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On the stability of a sum form functional equation related to entropies of type (α, β)

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Abstract

In this paper, we discuss the stability of the sum form functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} f(q_j) + \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} q_j^{\beta}$$

for all complete probability distributions $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$, $n \geqslant 3$, $m \geqslant 3$ are fixed integers, f, g are real valued mappings each having the domain $I = [0, 1]$ and β is a fixed positive real power such that $\beta \neq 1$, $0^{\beta} := 0$, $1^{\beta} := 1$.

Keywords: Stability, additive mapping, logarithmic mapping, multiplicative mapping, bounded mapping, entropies of type (α, β).

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1. Introduction

Over the last few years, functional equations with reference to the stability problem has emerged as a new branch of research. Indeed, one of the stimulating aspect considered in this direction is to examine the stability of those functional equations whose general solutions exist and are useful in characterizing entropies. Captivated by the same here we have identified and discussed the stability of a Pexiderized functional equation which characterizes an entropy of type (α, β) and whose general solutions have been obtained.

Let $\mathbb{R}$ denotes the set of real numbers and I denotes the closed interval $[0, 1]$. For $n = 1, 2, \ldots$, let

$$\Gamma_n = \left\{ (p_1, \ldots, p_n) : p_i \geqslant 0,\ i = 1, \ldots, n; \sum_{i=1}^{n} p_i = 1 \right\}$$

denote the set of all finite n-component discrete probability distributions.

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A mapping $a : I \to \mathbb{R}$ is said to be additive on I or on the unit triangle

$$\Delta = \{(x, y) : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1, 0 \leqslant x + y \leqslant 1\}$$

if it satisfies the equation $a(x + y) = a(x) + a(y)$ for all $(x, y) \in \Delta$. A mapping $A : \mathbb{R} \to \mathbb{R}$ is said to be additive on $\mathbb{R}$ if it satisfies the equation $A(x + y) = A(x) + A(y)$ for all $x \in \mathbb{R}$, $y \in \mathbb{R}$. It is known [3] that if a mapping $a : I \to \mathbb{R}$ is additive on I, then it has a unique additive extension $A : \mathbb{R} \to \mathbb{R}$ in the sense that A is additive on $\mathbb{R}$ and $A(x) = a(x)$ for all $x \in I$.

A mapping $\ell : I \to \mathbb{R}$ is said to be logarithmic on I if it satisfies $\ell(0) = 0$ and $\ell(xy) = \ell(x) + \ell(y)$ for all $x \in]0, 1]$, $y \in]0, 1]$.

A mapping $m : I \to \mathbb{R}$ is said to be multiplicative on I if it satisfies $m(0) = 0$, $m(1) = 1$, and $m(xy) = m(x)m(y)$ for all $x \in]0, 1[$, $y \in]0, 1[$.

The concept of entropy of type (α, β) was introduced by Behara and Nath [2]. For a probability distribution $(p_1, \ldots, p_n) \in \Gamma_n$, an entropy of type (α, β) is defined as follows:

$$H_n^{(\alpha,\beta)}(p_1, \ldots, p_n) = \begin{cases} (2^{1-\alpha} - 2^{1-\beta})^{-1} \left(\sum_{i=1}^{n} p_i^{\alpha} - \sum_{i=1}^{n} p_i^{\beta} \right), & \text{if } \alpha \neq \beta, \\ -2^{\beta-1} \sum_{i=1}^{n} p_i^{\beta} \log_2 p_i, & \text{if } \alpha = \beta, \end{cases} \tag{1.1}$$

where $H_n^{(\alpha,\beta)}$ is a real valued mapping with domain Γ_n, $n = 1, 2, \ldots$, α and β are fixed positive real powers satisfying the conventions

$$0^{\alpha} := 0, \; 0^{\beta} := 0, \; 1^{\alpha} := 1, \; 1^{\beta} := 1, \tag{1.2}$$

and $0 \log_2 0 := 0$. The functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} p_i^{\alpha} \sum_{j=1}^{m} f(q_j) + \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} q_j^{\beta}, \tag{1.3}$$

where $f : I \to \mathbb{R}$, $(p_1, \ldots, p_n) \in \Gamma_n$, and $(q_1, \ldots, q_m) \in \Gamma_m$ plays a significant role in characterizing the entropies of type (α, β) given by (1.1).

A wide variety of literature is available on entropies of type (α, β) which are axiomatically characterized by the functional equation (1.3). Many authors (Behara and Nath [2] and Kannappan [6, 7]) studied the equation (1.3) and obtained its solutions by presuming some regularity conditions on the mapping $f : I \to \mathbb{R}$. In 1981, Losonczi and Maksa [10] obtained the general solutions of (1.3) for $n \geqslant 3$, $m \geqslant 2$ being fixed integers and $\alpha \neq 1$, $\beta \neq 1$. Recently few generalizations of (1.3) were observed by Kocsis [8], Nath and Singh [14–16], and Singh and Dass [18].

Once the general solution of a sum form functional equation is obtained, the next open problem concerning it is to check its stability. The problem of stability was raised by Ulam [19]. One of the intriguing questions in reference to the stability problem for functional equations originated from a fundamental question:

"*When it is true that a mapping which is approximately satisfying a functional equation is in close proximity of an exact solution of the same*?"

Hyers [5] provided a partial affirmative answer to the Ulam's open problem. Captivated by same, the stability problem for the sum form functional equations was addressed affirmatively by Maksa [12]. The essence of the paper of Maksa is that it bridges the gap between the stability problem for the functional equations and the sum form functional equations (mentioned as Result 2.2 in Section 2).

The study of stability problem for functional equations is interesting as well as demanding. The stability of sum form functional equations have been discussed by Maksa [12], Kocsis and Maksa [9], Kocsis [8], and Nath and Singh [17]. The Stability of some of the generalizations of (1.3) have been examined but for many it still remains an open problem. These open problems seems to have missed the attention of researchers working in this field. Hence this has inspired us to identify and examine the stability of one of these generalizations whose general solutions exist. Our objective is to discuss the stability of a Pexiderized form of (1.3), i.e.,

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} f(q_j) + \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} q_j^{\beta}, \tag{1.4}$$

where $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$ and $\beta \neq 1$ is a fixed positive real power satisfying (1.2). The general solution of (1.4) was obtained by Nath and Singh [14].

This paper is structured as follows. Section 1, briefly touches upon the notion of entropies of type (α, β), sum form of functional equations characterizing them and stability of sum form functional equations followed by specifying the objective of this paper. In Section 2, we present few auxiliary results which will be used in the upcoming section. In Section 3, we discuss the stability of the functional equation (1.4) for $n \geqslant 3$, $m \geqslant 3$ being fixed integers.

