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Results on solvability of nonlinear quadratic integral equations of fractional orders in Banach algebra

Sh. M. Al-Issa[a,b,*], N. M. Mawed[b]

[a]Department of Mathematics, faculty of Art and Science, Lebanese International University, Beirut, Lebanon.

[b]Department of Mathematics, faculty of Art and Science, The International University of Beirut, Saida, Lebanon.

Abstract

Here, we investigate the existence result for a nonlinear quadratic functional integral equation of fractional order using a fixed point theorem of Dhage. The continuous dependence of solution on the delay functions will be studied. As an application, an existence theorem for the fractional hybrid differential equations is proved. Also, we study a general quadratic integral equation of fractional order.

Keywords: Dhage fixed point theorem, continuous dependence of solutions, hybrid differential equations, general quadratic integral equation.

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1. Introduction

Fractional Calculus is a mathematical field that deals with derivatives and integrals of arbitrary orders. The fractional differential equations were of great importance due to their extensive development of fractional calculus and its applications [22–25].

Recently, there has been great interest for many authors to study quadratic functional integral equations, which has become one of the most attractive and interesting research areas of integral equations and functional integral equations and there are many significant existence results, we refer the reader to [1–3, 14, 21] for some of very recent results. Quadratic integral equations have many useful applications in describing numerous events and problems of the real world, for example, the theory of radiative transfer, kinetic theory of gases, the theory of neutron transport, the traffic theory, plasma physics, and numerous branches of mathematical physics.

The aim of this paper is to study the existence of solutions for a nonlinear quadratic functional-integral equations of fractional order

$$x(t) = k(t, x(\varphi_1(t))) + g(t, x(\varphi_2(t)))I^{\alpha} f(t, I^{\beta}\, u(t, x(\varphi_3(t)))), \tag{1.1}$$

*Corresponding author
Email addresses: shorouk.alissa@liu.edu.lb (Sh. M. Al-Issa), 31430473@students.liu.edu.lb (N. M. Mawed)

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where $\alpha, \beta \in (0,1)$, with $g \in C([0,T] \times R, R \setminus \{0\})$, $f, u \in C([0,T] \times R, R)$, and $k \in C([0,T] \times R, R)$. To obtain our existence results we use a fixed-point theorem in a Banach algebra due to Dhage [5]. Further, the continuous dependence of a unique solution on delay functions will be considered. Many authors use fixed point theorems to prove the existence of the solution to nonlinear integral equations, see [9–11, 13].

As application, we establish the existence results for the fractional hybrid differential equations (in short FHDE) involving D^α the Riemann-Liouville differential operators of order $\alpha \in (0,1)$

$$\begin{cases} D^\alpha\left(\frac{x(t)-k(t,x(\varphi_1(t)))}{g(t,x(\varphi_2(t)))}\right) = f(t, I^\beta\ u(t, x(\varphi_3(t)))), \quad t \in I = [0,T], \\ x(0) = k(0, x(0)), \end{cases}$$

where α, β, functions f, u, and g are defined like as in problem (1.1). The importance of studying hybrid differential equations lies in dynamic systems include special situations. The consideration of hybrid differential equations is implicit in the work of Krasnoselskii [18] and is extensively covered in many papers on hybrid differential equations with various disturbances, see [4, 7, 15, 16, 20, 26, 27].

As a second problem, we discuss the general quadratic integral equation of fractional orders

$$x(t) = \sum_{i=1}^{n} g_i(t, x(t)) I^{\alpha_i} f_i(t, I^{\beta_i}\ u_i(t, x(t))), \quad t \in I = [0,T],$$

by applying the fixed point theorem due to Dhage [19], where α_i, $\beta_i \in (0,1)$, functions f_i, u_i, and $g_i(t, x(t))$ are defined like functions f, u and g as in problem (1.1), $(i = 1, 2, \ldots, n)$. Some remarks and applications are given.

