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On some new scenario of Δ-spaces

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Abstract

The structure of the Cesàro spaces were investigated by various authors as cited in the text. The scenario of this manuscript is to bring out the spaces $\mathfrak{C}_1\left(\triangle_g^s\right)$ and $\mathfrak{C}_\infty\left[\triangle_g^s\right]$ of Cesàro type for $s \in \mathbb{N} = \{0,1,2,\ldots\}$. We will study some of their basic topological properties and obtain some inclusion relations concerning these spaces.

Keywords: Cesàro sequence space, difference operator, BK-space.

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1. Introduction

By Λ we write the set of all real or complex sequences and by sequence space we mean a subspace of Λ. By $\mathbf{N}$, we represent the set $\{0,1,2,\ldots\}$; by $\mathbf{R}$ we mean $Q \cup Q^c$, where Q is set of rational numbers where as Q^c set of irrational numbers and $\mathbf{C}$ will represent the set of all complex numbers as can be seen in [7, 9]. We denote bounded sequences by l_∞; convergent sequences by c and those sequences with limit as zero by $\mathfrak{C}_0$ as can be seen in [8, 10–13]. Also, let $e = (1,1,\ldots)$.

We call a space $\mathcal{Y}$ to be FK space if it is a complete metric space with continuous coordinated $p_r : \mathcal{Y} \to \mathbb{C}$ where $p_r(u) = u_r$ for all $u \in \mathcal{Y}$ and $r \in \mathbb{N}$. A normed FK space is called a BK space as defined in [16, 23] and etc.

We call a space $\mathcal{V}$ with a linear topology as a K-space provided each of the maps $p_j : \mathcal{V} \to \mathbb{C}$ given by $p_j(v) = v_j$ is continuous for each $j \in \mathbb{N}$. A K-space $\mathcal{V}$ is said to be an FK-space provided it is complete linear metric space. An FK-space whose topology is normable is called a BK-space. We say that an FK-space $\mathcal{V}$ has AK (or has the AK property), if $e(k) = (1,1,\ldots)$ is a Schauder bases for $\mathcal{V}$. Note that FK spaces play an important role in the theory of sequence spaces and matrix transformations the reason being that matrix maps between FK space are continuous.

The spaces $T(\triangle)$, where

$$T(\triangle) = \{v = (v_i) \in \Lambda : (\triangle v_i) \in T\},$$

were introduced by Kizmaz [15], where $T \in \{l_\infty, c, \mathfrak{C}_0\}$ and $\triangle v_i = v_i - v_{i-1}$.

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Next the author in [23] had studied it and considered it as follows. Consider the integer $j \geqslant 0$, then

$$\Delta^j(\mathcal{H}) = \left\{ v = (v_k) : \left(\Delta^j v\right) \in \mathcal{H} \right\}, \text{ for } \mathcal{H} = l_\infty, c \text{ and } \mathfrak{C}_0,$$

where $\Delta^j v_i = \Delta^{j-1} v_i - \Delta^{j-1} v_{i+1}$ for all $i \in \mathbf{N}$.

Also, in [24] the above space were generalized to the following spaces:

$$\Delta^s_g(\mathcal{H}) = \left\{ v = (v_j) \in \Lambda : (\triangle^s_g v_j) \in \mathcal{H} \right\},$$

where

$$\triangle^s_g v_j = \Delta^{s-1}_g v_j - \Delta^{s-1}_g v_{j+1} = \sum_{\mu=0}^{s} (-1)^\mu \begin{pmatrix} m \\ \mu \end{pmatrix} g_{j+\mu} v_{j+\mu} \ \forall\, j \in \mathbb{N}.$$

The sequence spaces $\triangle^s_g(\mathcal{H})$ are Banach spaces normed by

$$\|v\|_\Delta = \sum_{i=1}^{s} |g_i v_i| + \|\Delta^s_g v\|_\infty.$$

This space was further studied by many authors as can be seen in [5, 11] and many others.