2. Auxiliary results

In this section, we state few known results which will be used in the upcoming Section 3.

Result 2.1 ([11]). *Suppose a mapping $\phi : I \to \mathbb{R}$ satisfies the functional equation $\sum_{i=1}^{n} \phi(p_i) = c$ for all $(p_1, \ldots, p_n) \in \Gamma_n$, $n \geqslant 3$ be a fixed integer and c a real constant. Then there exists an additive mapping $a : \mathbb{R} \to \mathbb{R}$ such that $\phi(p) = a(p) - \frac{1}{n}a(1) + \frac{c}{n}$ for all $p \in I$.*

Result 2.2 ([12]). *Let $n \geqslant 3$ be a fixed integer and ε be a fixed positive real number. Suppose a mapping $\bar{\phi} : I \to \mathbb{R}$ satisfies the functional inequality $\left|\sum_{i=1}^{n} \bar{\phi}(p_i)\right| \leqslant \varepsilon$ for all $(p_1, \ldots, p_n) \in \Gamma_n$. Then there exist an additive mapping $a_1 : \mathbb{R} \to \mathbb{R}$ and a bounded mapping $b_1 : \mathbb{R} \to \mathbb{R}$ such that $|b_1(p)| \leqslant 18\varepsilon$, $b_1(0) = 0$ and $\bar{\phi}(p) - \bar{\phi}(0) = a_1(p) + b_1(p)$ for all $p \in I$.*

Result 2.3 ([20]). *If a real additive mapping f is bounded over an interval $[a, b]$, then it must be linear, i.e., there exists some real number c' such that $f(p) = c'p$ for all $p \in \mathbb{R}$.*

Result 2.4 ([9]). *Let $n \geqslant 3$, $m \geqslant 3$ be fixed integers and ε be a fixed positive real constant. If a mapping $f : I \to \mathbb{R}$ satisfies the functional inequality*

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) - \sum_{i=1}^{n} p_i^{\beta} \sum_{j=1}^{m} f(q_j) - \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} q_j^{\beta}\right| \leqslant \varepsilon \tag{2.1}$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$, $\beta \neq 1$ being fixed positive real power satisfying (1.2), then for all $p \in I$, $f(p) = p^{\beta}\ell(p) + a_2(p) + b_2(p)$, $\ell : I \to \mathbb{R}$ is a logarithmic mapping, $a_2 : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $a_2(1) = 0$, and $b_2 : \mathbb{R} \to \mathbb{R}$ is a bounded mapping.

3. The stability of functional equation (1.4)

In this section, the main result is as follows.

Theorem 3.1. *Let $n \geqslant 3$, $m \geqslant 3$ be fixed integers and ε be a nonnegative real number. Suppose $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ be mappings satisfying the functional inequality*

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) - \sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} f(q_j) - \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} q_j^{\beta}\right| \leqslant \varepsilon \tag{A}$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$, $\beta \neq 1$ being fixed positive real power such that (1.2) holds. Then for *all* $p \in I$, *either*

(a) (i) $f(p) - f(0) = A_0(p),\ A_0(1) = -mf(0)$;
(ii) g *is an arbitrary real valued mapping;*

or

(b) (i) $f(p) - f(0) = c_1 p^{\beta} + A_1(p) + B_1(p)$ *with* $|B_1(p)| \leqslant 36\varepsilon\bar{c}$, $B_1(0) = 0$;
(ii) $g(p) - g(0) = \bar{A}_1(p),\ \bar{A}_1(1) = 1 - ng(0)$;

or

(c) (i) $f(p) - f(0) = A_2(p), A_2(1) = f(1) - f(0), f(1) + (m-1)f(0) \neq 0$;
(ii) $g(p) - g(0) = \bar{A}_2(p) + \bar{B}_1(p)$ *with* $|\bar{B}_1(p)| \leqslant 18\varepsilon c$, $\bar{B}_1(0) = 0$;

or

(d) (i) $f(p) = p^{\beta}\ell(p) + A_3(p) + B_2(p),\ A_3(1) = 0$;
(ii) $g(p) - g(0) = p^{\beta} + \bar{A}_3(p),\ \bar{A}_3(1) = -ng(0)$;

or

(e) (i) $f(p) - f(0) = c'p^{\beta} + A_4(p) + B_3(p),\ \ d \neq 0$ *with* $|B_3(p)| \leqslant 18\varepsilon(2c_0 + d^{-1})$, $B_3(0) = 0$;
(ii) $g(p) - g(0) = \bar{A}_4(p) + \bar{B}_2(p)$ *with* $|\bar{B}_2(p)| \leqslant 18\varepsilon$, $\bar{B}_2(0) = 0$;

or

(f) (i) $f(p) - f(0) = d^{-1}M(p) + c'p^{\beta} + A_5(p) + B_4(p),\ \ d \neq 0$ *with* $|B_4(p)| \leqslant 36\varepsilon c_0$, $B_4(0) = 0$;
(ii) $g(p) - g(0) = M(p) + \bar{A}_5(p),\ \bar{A}_5(1) = -[1 - g(1) + g(0)]$;

where $A_i : \mathbb{R} \to \mathbb{R}$ $(i = 0, 1, 2, 3, 4, 5)$, $\bar{A}_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, 3, 4, 5)$ *are additive mappings,* $B_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, 3, 4)$, $\bar{B}_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ *are bounded mappings,* $\ell : I \to \mathbb{R}$ *is a logarithmic mapping,* $M : I \to \mathbb{R}$ *is a nonconstant nonadditive multiplicative mapping,* $0 \neq c$, $0 \neq \bar{c}$, $0 \neq c_0$, $0 \neq d$, c' *and* c_1 *are arbitrary real constants.*

Before giving the proof of this theorem we need to prove the following lemma:

Lemma 3.2. *Let* $n \geqslant 3$, $m \geqslant 3$ *be fixed integers and* ε *be a nonnegative real number. Suppose a mapping* $f : I \to \mathbb{R}$ *satisfies the functional inequality*

$$\left| \sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) - \sum_{j=1}^{m} f(q_j) - \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} q_j^{\beta} \right| \leqslant \varepsilon \tag{3.1}$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$, $\beta \neq 1$ *being fixed positive real power such that* (1.2) *is satisfied. Then for all* $p \in I$

$$f(p) - f(0) = c_1 p^{\beta} + A_1(p) + B_1(p), \tag{3.2}$$

where $A_1 : \mathbb{R} \to \mathbb{R}$ *is an additive mapping,* $B_1 : \mathbb{R} \to \mathbb{R}$ *is a bounded mapping with* $|B_1(p)| \leqslant 36\varepsilon\bar{c}$, $B_1(0) = 0$, $0 \neq \bar{c}$ *and* c_1 *are arbitrary real constants.*