The main points of this paper is as follows. In Sect. 2, we recall a few important definitions and lemmas from fractional calculus used throughout this article. In Sect. 3, we state sufficient conditions which guarantee the existence of solutions to the Eq.(1.1). While Sect. 4, deals with the existence of continuous dependence of unique solutions for Eq. (1.1) on delay functions. In Sect. 5, we discuss existence result for initial value problem for fractional hybrid differential equation. The existence theorem of the general quadratic integral equation of fractional orders is presented in Sect. 6, where some remarks and applications are present. Our conclusion is presented in Sect. 7.

2. Preliminaries

In this section, we introduce some basic definitions and preliminary facts which we need in the sequel. Denote by $L^1(I)$ be the class of Lebesgue integrable functions on the interval $I = [0,T]$.

Definition 2.1. The Riemann-Liouville of fractional integral of the function $f \in L^1(I)$ of order $\alpha \in R^+$ is defined by (see [22–25])

$$I_a^\alpha\ f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ f(s)\ ds,$$

where $\Gamma(.)$ is Euler's Gamma function.

Definition 2.2. The (Caputo) fractional-order derivative D^α, $\alpha \in (0,1]$ and $t \in [a,b]$ of the absolutely continuous function g is defined as

$$D_a^\alpha\ g(t) = I_a^{1-\alpha}\ \frac{d}{dt}\ g(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\ \frac{d}{ds} g(s)\ ds.$$

For further properties of fractional calculus operator (see [22–25]).

Definition 2.3 ([6])**.** An algebra X is a vector space endowed with an internal composition law noted by

$$(.) : X \times X \to X,\ (x, y) \to x \cdot y,$$

which is associative and bilinear. A normed algebra is an algebra endowed with a norm satisfying the following property: For all $x, y \in X$, we have

$$\|x \cdot y\| \leqslant \|x\| \cdot \|y\|.$$

A complete normed algebra is called a Banach algebra.

Definition 2.4 ([6]). Let X be a normed vector space. A mapping $T : X \to X$ is called D-Lipschitzian, if there exists a continuous and nondecreasing function $\phi : R^+ \to R^+$, such that

$$\|Tx - Ty\| \leqslant \phi(\|x - y\|),$$

for all $x, y \in X$, where $\phi(0) = 0$.

Sometimes, we call the function ϕ to be a D-function of the mapping T on X. Obviously, every Lipschitzian mapping is D-Lipschitzian. Further, if $\phi(r) < r$, then T is called nonlinear contraction on X.

Now we state a useful lemma which are helpful in transforming the fractional differential equation into an equivalent Riemann-Louville integral equation.

Lemma 2.5 ([17]). *Let us consider* $0 < \alpha < 1$ *and* $w \in L^1(0, 1)$. *Then*

(H_1) *the equality* $D^\alpha I^\alpha w(t) = w(t)$ *holds;*

(H_2) *the equality*

$$I^\alpha D^\alpha w(t) = w(t) - \frac{I^{1-\alpha} w(t)|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1}$$

holds almost everywhere on I.

Lemma 2.6 ([5]). *Let* S *be a nonempty, closed convex and bounded subset of a Banach algebra* X *and let* $A, C : X \to X$ *and* $B : S \to X$ *be three operators such that:*

(a) A *and* C *are Lipschitzian with Lipschitz constants* δ *and* ρ, *respectively;*
(b) B *is completely continuous; and*
(c) $x = AxBy + Cx \Rightarrow x \in S$ *for all* $y \in S$;
(d) $\delta M + \rho < r$, *for* $r > 0$, *where* $M = \|B(S)\|$.

Then the operator equation $AxBx + Cx = x$ *has a solution in* S.

Let $X = C(I, R)$ be the vector of all real-valued continuous functions on $I = [0, T]$. We equip the space X with the norm $\|x\| = \sup_{t \in I} |x(t)|$. Clearly, $C(I, R)$ is a complete normed algebra with respect to this supremum norm.

By a solution of the quadratic functional integral equation of fractional order (1.1) we mean a function $x \in C(I, R)$ that satisfies Eq. (1.1).

3. Main results

In this section, we will study Eq. (1.1), using the following fixed point theorem for three operators in a Banach algebra X, due to Dhage [5].

