| \leqslant \left[|\Psi|(n\xi)(T^\rho/\rho)^{\alpha+\gamma-1} + (T^\rho/\rho)^\alpha \right] \frac{\delta}{\Gamma(1+\alpha)}, \qquad (3.15)$$

where,

$$\Xi_y = \frac{\Psi \, \rho^{1-\alpha}}{\Gamma(\alpha)} (t^\rho/\rho)^{\gamma-1} \sum_{i=1}^n \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_y(\tau)}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} d\tau, \qquad \xi = \max_{i=1,2,\dots,n} \{\xi_i\}.$$

Proof. In view of Remark 3.8, we have

$$_\rho D_{0+}^{\alpha,\beta} y(t) = \varphi(t, y(t), \, _\rho D_{0+}^{\alpha,\beta} y(t)) + z(t) = \Lambda_y(t) + z(t).$$

Hence,

$$\begin{aligned}
y(t) &= \frac{\Psi \, \rho^{1-\alpha}}{\Gamma(\alpha)} (t^\rho/\rho)^{\gamma-1} \sum_{i=1}^n \xi_i \left[\int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_y(\tau)}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} d\tau + \int_0^{\eta_i} \frac{\tau^{\rho-1} z(\tau)}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} d\tau \right] \\
&\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left[\int_0^t \frac{\tau^{\rho-1} \Lambda_y(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau + \int_0^t \frac{\tau^{\rho-1} z(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau \right].
\end{aligned} \qquad (3.16)$$

It follows from (3.15) and (3.16) that

$$\begin{aligned}
& \left| y(t) - \Xi_y - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1} \Lambda_y(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau \right| \\
&= \left| \frac{\Psi \, \rho^{1-\alpha}}{\Gamma(\alpha)} (t^\rho/\rho)^{\gamma-1} \sum_{i=1}^n \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} z(\tau)}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} d\tau + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1} z(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau \right| \\
&\leqslant \frac{|\Psi| \, \rho^{1-\alpha}}{\Gamma(\alpha)} (t^\rho/\rho)^{\gamma-1} \sum_{i=1}^n \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1}}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} |z(\tau)| d\tau + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} |z(\tau)| d\tau \\
&\leqslant \frac{|\Psi| \, \rho^{1-\alpha}}{\Gamma(\alpha)} (t^\rho/\rho)^{\gamma-1} \sum_{i=1}^n \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1}\delta}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} d\tau + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1}\delta}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau \\
&\leqslant \left[|\Psi|(n\xi)(T^\rho/\rho)^{\alpha+\gamma-1} + (T^\rho/\rho)^\alpha \right] \frac{\delta}{\Gamma(1+\alpha)}.
\end{aligned}$$

$\square$

Lemma 3.10. *Assume that $\rho > 0$, $0 < \alpha < 1$, and $\beta \in [0,1]$. If the function $y \in C^{\gamma}_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}]$ is the solution of the inequality* (3.13), *then y is the solution of the following integral inequality*

$$\left| y(t) - \Xi_y - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1} \Lambda_y(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau \right| \leqslant \left[|\Psi|(n\xi)(t^\rho/\rho)^{\gamma-1} + 1 \right] \delta \zeta_h h(t), \qquad (3.17)$$

where,

$$\Xi_y = \frac{\Psi \, \rho^{1-\alpha}}{\Gamma(\alpha)} (t^\rho/\rho)^{\gamma-1} \sum_{i=1}^n \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_y(\tau)}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} d\tau, \qquad \xi = \max_{i=1,2,\dots,n} \{\xi_i\},$$

and ζ_h is a positive constant.

Proof. Proof of this Lemma is directly follows from Remark 3.8 and Lemma 3.9 with the help of supposition $(\mathcal{H}_4)$. $\square$

Notice that, we can find a similar remark for the inequalities (3.14).
Now, We will state a generalization of Gronwall's lemma for singular kernels which we are using to discuss our stability results for the problem (1.2).

