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On the solution linear and nonlinear fractional beam equation

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Wanchak Satsanit

Department of Mathematics, Faculty of Science, Maejo University, Chiang Mai, 50290, Thailand.

Abstract

In this paper, we combined the fractional Laplace transform and Homotopy perturbation method (LHPM) and applied it to find an exact and approximation solution of different types of fractional beam equation. The fractional derivatives are considered in sense of Caputo. It was found that this method obtained the rapid convergence of the series solution. Four examples are illustrated to show the efficiency of this method.

Keywords: Beam equation, homotopy perturbation method, fractional derivatives.

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1. Introduction

It is well known the beam equation as follow

$$\frac{\partial^2}{\partial t^2} u(x,t) + \frac{\partial^4}{\partial x^4} u(x,t) = 0, \tag{1.1}$$

is widely used on applied mathematics and engineering. There are various types of integral transform method, that are used to solve (1.1). These include, the Laplace transform method [8], the Fourier transform [4], the Sumudu transform [3], and the Mellin transform. Among all of the integral transform methods, the Laplace transform method is the most popular. It has become a tradition that every new integral transform links with the Laplace transform.

Fractional differential equations have been of great interest and attracted many researchers in recent years. The exact solution for the majority of fractional differential cannot be found easily. The homotopy perturbation method (HPM) proposed by HE [5–7], for solving differential equation and integral equation. In 2002, the Adomian decomposition method (ADM) was suggested to solve fractional differential equation [2], but many researchers found it very difficulty to calculate the Adomian polynomials. In 2007, Monami and Odibat [9–11] applied homotopy perturbation method (HPM) combined Laplace transform to fractional differential equation and showed that the method is an alternative analytical method for fractional differential equation. The advantage of this method is its capability for obtaining exact solutions

for linear and nonlinear fractional partial differential equation. Furthermore, singe and Kumar [12], studied HPM and Laplace transform for solving fractional heat and wave-like equation. Xu and Cang [13], solved the fractional heat like and Wave-like equation with variable coefficients using homotopy analysis method.

The purpose of this work is studied the solution of a linear fractional beam equation as the following form

$$D_t^\alpha u(x,t) + \frac{\partial^4}{\partial x^4} u(x,t) = 0,$$

and a nonlinear fractional beam equation

$$D_t^\alpha u(x,t) + \frac{\partial^4}{\partial x^4} u(x,t) + 2uu_x - u_x^2 = 0,$$

where $1 < \alpha \leqslant 2, 0 < x \leqslant a$ and $t > 0$. By homotopy perturbation method and Laplace transform of fractional of derivatives in sense of Caputo, we obtain a very rapid convergence of the series solutions. Four illustrative examples are given to demonstrate the efficiency of the method. Before going to that point, the following definition and some important concepts are needed.

2. Preliminaries

In this section, we give some basic definition and proportion of fractional calculus which shall be used in this paper.

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f(t) = C_\mu, \mu \geqslant -1$ is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad J^0 f(t) = f(t).$$

For the Riemann-Liouville fractional integral we have:

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\alpha+\gamma}.$$

Definition 2.2. The fractional derivative of $f(t)$ the Caputo is defined by

$$D^\alpha t^\gamma = J^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau$$

for $n-1 < \alpha \leqslant n, n \in N, x > 0$.

Definition 2.3. The Laplace transform of a functional $f(t), t > 0$ is defined by

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t) dt,$$

where $f(t)$ is piece-wise continuous and of the exponential order (i.e., $|e^{-at} f(t)| < M$) for some constants a, M and complex parameter s.

Definition 2.4. The Laplace transform of the Caputo derivative is given by Caputo; see also Killbas et al. in the form

$$\mathcal{L}[D^\alpha f(t)] = s^\alpha \mathcal{L}[f(t)] - \sum_{r=0}^{n-1} s^{\alpha-r-1} f^{(r)}(0+), (n-1 < \alpha \leqslant n).$$

Definition 2.5. The Mittag-Leffler is defined by

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+1)}, \ (\alpha > 0, z \epsilon C), \quad \text{and} \quad E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)}, \ (\alpha,\beta > 0, z \epsilon C).$$

3. Homotopy perturbation method

Consider the following general nonlinear differential equation:

$$Lu + Nu = g(x, t), \tag{3.1}$$

with initial conditions

$$u(x, 0) = k_1 \ , \ u_t(x, 0) = k_1,$$

where u is a function of x and t and c_1, c_2 are constants or functions of x, and L and N are the linear and nonlinear operators, respectively.