Consider the following assumptions.

(A_1) The functions $g : [0, T] \times R \to R \setminus \{0\}$, and $k : [0, T] \times R \to R$ are continuous and there exist two positive functions $k(t)$, $L(t)$, with norms $\|k\|$ and $\|L\|$, respectively, such that

$$|k(t, x) - k(t, y)| \leqslant k(t)|x - y|, \qquad |g(t, x) - g(t, y)| \leqslant L(t)|x - y|,$$

for all $t \in I$ and $x, y \in R$.

(A_2) The functions $f : [0,T] \times R \to$ and $u : [0,T] \times R \to$ satisfy Caratheodory condition, i.e., f and u are measurables in t for any $x \in R$ and continuous in x for almost all $t \in [0,T]$. There exists three functions $t \to a(t)$, $t \to b(t)$ and $t \to m(t)$, such that

$$|f(t,x)| \leqslant a(t) + b(t)|x|, \ \forall \ (t,x) \in I \times R, \qquad |u(t,x)| \leqslant m(t), \ \forall \ (t,x) \in I \times R,$$

where $a(.)$, $m(.) \in L^1$ and $b(.)$ are measurable and bounded. And $I_c^{\gamma} a(.) \leqslant M_1$, $I_c^{\gamma} m(.) \leqslant M_2$, $\forall \ \gamma \leqslant \alpha$, $c \geqslant 0$.

(A_3) $\varphi_i : I \to I$, are continuous functions with $\varphi_i(0) = 0$, $i = 1,2,3$.

(A_4) There exists a number $r > 0$ such that

$$r \geqslant \frac{G + H + \frac{G\ M_1\ T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{G\ \|b\| M_2\ T^{\alpha+\beta-\gamma}}{\Gamma(\alpha+\beta-\gamma+1)}}{1 - [q\|L\| + \|k\| + \frac{\|L\|\ M_1 T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{\|L\|\ \|b\| M_2 T^{\alpha+\beta-\gamma}}{\Gamma(\alpha+\beta-\gamma+1)}]}, \tag{3.1}$$

where $H = \sup_{t\in I} |k(t,0)|$, $G = \sup_{t \in I} |g(t,0)|$, and

$$\|L\| + \|k\| + \frac{\|L\|\ M_1\ T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{\|L\|\ \|b\| M_2\ T^{\alpha+\beta-\gamma}}{\Gamma(\alpha+\beta-\gamma+1)} < 1.$$

At this stage, our target is to prove the following existence theorem

Theorem 3.1. *Assume that the hypotheses* (A_1)-(A_4) *hold. Then the quadratic functional integral equation* (1.1) *has at least one solution defined on* I.

Proof. Set $X = C(I,R)$ and define a subset S of X as

$$S := \{x \in X,\ \|x\| \leqslant r\},$$

where r satisfies inequality (3.1). Clearly S is closed, convex, and bounded subset of the Banach space X. Now we define three operators; $A : X \to X$, $B : S \to X$ and $C : X \to X$ defined by:

$$Ax(t) = g(t, x(\varphi_2(t))), \quad t \in I, \quad Bx(t) = I^{\alpha}\ f(t, I^{\beta}\ u(t, x(\varphi_3(t))), \ t \in I, \quad Cx(t) = k(t, x(\varphi_1(t))).$$

Then the integral Eq. (1.1) can be written as:

$$x(t) = Ax(t) \cdot Bx(t) + Cx(t), \quad t \in I.$$

We shall show that A, B, and C satisfy all the conditions of Lemma 2.6. This will be achieved in the following series of steps.

Step 1. We first show that A and C are Lipschitzian on X. To see this, let $x, y \in X$, so

$$|Ax(t) - Ay(t)| = |g(t, x(t)) - g(t, y(t))| \leqslant L(t)\ |x(t) - y(t)| \leqslant \|L\|\ \|x - y\|,$$

which implies $\|Ax - Ay\| \leqslant \|L\|\ \|x - y\|$, for all $x, y \in X$. Therefore, A is a Lipschitzian on X with Lipschitz constant $\|L\|$.