Proof. Without any loss of generality we may assume that $n \geqslant m$. Let us put $p_{m+1} = \cdots = p_n = 0$ in (3.1). We obtain

$$\left| \sum_{i=1}^{m} \sum_{j=1}^{m} f(p_i q_j) - \sum_{j=1}^{m} f(q_j) - \left[\sum_{i=1}^{m} f(p_i) + (n-m)f(0) \right] \sum_{j=1}^{m} q_j^{\beta} + m(n-m)f(0) \right| \leqslant \varepsilon \tag{3.3}$$

for all $(p_1, \ldots, p_m) \in \Gamma_m$, $(q_1, \ldots, q_m) \in \Gamma_m$. Now on interchanging p_i and q_j, $i = 1, \ldots, m$, $j = 1, \ldots, m$ in functional inequality (3.3), we have

$$\left| \sum_{i=1}^{m} \sum_{j=1}^{m} f(p_i q_j) - \sum_{i=1}^{m} f(p_i) - \left[\sum_{j=1}^{m} f(q_j) + (n-m)f(0) \right] \sum_{i=1}^{m} p_i^{\beta} + m(n-m)f(0) \right| \leqslant \varepsilon. \tag{3.4}$$

Applying triangle inequality to inequalities (3.3) and (3.4), we get

$$\left|\left(1-\sum_{j=1}^{m} q_j^{\beta}\right)\sum_{i=1}^{m} f(p_i)-\left(1-\sum_{i=1}^{m} p_i^{\beta}\right)\sum_{j=1}^{m} f(q_j)+(n-m)f(0)\left(\sum_{i=1}^{m} p_i^{\beta}-\sum_{j=1}^{m} q_j^{\beta}\right)\right| \leqslant 2\varepsilon.$$

In view of our presumption that $m \geqslant 3$ and $\beta \neq 1$, it follows that $1-\sum_{j=1}^{m} q_j^{\beta}$ does not vanish identically on Γ_m. Hence there exists a probability distribution $(q_1^*, \ldots, q_m^*) \in \Gamma_m$ such that $1-\sum_{j=1}^{m} q_j^{*\beta} \neq 0$. Now suppose $\bar{c} = \left[1-\sum_{j=1}^{m} q_j^{*\beta}\right]^{-1} \in \mathbb{R}$ and using this in the above functional inequality, we get

$$\left|\sum_{i=1}^{m} f(p_i) - c_1 \sum_{i=1}^{m} p_i^{\beta} - c_2\right| \leqslant 2\varepsilon\bar{c},$$

where $c_1 \in \mathbb{R}$, $c_2 \in \mathbb{R}$ and $(p_1, \ldots, p_m) \in \Gamma_m$. By Result 2.2, there exists an additive mapping $A_1^* : \mathbb{R} \to \mathbb{R}$ and a mapping $B_1 : \mathbb{R} \to \mathbb{R}$ bounded by $36\varepsilon\bar{c}$ with $B_1(0) = 0$, such that $f(p) - c_1 p^{\beta} - c_2 p - f(0) = A_1^*(p) + B_1(p)$ for all $p \in I$. Thus, (3.2) holds where an additive mapping $A_1 : \mathbb{R} \to \mathbb{R}$ is defined as $A_1(p) = A_1^*(p) + c_2 p$. □

It is noteworthy to mention that functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) - \sum_{j=1}^{m} f(q_j) - \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} q_j^{\beta} = 0, \tag{3.5}$$

where $f : I \to \mathbb{R}$, $(p_1, \ldots, p_n) \in \Gamma_n$ and $(q_1, \ldots, q_m) \in \Gamma_m$, characterizes the nonadditive entropies of degree β, which was introduced by Havrda and Charvát [4] as:

$$H_n^{\beta}(p_1, \ldots, p_n) = (1-2^{1-\beta})^{-1}\left(1-\sum_{i=1}^{n} p_i^{\beta}\right),$$

where $H_n^{\beta} : \Gamma_n \to R$, $n = 1, 2, \ldots$, $0 < \beta \in \mathbb{R}$, $\beta \neq 1$ satisfying the conventions (1.2). The functional equation (3.5) has been discussed by Nath [13] but the stability was yet to be discussed. Coincidentally, in this paper while discussing the stability of functional equation (1.4) we have examined the stability of (3.5) also (mentioned as Lemma 3.2).

Proof of Theorem 3.1. We divide our discussion into three cases.

Case 1. $\sum_{j=1}^{m} f(q_j)$ vanishes identically on Γ_m.

In this case, $\sum_{j=1}^{m} f(q_j) = 0$ for all $(q_1, \ldots, q_m) \in \Gamma_m$. By Result 2.1, there exists an additive mapping $A_0 : \mathbb{R} \to \mathbb{R}$ such that (a)(i) in Theorem 3.1 follows. Now on substituting (a)(i) in Theorem 3.1 in (A), we observe that "g is an arbitrary real valued mapping". Hence solution (a) is obtained.

Case 2. $1 - \sum_{i=1}^{n} g(p_i)$ vanishes identically on Γ_n.

In this case, $1 - \sum_{i=1}^{n} g(p_i) = 0$ for all $(p_1, \ldots, p_n) \in \Gamma_n$. By Result 2.1, there exists an additive mapping $\bar{A}_1 : \mathbb{R} \to \mathbb{R}$ such that (b)(ii) follows. Now on substituting (b)(ii) from Theorem 3.1 in (A), we obtain inequality (3.1). By using Lemma 3.2, the solution (b)(i) from Theorem 3.1 holds and thus solution (b) of (A) has been obtained.

Case 3. Neither $\sum_{j=1}^{m} f(q_j)$ vanishes identically on Γ_m nor $1-\sum_{i=1}^{n} g(p_i)$ vanishes identically on Γ_n.