According to HPM, we construct a homotopy which satisfies the following relation

$$H(u, p) = (1 - p)\big[Lu - Lu_0\big] + p\big[Lu + Nu - g(x, t)\big] = 0, \tag{3.2}$$

where $p \in [0, 1]$ is an embedding parameter and u_0 is an arbitrary initial approximation satisfied the given initial conditions. When we put $p = 0$ and $p = 1$ in Eq. (3.2), we obtain

$$H(u, 0) = Lu - Lu_0 = 0 \quad \text{and} \quad H(u, 1) = Lu + Nu - g(x, t) = 0.$$

In HPM, the solution of Eq. (3.2) is expressed as

$$u(x, t) = u_0(x, t) + pu_1(x, t) + p^2 u_2(x, t) + \cdots.$$

Hence, the approximate solution of Eq. (3.1) can be expressed as a series of the powers of p, i.e.,

$$u(x, t) = \lim_{p \to 1} \big(u_0(x, t) + pu_1(x, t) + p^2 u_2(x, t) + \cdots\big) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \cdots.$$

4. Laplace homotopy perturbation method (LHPM)

To illustrate the idea of this method, we consider a general fractional nonlinear non-homogeneous partial differential equation with the initial conditions of the form:

$$D_t^\alpha u(x, t) + Ru(x, t) + Nu(x, t) = g(x, t), \tag{4.1}$$
$$u(x, 0) = h(x), u_t(x, 0) = f(x),$$

where $D_t^\alpha u(x, t)$ is the Caputo fractional derivative of the function $u(x, t)$, R is the linear differential operator, N represents the general nonlinear differential operator and $g(x, t)$ is the source term. Taking the Laplace transform on both sides of (4.1), we get

$$\mathcal{L}[D_t^\alpha u(x, t)] + \mathcal{L}[Ru(x, t)] + \mathcal{L}[Nu(x, t)] = \mathcal{L}[g(x, t)].$$

Using the property of the Laplace transform, we have

$$\mathcal{L}[u(x, t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} + \frac{1}{s^\alpha}\mathcal{L}[gu(x, t)] - \frac{1}{s^\alpha}\mathcal{L}[Ru(x, t)] - \frac{1}{s^\alpha}\mathcal{L}[Nu(x, t)]. \tag{4.2}$$

Taking inverse Laplace inverse on both sides of (4.2), we obtain

$$u(x, t) = G(x, t) - \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}[Ru(x, t) + Nu(x, t)]\right], \tag{4.3}$$

where $G(x,t)$ represents the term arising form source term and the prescribed initial condition. Now we apply the HPM

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{4.4}$$

and the nonlinear term can be decomposed as

$$N(u) = \sum_{n=0}^{\infty} p^n H_n(u) \tag{4.5}$$

for some He's polynomials $H_n(u)$ are given by

$$H_n(u_0, u_1, \ldots, u_n) = \frac{1}{n!} \frac{d^n}{dp^n} \left[N \left(\sum_{i=0}^{\infty} p^i u_i(x,t) \right) \right]_{p=0} ; n = 0, 1, 2, \ldots.$$

The first few components He's are given by

$$H_0 = N(u_0),$$
$$H_1 = u_1 N'(u_0),$$
$$H_2 = u_2 N'(u_0) + \frac{1}{2} u_1^2 N''(u_0),$$
$$H_3 = u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{u_1^3}{3!} N^{(3)}(u_0),$$
$$\vdots$$

Substituting (4.4) and (4.5) into (4.3), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - p \left(\mathcal{L}^{-1} \left[\frac{1}{s^\alpha} \mathcal{L}[R \sum_{n=0}^{\infty} p^n u_n(x,t) + N \sum_{n=0}^{\infty} p^n u_n(x,t)] \right] \right).$$

Comparing the coefficients of like powers of p, the following approximations are obtained.

$$p^0 : u_0(x,t) = G(x,t),$$
$$p^1 : u_1(x,t) = -\mathcal{L}^{-1} \left[\frac{1}{s^\alpha} \mathcal{L} \left[Ru_0(x,t) - H_0(u) \right] \right],$$
$$p^2 : u_2(x,t) = -\mathcal{L}^{-1} \left[\frac{1}{s^\alpha} \mathcal{L} \left[Ru_1(x,t) - H_1(u) \right] \right],$$
$$p^3 : u_3(x,t) = -\mathcal{L}^{-1} \left[\frac{1}{s^\alpha} \mathcal{L} \left[Ru_2(x,t) - H_2(u) \right] \right],$$
$$\vdots$$

Proceeding in this same manner, the rest of the components $u_n(x,t)$ can be completely obtained and the series solution is thus entirely determined.