In a similar way, we can deduce that

$$\|Cx - Cy\| \leqslant \|k\| \|x - y\|,$$

for all $x, y \in S$. This shows that C is a Lipschitz mapping on X with the Lipschitz constant $\|k\|$.

Step 2. We show that B is a compact and continuous operator on S into X.

First we show that B is continuous on X. Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$.

Then by the Lebesgue dominated convergence theorem, let us assume that $t \in I$ and since $u(t, x(t))$ is continuous in X, then $u(t, x_n(t))$ converges to $u(t, x(t))$, (see assumption (A_2)) applying Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty} I^{\beta} u(s, x_n(\varphi_3(s))) = I^{\beta} u(s, x(\varphi_3(s))).$$

Also, since $f(t, x(t))$ is continuous in x, then using the properties of the fractional-order integral and applying Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty} Bx_n(t) = \lim_{n\to\infty} I^{\alpha} f(t, I^{\beta} u(t, x_n(\varphi_3(t)))) = I^{\alpha} f(t, I^{\beta} u(t, x(\varphi_3(t)))) = Bx(t).$$

Thus, $Bx_n \to Bx$ as $n \to \infty$ uniformly on R^+, and hence B is a continuous operator on S into S. Now, we show that B is a compact operator on S. It is enough to show that $B(S)$ is a uniformly bounded and equicontinuous set in X. Let $x \in S$ be arbitrary. Then by hypothesis (A_2),

$$\begin{aligned}
|Bx(t)| &\leqslant \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, I^{\beta}u(s, x(\varphi_3(s))))| ds \\
&\leqslant \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [a(s) + b(s) I^{\beta}|u(s, x(\varphi_3(s))|] ds \\
&\leqslant \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} a(s) ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |b(s)| I^{\beta} |u(s, x(\varphi_3(s)))|] ds \\
&\leqslant I^{\alpha} a(t) + \|b\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} I^{\beta} m(s) ds \\
&\leqslant I^{\alpha} a(t) + \|b\| I^{\alpha+\beta} m(t) \\
&\leqslant I^{\alpha-\gamma} I^{\gamma} a(t) + \|b\| I^{\alpha+\beta-\gamma} I^{\gamma} m(t) \\
&\leqslant M_1 \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} ds + \|b\| M_2 \int_0^t \frac{(t-s)^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha+\beta-\gamma)} ds \\
&\leqslant M_1 \frac{T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \|b\| M_2 \frac{T^{\alpha+\beta-\gamma}}{\Gamma(\alpha+\beta-\gamma+1)},
\end{aligned}$$

for all $t \in I$. Taking supermom over t,

$$\|Bx(t)\| \leqslant M_1 \frac{T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \|b\| M_2 \frac{T^{\alpha+\beta-\gamma}}{\Gamma(\alpha+\beta-\gamma+1)} = K.$$

Then

$$\|Bx(t)\| \leqslant K,$$

for all $x \in S$. This shows that B is uniformly bounded on S.

Now, we proceed to show that $B(S)$ is also equicontinuous set in X. Let $t_1, t_2 \in I$, and $x \in S$. Without loss of generality assume that $t_1 < t_2$, then we have

$$\begin{aligned}
&(Bx)(t_2) - (Bx)(t_1) \\
&\quad \leqslant \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, I^{\beta}u(s, x(\varphi_3(s))))) ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, I^{\beta}u(s, x(\varphi_3(s))))) ds \\
&\quad \leqslant \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, I^{\beta}u(s, x(\varphi_3(s))))) ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, I^{\beta}u(s, x(\varphi_3(s))))) ds \\
&\qquad - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, I^{\beta}u(s, x(\varphi_3(s)))) ds
\end{aligned}$$

$$\leqslant \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, I^\beta u(s, x(\varphi_3(s))))ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, I^\beta u(s, x(\varphi_3(s))))\, ds,$$