Now in this case, By Result 2.2 on functional inequality (A), we get

$$\begin{aligned}\sum_{i=1}^{n} f(p_i q) - \sum_{i=1}^{n} g(p_i)f(q) - \sum_{i=1}^{n} f(p_i)q^{\beta} - nf(0) + \sum_{i=1}^{n} g(p_i)f(0) \\ = A(p_1,\dots,p_n;q) + B(p_1,\dots,p_n;q),\end{aligned} \tag{3.6}$$

where mapping $A : \Gamma_n \times \mathbb{R} \to \mathbb{R}$ is additive in its second variable and mapping $B : \Gamma_n \times \mathbb{R} \to \mathbb{R}$ is bounded in its second variable by 18ε with $B(p_1,\dots,p_n;0) = 0$. Let $x \in I$ and $(r_1,\dots,r_n) \in \Gamma_n$. Substituting $q = r_t x$, $t = 1,\dots,n$ successively in (3.6), adding the resulting n equations so obtained and then putting the expression $\sum_{t=1}^{n} f(r_t x)$ obtained from (3.6), we have

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{t=1}^{n} f(p_i r_t x) - (f(x)-f(0))\sum_{i=1}^{n} g(p_i)\sum_{t=1}^{n} g(r_t) - n^2 f(0) \\ &= A(p_1,\dots,p_n;x) + \sum_{t=1}^{n} B(p_1,\dots,p_n;r_t x) + A(r_1,\dots,r_n;x)\sum_{i=1}^{n} g(p_i) \\ &\quad + B(r_1,\dots,r_n;x)\sum_{i=1}^{n} g(p_i) + x^{\beta}\left[\sum_{i=1}^{n} g(p_i)\sum_{t=1}^{n} f(r_t) + \sum_{i=1}^{n} f(p_i)\sum_{t=1}^{n} r_t^{\beta}\right].\end{aligned}$$

The left hand side of the above equation is commutative in p_i and r_t, $i = 1,\dots,n$, $t = 1,\dots,n$ (see Aczél [1, p. 59]). So should be its right hand side. Consequently, we get

$$\begin{aligned}&A(p_1,\dots,p_n;x)\left[1-\sum_{t=1}^{n} g(r_t)\right] - A(r_1,\dots,r_n;x)\left[1-\sum_{i=1}^{n} g(p_i)\right] \\ &= \sum_{i=1}^{n} B(r_1,\dots,r_n;p_i x) + B(p_1,\dots,p_n;x)\sum_{t=1}^{n} g(r_t) - \sum_{t=1}^{n} B(p_1,\dots,p_n;r_t x) \\ &\quad - B(r_1,\dots,r_n;x)\sum_{i=1}^{n} g(p_i) + x^{\beta}\left[\sum_{t=1}^{n} g(r_t)\sum_{i=1}^{n} f(p_i) - \sum_{i=1}^{n} g(p_i)\sum_{t=1}^{n} f(r_t)\right. \\ &\quad \left. + \sum_{t=1}^{n} f(r_t)\sum_{i=1}^{n} p_i^{\beta} - \sum_{i=1}^{n} f(p_i)\sum_{t=1}^{n} r_t^{\beta}\right].\end{aligned} \tag{3.7}$$

For fixed $(p_1,\dots,p_n) \in \Gamma_n$ and $(r_1,\dots,r_n) \in \Gamma_n$, the right hand side of (3.7) is bounded on I while the left hand side is additive in x. Thus by Result 2.3, it follows that

$$\begin{aligned}&[A(p_1,\dots,p_n;x) - xA(p_1,\dots,p_n;1)]\left[1-\sum_{t=1}^{n} g(r_t)\right] \\ &= [A(r_1,\dots,r_n;x) - x\,A(r_1,\dots,r_n;1)]\left[1-\sum_{i=1}^{n} g(p_i)\right]\end{aligned} \tag{3.8}$$

for all $x \in I$, $(p_1,\dots,p_n) \in \Gamma_n$ and $(r_1,\dots,r_n) \in \Gamma_n$. Since $1-\sum_{i=1}^{n} g(p_i)$ does not vanish identically on Γ_n, there exists a probability distribution $(p_1^*,\dots,p_n^*) \in \Gamma_n$ such that $1-\sum_{i=1}^{n} g(p_i^*) \neq 0$. Putting $p_i = p_i^*$,

$i = 1, \ldots, n$ in (3.8), we get

$$A(r_1, \ldots, r_n; x) = a_0(x)\left[1 - \sum_{t=1}^{n} g(r_t)\right] + xA(r_1, \ldots, r_n; 1), \tag{3.9}$$

where $a_0 : \mathbb{R} \to \mathbb{R}$ is an additive mapping defined as $a_0(x) = [A(p_1^*, \ldots, p_n^*; x) - xA(p_1^*, \ldots, p_n^*; 1)] \times \left[1 - \sum_{i=1}^{n} g\left(p_i^*\right)\right]^{-1}$ with $a_0(1) = 0$. Replacing q by 1 and p_i by r_t, $i = 1, \ldots, n$, $t = 1, \ldots, n$ in (3.6), using the fact that $1^\beta := 1$ and then substituting '$A(r_1, \ldots, r_n; 1)$' calculated from (3.6) in (3.9). We obtain

$$\begin{aligned} A(r_1, \ldots, r_n; x) = a_0(x)\left[1 - \sum_{t=1}^{n} g(r_t)\right] - x\Bigg[f(1)\sum_{t=1}^{n} g(r_t) \\ -f(0)\sum_{t=1}^{n} g(r_t) + nf(0) + B(r_1, \ldots, r_n; 1)\Bigg] \end{aligned} \tag{3.10}$$

for all $x \in I$ and $(r_1, \ldots, r_n) \in \Gamma_n$. In order to obtain some more information regarding the bounded mapping B, we substitute (3.10) in (3.7) and on performing required calculations, we obtain

$$\begin{aligned} &\left[B(r_1, \ldots, r_n; x) + x^\beta \sum_{t=1}^{n} f(r_t) - x[f(1) + (n-1)f(0) + B(r_1, \ldots, r_n; 1)]\right]\sum_{i=1}^{n} g(p_i) \\ &= \left[B(p_1, \ldots, p_n; x) + x^\beta \sum_{i=1}^{n} f(p_i) - x[f(1) + (n-1)f(0) + B(p_1, \ldots, p_n; 1)]\right]\sum_{t=1}^{n} g(r_t) \\ &\quad + \sum_{i=1}^{n} B(r_1, \ldots, r_n; p_i x) - \sum_{t=1}^{n} B(p_1, \ldots, p_n; r_t x) + xB(p_1, \ldots, p_n; 1) \\ &\quad - xB(r_1, \ldots, r_n; 1) + x^\beta\left[\sum_{t=1}^{n} f(r_t)\sum_{i=1}^{n} p_i^\beta - \sum_{i=1}^{n} f(p_i)\sum_{t=1}^{n} r_t^\beta\right] \end{aligned} \tag{3.11}$$

for all $x \in I$, $(p_1, \ldots, p_n) \in \Gamma_n$ and $(r_1, \ldots, r_n) \in \Gamma_n$. Functional equation (3.11) indicates that the proof strongly depends on the coefficient of $\sum_{i=1}^{n} g(p_i)$. So we divide our discussion into two cases.

Case 3.1. Coefficient of $\sum_{i=1}^{n} g(p_i)$ vanishes identically.