A sequence space $v = (v_j)$ of complex numbers is said to be Cesàro summable of order 1 or $(\mathfrak{C}, 1)$ summable to $\eta \in \mathbb{C}$ if

$$\lim_j \tau_j = \eta, \quad \text{where } \tau_j = \frac{1}{j} \sum_{i=1}^{j} v_i.$$

By $\mathfrak{C}_1$ we shall denote the linear space of all $(\mathfrak{C}, 1)$ summable sequences of complex numbers over $\mathfrak{C}$, i.e.,

$$\mathfrak{C}_1 = \left\{ v = (v_k) : \left(\frac{1}{i} \sum_{j=1}^{i} v_j \right) \in c \right\}.$$

It is easy to see that $\mathfrak{C}_1$ is a BK space normed by

$$\|v\| = \sum_i \left| \frac{1}{i} \sum_{j=1}^{i} v_j \right|.$$

It was further studied by several authors as can be found in [3, 5, 18]. In [17], the author has introduced the Cesàro sequence spaces X_p and X_∞ of non-absolute type and has shown that $Ces_p \subset X_p$ is strict for $1 \leqslant p \leqslant \infty$.

As in [23], we call a sequence space $\mathfrak{V}$ to be

(i) normal (or solid) if $v = (v_j) \in \mathfrak{V}$ whenever $|v_j| \leqslant |u_j|$, $j \geqslant 1$ for some $u = (u_j) \in \mathfrak{V}$;
(ii) monotone if it contains the canonical preimages of all its step spaces;
(iii) sequence algebra if $uv \in \mathfrak{V}$ whenever $u, v \in \mathfrak{V}$;
(iv) convergence free when, if $v = (v_j) \in \mathfrak{V}$ whenever $u = (u_j) \in \mathfrak{V}$ and $v_j = 0$ whenever $u_j = 0$.

2. Main section

In this division of the paper, we define the space $\mathfrak{C}_1\left(\triangle^s_g\right)$ and $\mathfrak{C}_\infty\left[\triangle^s_g\right]$, where $g = (g_j)$ is a sequence such that $g_j \neq 0\ \forall j \in \mathbf{N}$.

Following the authors cited in [1, 2, 6, 14, 19–22], we introduce the following spaces:

$$\mathfrak{C}_1\left(\triangle^s_g\right) = \left\{ v = (v_k) : \lim_i \frac{1}{i} \sum_{i=1}^{n} (\triangle^s_g v_k - \alpha) = 0 \right\},$$

where $\alpha \in \mathbb{R}$ and

$$\mathfrak{C}_\infty\left[\triangle_g^s\right] = \left\{ \nu = (\nu_k) : \sup_i \left(\frac{1}{i} \sum_{i=1}^{n} \triangle_g^s \nu_k \right) < \infty \right\}.$$

We now begin with the following theorem without proof.

Theorem 2.1. *The spaces* $\mathfrak{C}_1\left(\triangle_g^s\right)$ *and* $\mathfrak{C}_\infty\left[\triangle_g^s\right]$ *are* BK*-spaces with the norm*

$$\|\nu\|_{\Delta_\infty} = \sum_{j=1}^{s} |g_j \nu_j| + \sup_r \left(\frac{1}{r} \left| \sum_{i=1}^{r} \triangle_g^s \nu_i \right| \right).$$

Theorem 2.2.

(i) $\ell_\infty\left(\triangle_g^{s-1}\right) \subset \mathfrak{C}_\infty\left[\triangle_g^s\right]$ *and is sharp.*

(ii) $c\left(\triangle_g^s\right) \subset \mathfrak{C}_1\left(\triangle_g^s\right)$ *and is sharp.*

Proof. We only prove part (i) and part (ii) will follow on similar lines. Let $x \in \ell_\infty\left(\triangle_g^{m-1}\right)$, therefore we can find a constant β with $|\triangle_g^{s-1}\nu_j| \leqslant \beta$ for all $j \in \mathbb{N}$. But, we can write

$$\begin{aligned} \frac{1}{r}\left|\sum_{i=1}^{r} \triangle_g^s \nu_i\right| &= \frac{1}{r}\left|\triangle_g^s \nu_1 + \triangle_g^s \nu_2 + \cdots + \triangle_g^s \nu_r\right| \\ &= \frac{1}{r}\left|\left(\Delta_g^{m-1}\nu_1 - \Delta_g^{m-1}\nu_2\right) + \left(\Delta_g^{m-1}\nu_2 - \Delta_g^{m-1}\nu_3\right) + \cdots + \left(\Delta_g^{m-1}\nu_r - \Delta_g^{m-1}\nu_{r+1}\right)\right| \\ &= \frac{1}{r}\left|\Delta_g^{s-1}\nu_1 - \Delta_g^{s-1}\nu_{r+1}\right| \leqslant \frac{1}{r}\left|\Delta_g^{s-1}\nu_1\right| + \left|\Delta_g^{s-1}\nu_{r+1}\right| \leqslant \frac{2\beta}{r} \to 0 \quad (r \to \infty). \end{aligned}$$