and

$$\begin{aligned}
&|(Bx)(t_1)-(Bx)(t_2)| \\
&\leqslant \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, I^\beta u(s, x(\varphi_3(s))))|ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, I^\beta u(s, x(\varphi_3(s))))|ds \\
&\leqslant \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \left[a(s)+b(s)I^\beta|u(s, x(\varphi_3(s))|\right] ds \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \left[|a(s)|+|b(s)|I^\beta|u(s, x(\varphi_3(s))|\right]\, ds \\
&\leqslant \|a\| \left[\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right] \\
&\quad + \|b\| \left[\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} I^\beta|u(s, x(\varphi_3(s))|\, ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} I^\beta|u(s, x(\varphi_3(s)))|\, ds \right] \\
&\leqslant \|a\| \left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)} \right) \\
&\quad + \|b\| \left[\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} I^\beta m(s) ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} I^\beta m(s) ds \right] \\
&\leqslant \|a\| \left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)} \right) \\
&\quad + \|b\| \left[\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} I^{\beta-\gamma} I^\gamma m(s) ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} I^{\beta-\gamma} I^\gamma m(s)\, ds \right] \\
&\leqslant \|a\| \left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)} \right) + \|b\| M_2 \left[\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-\tau)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} d\tau ds \right. \\
&\quad \left. + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-\tau)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} d\tau ds \right] \\
&\leqslant \|a\| \left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)} \right) + \|b\| M_2 \left[\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{(s)^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} ds \right. \\
&\quad \left. + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{(s)^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} ds \right] \\
&\leqslant \|a\| \left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)} \right) + \|b\| M_2 \left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|\, T^{\beta-\gamma}}{\Gamma(\alpha+1)\Gamma(\beta-\gamma+1)} \right),
\end{aligned}$$

i.e.,

$$|(Bx)(t_2)-(Bx)(t_1)| \leqslant \|a\| \left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|}{\Gamma(\alpha+1)} \right) + \|b\| M_2 \left(\frac{|t_2^\alpha - t_1^\alpha - 2(t_2-t_1)^\alpha|\, T^{\beta-\gamma}}{\Gamma(\alpha+1)\Gamma(\beta-\gamma+1)} \right).$$

Hence, for $\epsilon > 0$, there exists $\delta > 0$ such that

$$|t_2 - t_1| < \delta \quad \Longrightarrow \quad |(Bx)(t_2)-(Bx)(t_1)| < \epsilon,$$

for all $t_1, t_2 \in I$ and for all $x \in S$. This shows that $B(S)$ is an equicontinuous set in X. Now, the set $B(S)$ is a uniformly bounded and equicontinuous set in X, so it is compact by the Arzela-Ascoli theorem. As a result, B is a complete continuous operator on S.

Step 3. The hypothesis (c) of Lemma 2.6 is satisfied. Let $x \in X$ and $y \in S$ be arbitrary elements such that $x = AxBy + Cx$. Then we have

$$\begin{aligned}
|x(t)| &\leqslant |Ax(t)||By(t)| + |Cx(t)| \\
&\leqslant |g(t, x(\varphi_2(t)))| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, I^\beta\, u(s, y(\varphi_3(s))))| ds + |k(t, x(\varphi_1(s)))| \\
&\leqslant \left[|g(t, x(\varphi_2(t))) - g(t,0)| + |g(t,0)|\right] \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, I^\beta\, u(s, y(\varphi_3(s))))| ds \\
&\quad + |k(t, x(\varphi_1(t))) - k(t,0)| + |k(t,0)| \\
&\leqslant \left[\|L\||x(\varphi_2(t))| + G\right] \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[a(s) + b(s) I^\beta |u(s, y(\varphi_3(t))|\right] ds + \|k\||x(\varphi_1(t))| + H \\
&\leqslant [\|L\||x(\varphi_2(t))| + G] + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [a(s) + |b(s)| I^\beta m(s)] ds + \|k\||x(\varphi_1(t))| + H \\
&\leqslant [\|L\|r + G] + I^\alpha a(t) + \|b\| I^{\alpha+\beta} m(t) + \|k\|r + H \\
&\leqslant [\|L\|r + G] \left(I^{\alpha-\gamma} I^\gamma a(t) + \|b\| I^{\alpha+\beta-\gamma} I^\gamma m(t)\right) + \|k\|r + H \\
&\leqslant [\|L\|r + G] \left(M_1 \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} ds + \|b\| M_2 \int_0^t \frac{(t-s)^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha+\beta-\gamma)} ds\right) + \|k\|r + H \\
&\leqslant [\|L\|r + G] \left(M_1 \frac{s^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \|b\| M_2 \frac{s^{\alpha+\beta-\gamma}}{\Gamma(\alpha+\beta-\gamma+1)}\right) + \|k\|\ r\ + H.
\end{aligned}$$