Finally, we approximate the analytical solution $u(x,t)$ by truncated series

$$u(x,t) = \lim_{N \to \infty} \sum_{n=0}^{N} u_n(x,t).$$

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by and Abbaoui and Cherruault [1].

5. Examples

In this section, we applied LHPM for solving fractional linear and nonlinear beam equation.

Example 5.1. Consider the following one-dimensional linear fractional beam equation

$$D_t^\alpha u + u_{xxxx} = 0, \quad 0 \leqslant x \leqslant 1, 1 < \alpha \leqslant 2, t > 0, \tag{5.1}$$

subject to the boundary condition

$$u(0,t) = 0, u(1,t) = \sinh(1)\sin t,$$

and the initial condition

$$u(x,0) = 0,, u_t(x,0) = \sinh x.$$

Applying the Laplace transform an both sides of (5.1) subject to the initial condition, we obtain

$$\mathcal{L}[u(x,t)] = \frac{\sinh x}{s^2} - \frac{1}{s^\alpha}\mathcal{L}[u_{xxxx}(x,t)]. \tag{5.2}$$

Taking inverse of Laplace transform an both sides of (5.2), we obtain

$$u(x,t) = t\sinh x - \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}[u_{xxxx}(x,t)]\right].$$

Now, we are applied the homotopy perturbation method, we obtain

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = t\sinh x - p\left(\mathcal{L}^{-1}\left(\frac{1}{s^\alpha}\mathcal{L}\sum_{n=0}^{\infty} p^n (u_n(x,t))_{xxxx}\right)\right).$$

Comparing the coefficients of like power of p, we have

$$p^0: \quad u_0(x,t) = t\sinh x,$$

$$p^1: \quad u_1(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\{u_0(x,t)\}_{xxxx}\right]$$

$$= -\mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\{t\sinh x\}\right] = -\sinh x\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha+1}}\right] = -\sinh x\frac{t^{\alpha+1}}{\Gamma(\alpha+2)},$$

$$p^2: \quad u_2(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\{u_1(x,t)\}_{xxxx}\right] = \sinh x\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)},$$

$$\vdots$$

$$p^n: \quad u_n(x,t) = (-1)^n \sinh x\mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\{u_{n-1}(x,t)\}_{xxxx}\right] = (-1)^n \sinh x\frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)},$$

$$\vdots$$

Using the above iteration the solution of $u(x,t)$ is given by

$$u(x,t) = \lim_{p\to 1}\sum_{n=0}^{\infty} p^n u_n(x,t),$$

$$u(x,t) = \sinh x\left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \cdots + (-1)^n\frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)} + \cdots\right) = \sinh x t E_{\alpha,2}(-t^\alpha).$$

If we put $\alpha = 2$ in (5.1), the equation is reduced to one dimensional fractional beam equation as follows:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0, \tag{5.3}$$

and the solution of (5.3) is given by

$$u(x, t) = \sinh x \sin t.$$

Example 5.2. Consider the following two dimensional fractional beam-like equation

$$D_t^\alpha u + \frac{1}{24} \sin x u_{xxxx} + \frac{1}{24} \sin y u_{yyyy} = 0, \tag{5.4}$$

where $0 \leqslant x, y \leqslant \frac{\pi}{2}$, $1 < \alpha \leqslant 2$, $t > 0$ subject to the boundary condition

$$u(0, y, t) = \sin y \sin t, \quad u(\frac{\pi}{2}, y, t) = \cos t + \sin y \sin t, \quad u(x, 0, t) = \sin x \cos t, \quad u(x, \frac{\pi}{2}, t) = \sin x \cos t + \sin t,$$

and the initial condition

$$u(x, y, 0) = \sin x, \quad u_t(x, y, 0) = \sin y.$$

In a similar method of Example 5.1, we obtain

$$\sum_{n=0}^{\infty} p^n u_n(x, y, t) = \sin x + t \sin y - p \left(\frac{1}{24} \sin x \mathcal{L}^{-1} \left[\frac{1}{s^\alpha} \mathcal{L} \left\{ \sum_{n=0}^{\infty} p^n (u_n(x, y, t))_{xxxx} \right\} \right] \right)$$
$$- p \left(\frac{1}{24} \sin y \mathcal{L}^{-1} \left[\frac{1}{s^\alpha} \mathcal{L} \left\{ \sum_{n=0}^{\infty} p^n (u_n(x, y, t))_{yyyy} \right\} \right] \right).$$