In this case, functional equation (3.11) yields

$$B(r_1, \ldots, r_n; x) = -x^\beta \sum_{t=1}^{n} f(r_t) + x[f(1) + (n-1)f(0) + B(r_1, \ldots, r_n; 1)] \tag{3.12}$$

for all $x \in I$ and $(r_1, \ldots, r_n) \in \Gamma_n$.

Now, with the help of (3.10) and (3.12), functional equation (3.6) reduces to

$$\sum_{i=1}^{n} H(p_i q) - H(q)\sum_{i=1}^{n} g(p_i) = 0, \tag{3.13}$$

where $H : I \to \mathbb{R}$ is defined as

$$H(x) = f(x) - f(0) - a_0(x) - x(f(1) - f(0)) \tag{3.14}$$

for all $x \in I$. Clearly $H(0) = 0$ and $H(1) = 0$. By Result 2.1, functional equation (3.13) implies $H(pq) - H(q)(g(p) - g(0)) = E(p; q)$ where $E : \mathbb{R} \times I \to \mathbb{R}$ is additive in first variable with $E(1; q) = ng(0)H(q)$. Clearly $E(1; 1) = 0$. Also for $q = 1$, it follows that $H(p) = E(p; 1)$ for all $p \in I$. Thus from this and (3.14) the solution (c)(i) from Theorem 3.1 follows by defining an additive mapping $A_2 : \mathbb{R} \to \mathbb{R}$ as $A_2(x) = E(x; 1) + a_0(x) + x(f(1) - f(0))$ with $A_2(1) = f(1) - f(0)$. Further we get, $\sum_{j=1}^{m} f(q_j) = f(1) + (m-1)f(0)$. Since $\sum_{j=1}^{m} f(q_j)$ does not vanish identically on Γ_m, we suppose $0 \neq f(1) + (m-1)f(0) = c^{-1} \in \mathbb{R}$. Moreover on putting $q_1 = 1$, $q_2 = \ldots = q_m = 0$ in (A), we obtain

$$\left| \sum_{i=1}^{n} \left[(m-1)f(0) - c^{-1} g(p_i) \right] \right| \leqslant \varepsilon$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$. By Result 2.2, there exists an additive mapping $\bar{A}_2^* : \mathbb{R} \to \mathbb{R}$ and a bounded mapping $\bar{B}_1^* : \mathbb{R} \to \mathbb{R}$ with $|\bar{B}_1^*(p)| \leqslant 18\varepsilon$ and $\bar{B}_1^*(0) = 0$, such that $-c^{-1}(g(p) - g(0)) = \bar{A}_2^*(p) + \bar{B}_1^*(p)$ for all $p \in I$. Hence solution (c)(ii) from Theorem 3.1 is attained by defining an additive mapping $\bar{A}_2 : \mathbb{R} \to \mathbb{R}$ as $\bar{A}_2(x) = -c\bar{A}_2^*(x)$ and a bounded mapping $\bar{B}_1 : \mathbb{R} \to \mathbb{R}$ as $\bar{B}_1(x) = -c\bar{B}_1^*(x)$ such that $|\bar{B}_1(x)| \leqslant 18\varepsilon c$ with $\bar{B}_1(0) = 0$.

Case 3.2. Coefficient of $\sum_{i=1}^{n} g(p_i)$ does not vanish identically.

In this case, there is no loss of generality in assuming $n \geqslant m$. Let $p_{m+1} = \ldots = p_n = 0$ in (A), we have

$$\begin{aligned} \Bigg| \sum_{i=1}^{m} \sum_{j=1}^{m} f(p_i q_j) - \left[\sum_{i=1}^{m} g(p_i) + (n-m)g(0) \right] \sum_{j=1}^{m} f(q_j) \\ - \left[\sum_{i=1}^{m} f(p_i) + (n-m)f(0) \right] \sum_{j=1}^{m} q_j^{\beta} + m(n-m)f(0) \Bigg| \leqslant \varepsilon \end{aligned} \tag{3.15}$$

for all $(p_1, \ldots, p_m) \in \Gamma_m$ and $(q_1, \ldots, q_m) \in \Gamma_m$. Now, on interchanging p_i and q_j, $i = 1, \ldots, m$, $j = 1, \ldots, m$ in functional inequality (3.15), we get

$$\begin{aligned} \Bigg| \sum_{i=1}^{m} \sum_{j=1}^{m} f(p_i q_j) - \left[\sum_{j=1}^{m} g(q_j) + (n-m)g(0) \right] \sum_{i=1}^{m} f(p_i) \\ - \left[\sum_{j=1}^{m} f(q_j) + (n-m)f(0) \right] \sum_{i=1}^{m} p_i^{\beta} + m(n-m)f(0) \Bigg| \leqslant \varepsilon. \end{aligned} \tag{3.16}$$

By applying triangle inequality to functional inequalities (3.15) and (3.16), we get

$$\begin{aligned} \Bigg\| \left[\sum_{j=1}^{m} g(q_j) + (n-m)g(0) - \sum_{j=1}^{m} q_j^{\beta} \right] \sum_{i=1}^{m} f(p_i) - \Bigg[\sum_{i=1}^{m} g(p_i) + (n-m)g(0) \\ - \sum_{i=1}^{m} p_i^{\beta} \Bigg] \sum_{j=1}^{m} f(q_j) + (n-m)f(0) \left[\sum_{i=1}^{m} p_i^{\beta} - \sum_{j=1}^{m} q_j^{\beta} \right] \Bigg\| \leqslant 2\varepsilon. \end{aligned} \tag{3.17}$$

Now, if $\sum_{j=1}^{m} g(q_j) + (n-m)g(0) - \sum_{j=1}^{m} q_j^{\beta}$ vanishes identically on Γ_m, then by Result 2.1 there exists an additive mapping $\bar{A}_3 : \mathbb{R} \to \mathbb{R}$ such that solution (d)(ii) from Theorem 3.1 holds. Further with the aid of (d)(ii) from Theorem 3.1 functional inequality (A) reduces to inequality (2.1). So, by Result 2.4 solution (d)(i) from Theorem 3.1 follows where $A_3 : \mathbb{R} \to \mathbb{R}$ is an additive mapping and $B_2 : \mathbb{R} \to \mathbb{R}$ is a bounded mapping. Hence solution (d) from Theorem 3.1 is attained.