Therefore

$$|x(t)| \leqslant \|k\|\ r\ + H + [\|L\|r + G] \left(M_1 \frac{T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \|b\| M_2 \frac{T^{\alpha+\beta-\gamma}}{\Gamma(\alpha+\beta-\gamma+1)}\right).$$

Taking supremum over t,

$$\|x(t)\| \leqslant \|k\|\ r\ + H + [\|L\|r + G] \left(\frac{M_1\, T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{\|b\| M_2 T^{\alpha+\beta-\gamma}}{\Gamma(\alpha+\beta-\gamma+1)}\right) \leqslant r.$$

Therefore, $x \in S$.

Step 4. Finally we show that $\delta M + \rho < 1$, that is, (d) of Theorem 2.6 holds. Since

$$M = \|B(S)\| = \sup_{x \in S} \left\{\sup_{t \in J} |Bx(t)|\right\} \leqslant \frac{M_1\, T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{\|b\| M_2\, T^{\alpha+\beta-\gamma}}{\Gamma(\alpha+\beta-\gamma+1)},$$

and by (A_4) we have

$$\|L\|\ M\ +\ \|k\|\ < 1,$$

with $\delta = \|L\|$ and $\rho = \|k\|$.

Thus all the conditions of lemma 2.6 are satisfied and hence the operator equation $x = AxBx + Cx$ has a solution in S. In consequence, Eq. (1.1) has a solution on I. This completes the proof. □

4. Continuous dependence

In this section, we give sufficient conditions for the uniqueness of the solution of Eq. (1.1) and study the continuous dependence of this solution on the delay functions $\varphi_i(t)$.

4.1. Uniqueness of the solution

Let us assume the following assumption

(A_2^*) Let $f : [0,T] \times R \to R$ and $u : [0,T] \times R \to R$ be a continuous functions satisfying the Lipschitz condition and there exists two positive functions $w(t)$, $\theta(t)$ with bounded $\|w\|$ and $\|\theta\|$, such that

$$|f(t,x) - f(t,y)| \leqslant w(t)|x-y|, \qquad |u(t,x) - u(t,y)| \leqslant \theta(t)|x-y|,$$

where $F = \sup_{t\in I} |f(t,0)|$, and $U = \sup_{t\in I} |u(t,0)|$.

Theorem 4.1. *Let the assumptions of Theorem 3.1 be satisfied with replace condition* (A_2) *by* (A_2^*)*. Then the solution* $x \in C[0,T]$ *of Eq.* (1.1) *is unique, if*

$$\|k\| + \frac{\|w\|\left[\|\theta\|\|x\| + U\right] \ \|L\| \ T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\|L\| \ F \ T^{\alpha}}{\Gamma(\alpha+1)} + \frac{\left[\|L\|\|x\| + G\right]\|w\|\|\theta\| \ T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} < 1.$$