Comparing the coefficients of the power p, we have

$$p^0: \quad u_0(x, y, t) = \sin x + t \sin y, \tag{5.5}$$

$$p^1: \quad u_1(x, y, t) = \mathcal{L}^{-1} \left(\frac{1}{s^\alpha} \mathcal{L} \left((u_0)_{xxxx} + (u_0) \right)_{yyyy} \right) = -\sin x \frac{t^\alpha}{\Gamma(\alpha+1)} - \sin y \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}, \tag{5.6}$$

$$p^2: \quad u_2(x, y, t) = \sin x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \sin y \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}, \tag{5.7}$$

$$\vdots$$

$$p^n: \quad u_n(x, y, t) = (-1)^n \sin x \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + (-1)^n \sin y \frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)}, \tag{5.8}$$

$$\vdots$$

By (5.5)-(5.8), the solution of (5.4) is given by

$$u(x, y, t) = \sin x \left(1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \cdots - (-1)^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots \right)$$
$$+ \sin y \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \cdots - (-1)^n \frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)} + \cdots \right)$$
$$= \sin x E_\alpha(-t^\alpha) + \sin y E_{\alpha,2}(-t^\alpha).$$

If we put $\alpha = 2$ in (5.4), we obtain two dimensional fractional beam equation as follows:

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{24} \sin x \frac{\partial^4 u}{\partial x^4} + \frac{1}{24} \sin y \frac{\partial^4 u}{\partial y^4} = 0, \tag{5.9}$$

the solution of (5.9) is given by

$$u(x, y, t) = \sin x \cos t + \sin y \sin t.$$

Example 5.3. Consider the following three dimensional in-homogeneous fractional beam-like equation

$$D_t^\alpha u + \frac{1}{360}\left(x^4 u_{xxxx} + y^4 u_{yyyy} + z^4 u_{zzzz}\right) = x^6 + y^6 + z^6, \tag{5.10}$$

where $0 \leqslant x, y, z \leqslant 1$, $1 < \alpha \leqslant 2$, $t > 0$, subject to the boundary condition

$$
\begin{aligned}
u(0, y, z, t) &= (y^6 + z^6)(1 - \cos t), & u(1, y, z, t) &= (1 + y^6 + z^6) + y^6 z^6 \sin t, \\
u(x, 0, z, t) &= (x^6 + z^6)(1 - \cos t), & u(x, 1, z, t) &= (1 + y^6 + z^6) + x^6 z^6 \sin t, \\
u(x, y, 0, t) &= (x^6 + y^6)(1 - \cos t)0, & u(x, y, 1, t) &= (1 + y^6 + z^6) + x^6 y^6 \sin t,
\end{aligned}
$$

and the initial condition

$$u(x, y, z, 0) = 0, \quad u_t(x, y, z, 0) = x^6 y^6 z^6.$$

In a similar way as above, we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty} p^n u_n(x, y, z, t) =&(x^6 + y^6 + z^6)\frac{t^\alpha}{\Gamma(\alpha+1)} + (x^6 y^6 z^6)t \\
& - p\left(\frac{1}{360}x^4 \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left\{\sum_{n=0}^{\infty} p^n (u_n(x, y, z, t))_{xxxx}\right\}\right]\right) \\
& - p\left(\frac{1}{360}y^4 \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left\{\sum_{n=0}^{\infty} p^n (u_n(x, y, z, t))_{yyyy}\right\}\right]\right) \\
& - p\left(\frac{1}{360}z^4 \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left\{\sum_{n=0}^{\infty} p^n (u_n(x, y, z, t))_{zzzz}\right\}\right]\right).
\end{aligned}
$$

Comparing the coefficients of like power of p, we have

$$
\begin{aligned}
p^0: \quad u_0(x, y, z, t) &= (x^6 + y^6 + z^6)\frac{t^\alpha}{\Gamma(\alpha+1)} + (x^6 y^6 z^6)t, \\
p^1: \quad u_1(x, y, z, t) &= -(x^6 + y^6 + z^6)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - (x^6 y^6 z^6)\frac{t^\alpha}{\Gamma(\alpha+1)}, \\
p^2: \quad u_2(x, y, z, t) &= (x^6 + y^6 + z^6)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + (x^6 y^6 z^6)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\
&\vdots \\
p^n: \quad u_n(x, y, z, t) &= (-1)^n(x^6 + y^6 + z^6)\frac{t^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} + (-1)^n(x^6 y^6 z^6)\frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)}, \\
&\vdots
\end{aligned}
$$