Hence, $\nu \in \mathfrak{C}_1\left(\triangle_g^s\right)$. To prove inclusion is sharp, choose $g = e$, then it is clear that $(t^s) \in \mathfrak{C}_1\left(\triangle_g^s\right)$ but $(t^s) \notin \ell_\infty\left(\triangle_g^{s-1}\right)$. For if $\nu_t = t^s$, then clearly $\triangle_g^{s-1}\nu_t = (-1)^s s!$, but $\triangle_g^s \nu_t = (-1)^{s+1} s!\left(t + \frac{s-1}{2}\right)$, $\forall\, t \in \mathbb{N}$. □

We have following corollaries.

Corollary 2.3. $\mathfrak{C}_1\left(\triangle_g^s\right)$ *is a closed subspace of* $\mathfrak{C}_\infty\left(\triangle_g^s\right)$.

Corollary 2.4. $\mathfrak{C}_1\left(\triangle_g^s\right)$ *is a nowhere dense subset of* $\mathfrak{C}_\infty\left(\triangle_g^s\right)$.

Corollary 2.5. $\mathfrak{C}_\infty\left(\triangle_g^s\right)$ *is not separable, in general and has no Schauder basis.*

Proof. Hint: We know that if a normed space has a Schauder basis, then it is separable. □

Corollary 2.6. $\mathfrak{C}_1\left(\triangle_g^s\right)$ *is not normal (solid) and hence neither perfect nor convergence free.*

Proof. To prove this result, we let $g = e$, $s = 1$ and define $\nu = (\nu_j) = (j-1)$ and $u = (u_j) = \left((-1)^j (j-1)\right)$, it is then trivial that

$$\nu \in \mathfrak{C}_1\left(\triangle_g^s\right) \quad \text{but} \quad u \notin \mathfrak{C}_1\left(\triangle_g^s\right),$$

although $|u_j| \leqslant |\nu_j|$, $j \geqslant 1$. Consequently, $\mathfrak{C}_1\left(\triangle_g^s\right)$ is not normal.

Since $\mathfrak{C}_1\left(\triangle_g^s\right)$ is not normal, hence we conclude that it is neither perfect nor convergence free because every perfect space and also every convergence free space should be normal [4]. □

Corollary 2.7. $\mathfrak{C}_1\left(\triangle_g^s\right)$ *is neither monotone nor a sequence algebra.*

Proof. To prove this result, we let $g = e$, $s = 1$ and define $v = (v_j) = (j) \in \mathfrak{C}_1(\triangle_g^s)$ and define $u = (u_j)$ by

$$u(j) = \begin{cases} v_j, & \text{for } j \text{ being even}, \\ 0, & \text{for } j \text{ is odd}, \end{cases}$$

that is, $u = (0, 2, 0, 4, \ldots)$. Then, $(\triangle_g^s u_j) = (-2, 2, -4, 4, -6, 6, \ldots)$ and thus $(\triangle_g^s u_j) \notin \mathfrak{C}_1$ this means that $(u_j) \notin \mathfrak{C}_1(\triangle_g^s)$ and $\mathfrak{C}_1(\triangle_g^s)$ is not monotone.

Now in order to prove it is not sequence algebra, we choose $g = e$, $s = 1$, $v = u = (j)$ and it is observed that $u, v \in \mathfrak{C}_1(\triangle_g^s)$ but $uv \in (j^2) \notin \mathfrak{C}_1(\triangle_g^s)$. □

Theorem 2.8. *The space $\mathfrak{C}_1(\triangle_g^s)$ does not attain AK property.*

Proof. For the sequence $v = (v_j) = (t^s) = (t^s) = (1^s, 2^s, \ldots) \in \mathfrak{C}_1(\triangle_g^s)$ with $g = e$, consider its j^{th} section as $v^{[j]} = (1^s, 2^s, \ldots, j^s, 0, 0, \ldots)$. Then it is clear that

$$\|v - v^j\|_{\Delta_\infty} = \|(0, 0 \ldots, j^{s+1}, j^{s+2}, \ldots)\|_{\Delta_\infty} = \frac{1}{j}\left[(-1)^{s+1} s! \left((j+1) + \frac{s-1}{2}\right)\right] \nrightarrow 0 \text{ as } j \to \infty. \qquad \square$$

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