Next, if $\sum_{j=1}^{m} g(q_j) + (n-m)\, g(0) - \sum_{j=1}^{m} q_j^{\beta}$ does not vanish identically on Γ_m, then there exist some probability distribution $(q_1^*, \ldots, q_m^*) \in \Gamma_m$ for which $\sum_{j=1}^{m} g(q_j^*) + (n-m) g(0) - \sum_{j=1}^{m} q_j^{*\beta} \neq 0$. Suppose $c_0 = \left[\sum_{j=1}^{m} g\left(q_j^*\right) + (n-m)\, g(0) - \sum_{j=1}^{m} q_j^{*\beta}\right]^{-1} \in \mathbb{R}$. With the help of this, inequality (3.17) can be rewritten to the form

$$\left|\sum_{i=1}^{m} f(p_i) - c_3 \sum_{i=1}^{m} g(p_i) - c' \sum_{i=1}^{m} p_i^{\beta} - c_4\right| \leqslant 2\varepsilon c_0,$$

where $0 \neq c_3 \in \mathbb{R}$, $c' \in \mathbb{R}$, $c_4 \in \mathbb{R}$ and $(p_1, \ldots, p_m) \in \Gamma_m$. By Result 2.2, there exists an additive mapping $A_4^* : \mathbb{R} \to \mathbb{R}$ and a bounded mapping $B_3^* : \mathbb{R} \to \mathbb{R}$ such that $|B_3^*(p)| \leqslant 36\varepsilon c_0$ with $B_3^*(0) = 0$ satisfying

$$f(p) - c_3 g(p) - c' p^{\beta} - c_4 p - f(0) + c_3 g(0) = A_4^*(p) + B_3^*(p) \tag{3.18}$$

for all $p \in I$. Moreover as $c_3 \neq 0$, then from (3.18) it follows that

$$d f(p) = g(p) + c^* p^{\beta} + \bar{A}(p) + \bar{B}(p), \tag{3.19}$$

where $0 \neq d = c_3^{-1} \in \mathbb{R}$, $c^* = c'd \in \mathbb{R}$, $\bar{A} : \mathbb{R} \to \mathbb{R}$ is an additive mapping defined as $\bar{A}(x) = d[A_4^*(x) + c_4 x]$ and $\bar{B} : \mathbb{R} \to \mathbb{R}$ is a bounded mapping defined as $\bar{B}(x) = d[f(0) - c_3 g(0) + B_3^*(x)]$. From (3.19), inequality (A) can be rewritten in the form

$$\begin{aligned}\Bigg|\sum_{i=1}^{n}\sum_{j=1}^{m} g(p_i q_j) - \sum_{i=1}^{n} g(p_i)\left[\sum_{j=1}^{m} g(q_j) + \hat{c}\sum_{j=1}^{m} q_j^{\beta} + \bar{A}(1) + \sum_{j=1}^{m}\bar{B}(q_j)\right] \\ - \left[\bar{A}(1) + \sum_{i=1}^{n}\bar{B}(p_i)\right]\sum_{j=1}^{m} q_j^{\beta} + \bar{A}(1) + \sum_{i=1}^{n}\sum_{j=1}^{m}\bar{B}(p_i q_j)\Bigg| \leqslant d\varepsilon,\end{aligned}$$

where $\hat{c} = c^* + 1 \in \mathbb{R}$. By Result 2.2, there exists a mapping $a : \Gamma_n \times \mathbb{R} \to \mathbb{R}$ additive in its second variable and a mapping $b : \Gamma_n \times \mathbb{R} \to \mathbb{R}$ bounded in its second variable by $18d\varepsilon$ with $b(0) = 0$ such that

$$\begin{aligned}&\sum_{i=1}^{n} g(p_i q) - \sum_{i=1}^{n} g(p_i)\left[g(q) + \hat{c} q^{\beta} + \bar{A}(1) q + \bar{B}(q) - g(0) - \bar{B}(0)\right] \\ &\quad - \left[\bar{A}(1) + \sum_{i=1}^{n}\bar{B}(p_i)\right] q^{\beta} + \bar{A}(1) q + \sum_{i=1}^{n}\bar{B}(p_i q) - n g(0) - n\bar{B}(0) \\ &\quad = a(p_1, \ldots, p_n; q) + b(p_1, \ldots, p_n; q)\end{aligned} \tag{3.20}$$

with

$$\begin{aligned}a(p_1, \ldots, p_n; 1) = &\sum_{i=1}^{n} g(p_i)[1 - g(1) - \hat{c} - \bar{A}(1) - \bar{B}(1) + g(0) + \bar{B}(0)] \\ &- n g(0) - n\bar{B}(0) - b(p_1, \ldots, p_n; 1).\end{aligned} \tag{3.21}$$

Let $x \in I$ and $(r_1,\ldots,r_n) \in \Gamma_n$. Replacing q by $r_t x$, $t = 1,\ldots,n$ successively in (3.20), summing the resulting n equations so obtained and substituting the value of $\sum_{t=1}^{n} g(r_t x)$ from (3.20), we get

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{t=1}^{n} g(p_i r_t x) - [g(x) + \hat{c}x^{\beta} + \bar{A}(1)x + \bar{B}(x) - g(0) - \bar{B}(0)] \sum_{i=1}^{n} g(p_i) \sum_{t=1}^{n} g(r_t)\\ &+ \bar{A}(1)x + \sum_{i=1}^{n}\sum_{t=1}^{n} \bar{B}(p_i r_t x) - n^2 g(0) - n^2 \bar{B}(0) = a(p_1,\ldots,p_n;x)\\ &+ \sum_{t=1}^{n} b(p_1,\ldots,p_n;r_t x) + x^{\beta}\left\{\sum_{i=1}^{n} g(p_i)\left[\bar{A}(1) + \sum_{t=1}^{n}\bar{B}(r_t) + \hat{c}\sum_{t=1}^{n} r_t^{\beta}\right]\right.\\ &\left.+\left[\bar{A}(1) + \sum_{i=1}^{n}\bar{B}(p_i)\right]\sum_{t=1}^{n} r_t^{\beta}\right\} + a(r_1,\ldots,r_n;x)\sum_{i=1}^{n} g(p_i) + b(r_1,\ldots,r_n;x)\sum_{i=1}^{n} g(p_i).\end{aligned}$$