Using the above iteration the solution of $u(x, y, z, t)$ is given by

$$
\begin{aligned}
u(x, y, z, t) &= \lim_{p \to 1} \sum_{n=0}^{\infty} p^n u_n(x, y, z, t) \\
&= (x^6 + y^6 + z^6)\left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots - \frac{t^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} + \cdots\right) \\
&\quad + x^6 y^6 z^6\left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \cdots - \frac{t^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} + \cdots\right)
\end{aligned}
$$

$$= (x^6 + y^6 + z^6)(E_\alpha(t^\alpha) - 1) + (x^6 y^6 z^6)(E_{\alpha,2}(-t^\alpha)).$$

If we put $\alpha = 2$ into (5.10), we obtain three dimensional in-homogeneous fractional beam-like equation as follows

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{360}(x^4 \frac{\partial^4 u}{\partial x^4} + y^4 \frac{\partial^4 u}{\partial y^4} + z^4 \frac{\partial^4 u}{\partial z^4}) = x^6 + y^6 + z^6, \tag{5.11}$$

and the solution of (5.11) is given by

$$u(x, y, z, t) = (x^6 + y^6 + z^6)(1 - \cos t) + (x^6 y^6 z^6) \sin t.$$

Example 5.4. Consider the following one dimensional fractional nonlinear beam equation

$$D_t^\alpha u + u_{xxxx} + 2uu_x - u_x^2 = 0, \tag{5.12}$$

where $0 \leqslant x \leqslant \frac{\pi}{2}$, $1 < \alpha \leqslant 2$, $t > 0$, subject the boundary condition

$$u(0, t) = 0, \quad u(\frac{\pi}{2}, t) = \cos t,$$

and the initial condition

$$u(x, 0) = \sin x, \quad u_t(x, 0) = 0.$$

In a similar way as above, we obtain

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = \sin x - p \left(\mathcal{L}^{-1} \left[\frac{1}{s^\alpha} \mathcal{L} \left(\sum_{n=0}^{\infty} p^n (u_n(x, t))_{xxxx} + 2 \sum_{n=0}^{\infty} p^n H_n(u) - \sum_{n=0}^{\infty} p^n u_{nx}^2 \right) \right] \right).$$

Comparing the coefficients of like power p, we have

$$p^0: \quad u_0(x, t) = \sin x,$$

$$p^1: \quad u_1(x, t) = -\mathcal{L}^{-1} \left[\frac{1}{s^\alpha} \mathcal{L} \left((u_0)_{xxxx} + 2H_0(u) - u_{0x}^2 \right) \right]$$

$$= -\mathcal{L}^{-1} \left[\frac{1}{s^\alpha} \mathcal{L} \left(\sin x + 2u_0 u_{0x} - u_{0x}^2 \right) \right]$$

$$= -\mathcal{L}^{-1} \left[\frac{1}{s^\alpha} \mathcal{L} (\sin x + 2 \sin x \cos x - 2 \sin x \cos x) \right]$$

$$= -\sin x \frac{t^\alpha}{\Gamma(\alpha + 1)},$$

$$p^2: \quad u_2(x, t) = -\mathcal{L}^{-1} \left[\frac{1}{s^\alpha} \mathcal{L} \left((u_1)_{xxxx} + 2H_1(u) - u_{1x}^2 \right) \right]$$

$$= -\mathcal{L}^{-1} \left[\frac{1}{s^\alpha} \mathcal{L} \left(\sin x + 2(u_1 u_{0x} + u_0 u_{1x}) - u_{1x}^2 \right) \right]$$

$$= \sin x \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)},$$

$$\vdots$$

$$p^n: \quad u_n(x, t) = \sin x \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)},$$

$$\vdots$$

Using the above iteration the solution $u(x, t)$ is given by

$$u(x, t) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n u_n(x, t) = \sin x \left(1 - \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} - \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} + \cdots \right) = \sin x E_\alpha(-t^\alpha).$$

If we put $\alpha = 2$ into (5.12), we obtain nonlinear fractional beam equation as follows:

$$u_{tt} + u_{xxxx} + 2uu_x - u_x^2 = 0 \tag{5.13}$$

and the solution of (5.13) is given by

$$u(x, t) = \sin x \cos t.$$

6. Conclusion

In this paper, the LHPM has been successfully applied to find the solution of the fractional linear and nonlinear beam equation with initial conditions. This method is reliable and easy to use. The result shows the LHPM is powerful and efficient technique to find exact and approximate solution for linear and nonlinear partial differential equation.

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