Proof. Firstly, we notice that condition (A_2^*) implies condition (A_2) for function f. Let x, y be two solutions of Eq. (1.1), then

$$\begin{aligned}
|x(t) - y(t)| &\leqslant |k(t, x(\varphi_1(t))) - k(t, y(\varphi_1(t)))| \\
&\quad + |g(t, x(\varphi_2(t)))I^{\alpha} f(t, I^{\beta}\, u(t, x(\varphi_3(t))))) - g(t, y(\varphi_2(t)))I^{\alpha} f(t, I^{\beta}\, u(t, y(\varphi_3(t))))| \\
&\leqslant k(t)|x(\varphi_1(t)) - y(\varphi_1(t))| \\
&\quad + \left|g(t, x(\varphi_2(t))) - g(t, y(\varphi_2(t)))\right| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, I^{\beta} u(s, x(\varphi_3(s)))| ds \\
&\quad + |g(t, x(\varphi_2(t)))| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|f(s, I^{\beta} u(s, x(\varphi_3(s))) - f(s, I^{\beta} u(s, y(\varphi_3(s)))\right| \, ds \\
&\leqslant |k(t)||x(\varphi_1(t)) - y(\varphi_1(t))| \\
&\quad + L(t)|x(\varphi_2(t)) - y(\varphi_2(t))| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[|f(s, I^{\beta} u(s, x(\varphi_3(s))) - f(s,0)| + |f(s,0)|\right] ds \\
&\quad + \left[|g(t, x(\varphi_2(t))) - g(t,0)| + |g(t,0)|\right] \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} w(s) \left|I^{\beta} u(s, x(\varphi_3(s))) - I^{\beta} u(s, y(\varphi_3(s)))\right| ds \\
&\leqslant \ |k(t)| \ |x(\varphi_1(t)) - y(\varphi_1(t))| \\
&\quad + |L(t)||x(\varphi_2(t)) - y(\varphi_2(t))| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[|w(s)| I^{\beta} |u(s, x(\varphi_3(s)))| + F\right] ds \\
&\quad + \left[L(t)|x(\varphi_2(t))| + G\right] \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |w(s)| \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} \left|u(\tau, x(\varphi_3(\tau))) - u(\tau, y(\varphi_3(\tau)))\right| \, ds \, d\tau \\
&\leqslant \ \|k\| \ \|x-y\| + \ \|L\|\|x-y\| \left[\|w\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} \left[|\theta(\tau)||x(\varphi_3(\tau)| + U\right] ds \, d\tau \right. \\
&\quad \left. + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F ds \right] + [\|L\|\|x\| + G]\|w\|\|\theta\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} |x(\varphi_3(\tau) - y(\varphi_3(\tau)| ds d\tau \\
&\leqslant \ \|k\|\|x-y\| + \ \|L\|\|x-y\| \left[\|w\|[\|\theta\|\|x\| + U] \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + F \ \frac{T^{\alpha}}{\Gamma(\alpha+1)}\right] \\
&\quad + [\|L\|\|x\| + G]\|w\|\|\theta\|\|x-y\| \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.
\end{aligned}$$

Taking supremum over t, we conclude that

$$\|x-y\| \leqslant \left(\|k\| + \ \|L\| \frac{\|w\|[\|\theta\|\|x\| + U] \ T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \|L\| \frac{F \ T^{\alpha}}{\Gamma(\alpha+1)} + \frac{[\|L\|\|x\| + G]\|w\|\|\theta\| \ T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right) \|x-y\|.$$

Then

$$\left[1-\left(\|k\|+\frac{\|w\|[\|\theta\|\|x\|+U]\ \ \|L\|\ T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{\|L\|\ F\ T^{\alpha}}{\Gamma(\alpha+1)}+\frac{[\|L\|\|x\|+G]\|w\|\|\theta\|\ T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)\right]\|x-y\|\leqslant 0.$$

This yields that the uniqueness of solution for Eq. (1.1). □

4.2. Continuous dependence on the delay functions

Next we prove the continuous dependence of the unique solutions on the delay functions $\varphi_i(t)$.