Lemma 3.11 ([34]). *Assume that* $f : [0, T] \to [0, \infty)$ *is a real function and* $g(\cdot)$ *is a non-negative locally integrable function on* $[0, T]$. *And let there exists* $\rho > 0$, $\omega > 0$, *and* $0 < \alpha < 1$, *such that*

$$f(t) \leqslant g(t) + \omega \int_0^t \tau^{\rho-1} \frac{\rho^\alpha}{(t^\rho - \tau^\rho)^\alpha} f(\tau) d\tau.$$

Then, there exists a constant $C^* = C^*(\alpha)$ *such that for* $t \in [0, T]$, *we have*

$$f(t) \leqslant g(t) + C^* \omega \int_0^t \tau^{\rho-1} \frac{\rho^\alpha}{(t^\rho - \tau^\rho)^\alpha} g(\tau) d\tau.$$

Theorem 3.12. *Assume that the hypotheses* $(\mathcal{H}_2)$ *and* $(\mathcal{H}_3)$ *are satisfied. Then, the problem* (1.2) *is U.H.S..*

Proof. Let $\delta > 0$ and let for any $t \in \mathbb{I}$ the function $y \in C^\gamma_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}]$ satisfies the inequality (3.12). In the light of Theorem 3.2, $x \in C^\gamma_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}]$ is a unique solution of implicit fractional differential equation satisfying

$$\begin{cases} {}_\rho D^{\alpha,\beta}_{0+} x(t) = \varphi(t, x(t), {}_\rho D^{\alpha,\beta}_{0+} x(t)), & t \in \mathbb{I} := [0, T], \ 0 < \alpha < 1, 0 \leqslant \beta \leqslant 1, \\ {}_\rho I^{1-\gamma}_{0+} x(0) = {}_\rho I^{1-\gamma}_{0+} y(0) = \sum_{i=1}^n \xi_i x(\eta_i), & \gamma = \alpha + \beta(1-\alpha). \end{cases} \tag{3.18}$$

By using Lemma 2.11, we get

$$x(t) = \Xi_x + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau,$$

where

$$\Xi_x = \frac{\Psi \rho^{1-\alpha}}{\Gamma(\alpha)} (t^\rho/\rho)^{\gamma-1} \sum_{i=1}^n \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} d\tau.$$

Now, if $x(\eta_i) = y(\eta_i)$ and ${}_\rho I^{1-\gamma}_{0+} x(0) = {}_\rho I^{1-\gamma}_{0+} y(0)$, then $\Xi_x = \Xi_y$. Clearly,

$$|\Xi_x - \Xi_y| \leqslant \frac{|\Psi| \rho^{1-\alpha}}{\Gamma(\alpha)} (t^\rho/\rho)^{\gamma-1} \sum_{i=1}^n \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1}}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} |\Lambda_x(\tau) - \Lambda_y(\tau)| d\tau$$

$$\leqslant \frac{|\Psi| \rho^{1-\alpha}}{\Gamma(\alpha)} (t^\rho/\rho)^{\gamma-1} \sum_{i=1}^n \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1}}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}} \left(\frac{\mu}{1-\nu}\right) |x(\tau) - y(\tau)| d\tau$$

$$\leqslant \left(\frac{\mu}{1-\nu}\right) \frac{|\Psi| \rho^{1-\alpha}}{\Gamma(\alpha)} (t^\rho/\rho)^{\gamma-1} \sum_{i=1}^n \xi_i \, {}_\rho I^\alpha_{0+} |x(\eta_i) - y(\eta_i)| = 0.$$

Thus, we have

$$x(t) = \Xi_y + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1} \Lambda_x(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau.$$

By applying Lemma 3.9 and integration of inequality (3.12) for any $t \in \mathbb{I}$, we have

$$|y(t) - x(t)| \leqslant \left| y(t) - \Xi_y - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1} \Lambda_y(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau \right| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} |\Lambda_y(\tau) - \Lambda_x(\tau)| d\tau$$

$$\leqslant \left| y(t) - \Xi_y - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1} \Lambda_y(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau \right| \tag{3.19}$$

$$+ \left(\frac{\mu}{1-\nu}\right) \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} |y(\tau) - x(\tau)| d\tau.$$

By the inequality (3.15), we can write the inequality (3.19) as follows

$$|y(t) - x(t)| \leqslant \left[|\Psi|(n\xi) (T^\rho/\rho)^{\alpha+\gamma-1} + (T^\rho/\rho)^\alpha \right] \frac{\delta}{\Gamma(1+\alpha)}$$

$$+ \left(\frac{\mu}{1-\nu}\right) \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} |y(\tau) - x(\tau)| d\tau.$$