The left hand side of the above equation is symmetric in p_i and r_t, $i = 1,\ldots,n$, $t = 1,\ldots,n$. So should be its right hand side. Consequently, we get

$$\begin{aligned}&a(p_1,\ldots,p_n;x)\left[1 - \sum_{t=1}^{n} g(r_t)\right] - a(r_1,\ldots,r_n;x)\left[1 - \sum_{i=1}^{n} g(p_i)\right]\\ &= \sum_{i=1}^{n} b(r_1,\ldots,r_n;p_i x) + x^{\beta}\left\{\sum_{t=1}^{n} g(r_t)\left[\bar{A}(1) + \sum_{i=1}^{n}\bar{B}(p_i) + \hat{c}\sum_{i=1}^{n} p_i^{\beta}\right]\right.\\ &\left.+\left[\bar{A}(1) + \sum_{t=1}^{n}\bar{B}(r_t)\right]\sum_{i=1}^{n} p_i^{\beta}\right\} + b\,(p_1,\ldots,p_n;x)\sum_{t=1}^{n} g(r_t)\\ &- \sum_{t=1}^{n} b\,(p_1,\ldots,p_n;r_t x) - x^{\beta}\left\{\sum_{i=1}^{n} g(p_i)\left[\bar{A}(1) + \sum_{t=1}^{n}\bar{B}(r_t) + \hat{c}\sum_{t=1}^{n} r_t^{\beta}\right]\right.\\ &\left.+\left[\bar{A}(1) + \sum_{i=1}^{n}\bar{B}(p_i)\right]\sum_{t=1}^{n} r_t^{\beta}\right\} - b\,(r_1,\ldots,r_n;x)\sum_{i=1}^{n} g(p_i).\end{aligned} \tag{3.22}$$

For fixed $(p_1,\ldots,p_n) \in \Gamma_n$ and $(r_1,\ldots,r_n) \in \Gamma_n$, the left hand side of (3.22) is additive in x but the right hand side is bounded on I. Thus by Result 2.3, it follows that left hand side must be linear. We obtain

$$[a(p_1,\ldots,p_n;x) - xa(p_1,\ldots,p_n;1)]\left[1 - \sum_{t=1}^{n} g(r_t)\right] = [a(r_1,\ldots,r_n;x) - xa(r_1,\ldots,r_n;1)]\left[1 - \sum_{i=1}^{n} g(p_i)\right].$$

Since $1 - \sum_{t=1}^{n} g(r_t) \neq 0$, there exists a probability distribution $(r_1^*,\ldots,r_n^*) \in \Gamma_n$ so that $1 - \sum_{t=1}^{n} g(r_t^*) \neq 0$. Let us replace r_t by r_t^*, $t = 1,\ldots,n$ in the above equation and using this, it follows that

$$a(p_1,\ldots,p_n;x) = \bar{a}_0(x)\left[1 - \sum_{i=1}^{n} g(p_i)\right] + xa(p_1,\ldots,p_n;1), \tag{3.23}$$

where $\bar{a}_0 : \mathbb{R} \to \mathbb{R}$ is an additive mapping defined as $\bar{a}_0(x) = [a(r_1^*,\ldots,r_n^*;x) - xa(r_1^*,\ldots,r_n^*;1)] \times \left[1 - \sum_{t=1}^{n} g\,(r_t^*)\right]^{-1}$ with $\bar{a}_0(1) = 0$. From (3.21) and (3.23), we have

$$\begin{aligned}a(p_1,\ldots,p_n;x) = \bar{a}_0(x)\left[1 - \sum_{i=1}^{n} g(p_i)\right] + x\Bigg\{&\sum_{i=1}^{n} g(p_i)[1 - g(1) - \hat{c} - \bar{A}(1)\\ &- \bar{B}(1) + g(0) + \bar{B}(0)] - ng(0) - n\bar{B}(0) - b(p_1,\ldots,p_n;1)\Bigg\}\end{aligned} \tag{3.24}$$

for all $x \in I$ and $(p_1, \ldots, p_n) \in \Gamma_n$. Now, on substituting (3.24) in (3.22), we obtain

$$\begin{aligned}
&\Big\{b(p_1,\ldots,p_n;x)+x^{\beta}\Big[\bar{A}(1)+\sum_{i=1}^{n}\bar{B}(p_i)+\hat{c}\sum_{i=1}^{n}p_i^{\beta}\Big]+x[1-g(1)\\
&\quad-(n-1)g(0)-\hat{c}-\bar{A}(1)-\bar{B}(1)-(n-1)\bar{B}(0)-b(p_1,\ldots,p_n;1)]\Big\}\sum_{t=1}^{n}g(r_t)\\
&=\Big\{b(r_1,\ldots,r_n;x)+x^{\beta}\Big[\bar{A}(1)+\sum_{t=1}^{n}\bar{B}(r_t)+\hat{c}\sum_{t=1}^{n}r_t^{\beta}\Big]+x[1-g(1)\\
&\quad-(n-1)g(0)-\hat{c}-\bar{A}(1)-\bar{B}(1)-(n-1)\bar{B}(0)-b(r_1,\ldots,r_n;1)]\Big\}\sum_{i=1}^{n}g(p_i)\\
&\quad+x[b(r_1,\ldots,r_n;1)-b(p_1,\ldots,p_n;1)]-\sum_{i=1}^{n}b(r_1,\ldots,r_n;p_ix)\\
&\quad+\sum_{t=1}^{n}b(p_1,\ldots,p_n;r_tx)-x^{\beta}\Big\{\Big[\bar{A}(1)+\sum_{t=1}^{n}\bar{B}(r_t)\Big]\sum_{i=1}^{n}p_i^{\beta}-\Big[\bar{A}(1)+\sum_{i=1}^{n}\bar{B}(p_i)\Big]\sum_{t=1}^{n}r_t^{\beta}\Big\}
\end{aligned} \tag{3.25}$$

for all $x \in I$, $(p_1, \ldots, p_n) \in \Gamma_n$ and $(r_1, \ldots, r_n) \in \Gamma_n$.

If the coefficient of $\sum_{t=1}^{n} g(r_t)$ in (3.25) does not vanish identically, then by the boundedness of the mappings b and $\bar{B}$, it follows that $\left|\sum_{t=1}^{n} g(r_t)\right| \leqslant \varepsilon$ for some positive real number ε. By Result 2.2, there exists an additive mapping $\bar{A}_4 : \mathbb{R} \to \mathbb{R}$ and a bounded mapping $\bar{B}_2 : \mathbb{R} \to \mathbb{R}$ such that $|\bar{B}_2(p)| \leqslant 18\varepsilon$ and $\bar{B}_2(0) = 0$ satisfying (e)(ii) from Theorem 3.1. Also, from (3.18) and (e)(ii) from Theorem 3.1, solution (e)(i) from Theorem 3.1 is attained by defining an additive mapping $A_4 : \mathbb{R} \to \mathbb{R}$ as $A_4(x) = A_4^*(x) + c_4x + c_3\bar{A}_4(x)$ and a bounded mapping $B_3 : \mathbb{R} \to \mathbb{R}$ as $B_3(x) = B_3^*(x) + c_3\bar{B}_2(x)$. Thus solution (e) from Theorem 3.1 is obtained.