Now, applying Lemma 3.11, we get

$$\left|y(t) - x(t)\right| \leqslant \left[|\Psi|(n\xi)\left(T^\rho/\rho\right)^{\alpha+\gamma-1} + \left(T^\rho/\rho\right)^\alpha\right]\left(1 + \frac{C_1\,\mu}{(1-\nu)\Gamma(\alpha)}\left(T^\rho/\rho\right)^\alpha\right)\frac{\delta}{\Gamma(1+\alpha)} := \lambda_\varphi\delta,$$

where $C_1 = C_1(\alpha) = \frac{1}{\alpha}$ be a constant. Furthermore, if we put $z(\delta) = \lambda_\varphi\delta$; $z(0) = 0$, then the problem (1.2) is generalized U.H.S.. $\qquad\qquad\qquad\square$

Theorem 3.13. *Suppose that the hypotheses* $(\mathcal{H}_2)$, $(\mathcal{H}_3)$, *and* $(\mathcal{H}_4)$ *are satisfied. Then, the problem* (1.2) *is U.H.R.S..*

Proof. Let $\delta > 0$ and let for any $t \in \mathbb{I}$ the function $y \in C^\gamma_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}]$ satisfies the inequality (3.13). In the light of Theorem 3.2, $x \in C^\gamma_{1-\gamma,\rho}[\mathbb{I}, \mathbb{R}]$ is a unique solution of implicit fractional differential equation satisfying the problem (3.18).

By using same arguments in the above theorem, we obtain

$$x(t) = \Xi_y + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{\tau^{\rho-1}\Lambda_x(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}}d\tau,$$

where

$$\Xi_y = \frac{\Psi\,\rho^{1-\alpha}}{\Gamma(\alpha)}\left(t^\rho/\rho\right)^{\gamma-1}\sum_{i=1}^n \xi_i \int_0^{\eta_i} \frac{\tau^{\rho-1}\Lambda_y(\tau)}{(\eta_i^\rho - \tau^\rho)^{1-\alpha}}d\tau.$$

Now, applying Lemma 3.10 and integration of inequality (3.13) for any $t \in \mathbb{I}$, we have

$$\begin{aligned}
\left|y(t) - x(t)\right| &\leqslant \left|y(t) - \Xi_y - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{\tau^{\rho-1}\Lambda_y(\tau)}{(t^\rho - \tau^\rho)^{1-\alpha}}d\tau\right| \\
&\quad + \left(\frac{\mu}{1-\nu}\right)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}}\left|y(\tau) - x(\tau)\right|d\tau.
\end{aligned} \qquad (3.20)$$

By the inequality (3.17), we can write the inequality (3.20) as follows

$$\left|y(t) - x(t)\right| \leqslant \left[|\Psi|(n\xi)\left(t^\rho/\rho\right)^{\gamma-1} + 1\right]\delta\zeta_h h(t) + \left(\frac{\mu}{1-\nu}\right)\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}}\left|y(\tau) - x(\tau)\right|d\tau.$$

Finally, applying Lemma 3.11, we get

$$\left|y(t) - x(t)\right| \leqslant \left(\left[|\Psi|(n\xi)\left(t^\rho/\rho\right)^{\gamma-1} + 1\right]\left[1 + \frac{C_2\,\mu\,\zeta_h}{(1-\nu)}\right]\zeta_h\right)\delta\,h(t) := \lambda_\varphi\delta\,h(t),$$

where $C_2 = C_2(\alpha) = \frac{1}{\Gamma(1+\alpha)}$ be a constant. $\qquad\qquad\qquad\square$

4. Examples

Example 4.1. Consider the following Hilfer-Katugampola fractional implicit differential equation

$$\begin{cases} {}_{\frac{1}{2}}D_{1^+}^{\frac{1}{3},\frac{1}{2}}x(t) = \varphi(t, x(t), {}_{\frac{1}{2}}D_{1^+}^{\frac{1}{3},\frac{1}{2}}x(t)), & t \in \mathbb{I} := (1, 2], \\ {}_{\frac{1}{2}}I_{1^+}^{1-\gamma}x(1) = \frac{2}{3}x(\frac{4}{5}), & \gamma = \alpha + \beta(1-\alpha). \end{cases} \qquad (4.1)$$

Here, $\rho = \frac{1}{2} > 0$, $\alpha = \frac{1}{3}, \beta = \frac{1}{2}, \gamma = \frac{2}{3}, \xi = \frac{2}{3}, \eta = \frac{4}{5}$, and let

$$\varphi(t, x(t), {}_{\frac{1}{2}}D_{1^+}^{\frac{1}{3},\frac{1}{2}}x(t)) = \left(17e^{2t-1}\left[1 + |x(t)| + |{}_{\frac{1}{2}}D_{1^+}^{\frac{1}{3},\frac{1}{2}}x(t)|\right]\right)^{-1}, \qquad t \in \mathbb{I}.$$

Clearly,

$$\varphi(t, p, q) = \left(17e^{2t-1}\left[1 + |p| + |q|\right]\right)^{-1}, \qquad t \in \mathbb{I}, \quad p, q \in \mathbb{R}^{+}$$

is a continuous function and for $p, q, \tilde{p}, \tilde{q} \in \mathbb{R}^{+}$, $t \in \mathbb{I}$, we have

$$\left|\varphi(t, p, q) - \varphi(t, \tilde{p}, \tilde{q})\right| \leqslant \frac{1}{17e}\left[|p - \tilde{p}| + |q - \tilde{q}|\right].$$

Thus, the suppositions $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ hold with $\mu = \nu = \frac{1}{17e}$ and we can see that $|\Psi| \simeq 1.242338$ and $\Theta \simeq 0.067165 < 1$. From Theorem 3.2, it follows that the problem (4.1) has unique solution for $t \in \mathbb{I}$. Here, if we let $h(t) = t$, then we have

$$_{\frac{1}{2}}I_{1+}^{\frac{1}{3}}h(t) = \frac{(1/2)^{2/3}}{\Gamma(1/3)}\int_1^t \frac{\tau^{-1/2}(\tau)}{(t^{1/2} - \tau^{1/2})^{2/3}}d\tau \leqslant \frac{(1/2)^{2/3}\,t}{\Gamma(1/3)}\int_1^t \frac{\tau^{-1/2}}{(t^{1/2} - \tau^{1/2})^{2/3}}d\tau \leqslant \frac{6}{\Gamma(1/3)}h(t),$$

so supposition $(\mathcal{H}_4)$ holds with $\zeta_h \simeq 2.239692$. Hence, from Theorem 3.13, it follows that the problem (4.1) is U.H.R.S..

Example 4.2. Consider the following Hilfer-Katugampola fractional implicit differential equation

$$\begin{cases} {}_{\frac{1}{2}}D_{1+}^{\frac{1}{3},\frac{1}{2}}x(t) = \varphi(t, x(t), {}_{\frac{1}{2}}D_{1+}^{\frac{1}{3},\frac{1}{2}}x(t)), & t \in \mathbb{I} := (1, 2], \\ {}_{\frac{1}{2}}I_{1+}^{1-\gamma}x(1) = \frac{2}{3}x(\frac{4}{5}) + 3x(\frac{3}{4}), & \gamma = \alpha + \beta(1 - \alpha). \end{cases} \tag{4.2}$$

In addition of previous example, we have $\xi_1 = \frac{2}{3}, \eta_1 = \frac{4}{5}, \xi_2 = 3, \eta_2 = \frac{3}{4}$, and let

$$\varphi(t, x(t), {}_{\frac{1}{2}}D_{1+}^{\frac{1}{3},\frac{1}{2}}x(t)) = \frac{1}{107e^{3t+2}}\left(\frac{|x(t)|}{1 + |x(t)|} + \frac{\left|{}_{\frac{1}{2}}D_{1+}^{\frac{1}{3},\frac{1}{2}}x(t)\right|}{1 + \left|{}_{\frac{1}{2}}D_{1+}^{\frac{1}{3},\frac{1}{2}}x(t)\right|}\right), \qquad t \in \mathbb{I}.$$

Clearly, for any $p, q \in \mathbb{R}^{+}$, we have

$$\varphi(t, p, q) = \frac{1}{107e^{3t+2}}\left(\frac{p}{1 + p} + \frac{q}{1 + q}\right)$$

is a continuous function satisfies the hypotheses of Theorem 3.1 and for $p, q, \tilde{p}, \tilde{q} \in \mathbb{R}^{+}$ and $t \in \mathbb{I}$, we have

$$\left|\varphi(t, p, q) - \varphi(t, \tilde{p}, \tilde{q})\right| \leqslant \frac{1}{107e^5}\left[|p - \tilde{p}| + |q - \tilde{q}|\right].$$

Thus, the suppositions $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ hold with $\mu = \nu = \frac{1}{107e^5}$ and we can see that $|\Psi| \simeq 0.5906272$ and $\Theta \simeq 0.0003054 < 1$. Hence, from Theorem 3.2, it follows that the problem (4.2) has unique solution for $t \in \mathbb{I}$. Furthermore, it implies from Theorem 3.12, that the problem (4.2) is U.H.S..

Acknowledgment

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