In the remaining case, functional equation (3.25) yields

$$\begin{aligned}
b(p_1,\ldots,p_n;x) = -x^{\beta}\Big[\bar{A}(1)+\sum_{i=1}^{n}\bar{B}(p_i)+\hat{c}\sum_{i=1}^{n}p_i^{\beta}\Big]-x[1-g(1)-ng(0)\\
+g(0)-\hat{c}-\bar{A}(1)-\bar{B}(1)-n\bar{B}(0)+\bar{B}(0)-b(p_1,\ldots,p_n;1)]
\end{aligned} \tag{3.26}$$

for all $x \in I$ and $(p_1, \ldots, p_n) \in \Gamma_n$. Also, from (3.20), (3.24), and (3.26), it follows that

$$\begin{aligned}
\sum_{i=1}^{n}M(p_iq)-\sum_{i=1}^{n}M(p_i)M(q)-\Big[\hat{c}\Big(1-\sum_{i=1}^{n}p_i^{\beta}\Big)-\sum_{i=1}^{n}\bar{B}(p_i)\\
+g(1)+(n-1)g(0)+\bar{B}(1)+(n-1)\bar{B}(0)-1\Big]M(q)=0,
\end{aligned} \tag{3.27}$$

where $M : I \to \mathbb{R}$ is a mapping defined as

$$M(x) = g(x)+\hat{c}x^{\beta}+\bar{B}(x)-g(0)-\bar{B}(0)-\bar{a}_0(x)+x[1-g(1)-\hat{c}-\bar{B}(1)+g(0)+\bar{B}(0)] \tag{3.28}$$

for all $x \in I$. By applying Result 2.1 on (3.27), there exists a mapping $\bar{E} : \mathbb{R} \times I \to \mathbb{R}$, additive in the first variable such that

$$\begin{aligned}
M(pq)-M(p)M(q)+\{\hat{c}p^{\beta}+\bar{B}(p)-\bar{B}(0)-p[\hat{c}+g(1)+(n-1)g(0)\\
+\bar{B}(1)+(n-1)\bar{B}(0)-1]\}M(q)=\bar{E}(p;q)
\end{aligned} \tag{3.29}$$

with $\bar{E}(1;q) = -n\bar{B}(0)M(q)$. Now let us put $q = 1$ in (3.29) and use $M(1) = 1$. We obtain

$$\hat{c}p^{\beta} + \bar{B}(p) - \bar{B}(0) - p[\hat{c} + g(1) + (n-1)g(0) + \bar{B}(1) + (n-1)\bar{B}(0) - 1] = \bar{E}(p;1). \tag{3.30}$$

The left hand side of (3.30) is bounded on I, while the right hand side is additive in p. Thus by applying Result 2.3, it follows that $\bar{E}(p;1) = p\bar{E}(1;1)$. As a result from (3.30), it follows that

$$\hat{c}p^{\beta} + \bar{B}(p) - \bar{B}(0) - p[\hat{c} + \bar{B}(1) - \bar{B}(0)] = 0. \tag{3.31}$$

Consequently from (3.28) and (3.31), it follows that

$$M(x) = g(x) - g(0) - \bar{a}_0(x) + x[1 - g(1) + g(0)] \tag{3.32}$$

for all $x \in I$. Clearly from (3.32), $M(1) = 1$ and $M(0) = 0$ as $\bar{a}_0(1) = 0$ and $\bar{a}_0(0) = 0$. Also on substituting (3.31) in (3.29), we get

$$M(pq) - M(p)M(q) = \bar{E}(p;q) + p[g(1) + (n-1)g(0) + n\bar{B}(0) - 1]M(q) \tag{3.33}$$

for all $p \in I$ and $q \in I$.

Now, if $\bar{E}(p;q) + p[g(1) + (n-1)g(0) + n\bar{B}(0) - 1]M(q) = 0$, then from (3.33) we conclude that M is a nonconstant nonadditive mapping satisfying $M(0) = 0$, $M(1) = 1$ and $M(xy) = M(x)M(y)$ for all $x \in]0,1[$, $y \in]0,1[$. As a result from (3.32), (f)(ii) from Theorem 3.1 holds where an additive mapping $\bar{A}_5 : \mathbb{R} \to \mathbb{R}$ is defined as $\bar{A}_5(x) = \bar{a}_0(x) - x[1 - g(1) + g(0)]$. Furthermore, from (3.18) and (f)(ii) from Theorem 3.1, solution (f)(i) from Theorem 3.1 follows where an additive mapping $A_5 : \mathbb{R} \to \mathbb{R}$ is defined as $A_5(x) = A_4^*(x) + c_4x + c_3\bar{A}_5(x)$ and a bounded mapping $B_4 : \mathbb{R} \to \mathbb{R}$ as $B_4(x) = B_3^*(x)$.

If $\bar{E}(p;q) + p[g(1) + (n-1)g(0) + n\bar{B}(0) - 1]M(q) \neq 0$, then there exists some $p^* \in I$ and $q^* \in I$ such that

$$\bar{E}(p^*;q^*) + p^*[g(1) + (n-1)g(0) + n\bar{B}(0) - 1]M(q^*) \neq 0. \tag{3.34}$$

Also, from (3.33) it can easily be verified that

$$\begin{aligned} M(pqr) - M(p)M(q)M(r) &= \bar{E}(r;pq) + M(r)\{\bar{E}(p;q) + p[g(1) + (n-1)g(0) + n\bar{B}(0) - 1]M(q)\} \\ &\quad + r[g(1) + (n-1)g(0) + n\bar{B}(0) - 1]M(pq) \\ &= \bar{E}(rp;q) + M(q)\{\bar{E}(r;p) + r[g(1) + (n-1)g(0) + n\bar{B}(0) - 1]M(p) \\ &\quad + rp[g(1) + (n-1)g(0) + n\bar{B}(0) - 1]\}. \end{aligned} \tag{3.35}$$

From (3.34) and (3.35), we get

$$\begin{aligned} M(r) = [\bar{E}(p^*;q^*) + p^*[g(1) + (n-1)g(0) + n\bar{B}(0) - 1]M(q^*)]^{-1} \Big\{ &\bar{E}(rp^*;q^*) \\ + M(q^*)\{\bar{E}(r;p^*) + r[g(1) + (n-1)g(0) + n\bar{B}(0) - 1]M(p^*) \\ + rp^*[g(1) + (n-1)g(0) + n\bar{B}(0) - 1]\} - \bar{E}(r;p^*q^*) \\ - r[g(1) + (n-1)g(0) + n\bar{B}(0) - 1]M(p^*q^*) \Big\}. \end{aligned}$$

This gives us that the mapping M is additive. Consequently (e)(ii) from Theorem 3.1 follows from (3.32) by suitably defining additive and bounded mappings. Similarly, (e)(i) from Theorem 3.1 follows from (e)(ii) from Theorem 3.1 and (3.18) by defining suitable additive and bounded mappings. This solution is already included in (e) from Theorem 3.1. □

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