Definition 4.2. The solution of Eq. (1.1) depends continuously on the delay functions $\varphi_i(t)$ if $\forall\ \epsilon>0$, $\exists\ \delta>0$, such that

$$|\varphi_i(t)-\varphi_i^*(t)|\leqslant\delta \quad\Longrightarrow\ \|x-x^*\|\leqslant\epsilon.$$

Theorem 4.3. *Let the assumptions of Theorem 4.1 be satisfied. Then the solution of the Eq. (1.1) depends continuously on the delay function $\varphi_1(t)$.*

Proof. For $x,\ x^*$ be two solutions of the Eq. (1.1). Let $\delta>0$ be given such that $|\varphi_1(t)-\varphi_1^*(t)|\leqslant\delta$, $\forall\ \epsilon\geqslant 0$. Then

$$\begin{aligned}
&|x(t)-x^*(t)|\\
&\leqslant |k(t,x(\varphi_1(t)))-k(t,x^*(\varphi_1^*(t)))|+\Big|g(t,x(\varphi_2(t)))\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,I^{\beta}u(s,x(\varphi_3(s)))ds\\
&\quad-g(t,x^*(\varphi_2(t)))\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,I^{\beta}u(s,x^*((\varphi_3(s)))ds\Big|\\
&\leqslant |k(t,x(\varphi_1(t)))-k(t,x^*(\varphi_1^*(t)))|+\Big|g(t,x(\varphi_2(t)))\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,I^{\beta}u(s,x(\varphi_3(s)))ds\\
&\quad-g(t,x^*(\varphi_2(t)))\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,I^{\beta}u(s,x(\varphi_3(s)))ds\Big|\\
&\quad+\Big|g(t,x^*(\varphi_2(t)))\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,I^{\beta}u(s,x(\varphi_3(s)))ds\\
&\quad-g(t,x^*(\varphi_2(t)))\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,I^{\beta}u(s,x^*(\varphi_3(s)))ds\Big|\\
&\leqslant k(t)|x(\varphi_1(t))\ -x^*(\varphi_1^*(t))|+\Big|g(t,x(\varphi_2(t)))-g(t,x^*(\varphi_2(t)))\Big|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,I^{\beta}\ u(s,x(\varphi_3(s)))|\ ds\\
&\quad+|g(t,x^*(\varphi_2(t)))|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big|f(s,I^{\beta}\ u(s,x(\varphi_3(s)))-f(s,I^{\beta}\ u(s,x^*(\varphi_3(s)))\Big|\ ds\\
&\leqslant\ |k(t)||x(\varphi_1(t))\ -x^*(\varphi_1(t))+x^*(\varphi_1(t))\ -x^*(\varphi_1^*(t))|\\
&\quad+L(t)|x(\varphi_2(t))-x^*(\varphi_2(t))|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big[|f(s,I^{\beta}u(s,x(\varphi_3(s)))-f(s,0)|+|f(s,0)|\Big]ds\\
&\quad+[L(t)|x^*(\varphi_2(t))|+G]\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}w(t)\Big|I^{\beta}u(s,x(\varphi_3(s)))-I^{\beta}u(s,x^*(\varphi_3(s)))\Big|\ ds\\
&\leqslant\ \|k\|\Big[|x(\varphi_1(t))\ -x^*(\varphi_1(t))|+|x^*(\varphi_1(t))\ -x^*(\varphi_1^*(t))|\Big]\\
&\quad+|L(t)||x(\varphi_2(t))-x^*(\varphi_2(t))|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[w(t)|I^{\beta}|u(s,x(\varphi_3(s)))|+F]ds\\
&\quad+(|L(t)||x^*(\varphi_2(t))|+G)\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|w(t)|\int_0^{\tau}\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\Big|u(\tau,x(\varphi_3(\tau)))-u(\tau,x^*(\varphi_3(\tau)))\Big|\ ds\ d\tau\\
&\leqslant\ \|k\|\Big[\|x\ -x^*\|+|x^*(\varphi_1(t))\ -x^*(\varphi_1^*(t))|\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\|L\|\|x-x^*\|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[|w(t)|I^{\beta}\left[|\theta(t)|u(s,x(\varphi_3(s)))-u(s,0)|+|u(s,0)|\right]+F\big]ds\\
&+(\|L\|\|x^*\|+G)\ \|w\|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\theta(t)|\int_0^{\tau}\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}|x(\varphi_3(\tau))-x^*(\varphi_3(\tau))|\ ds\ d\tau\\
\leqslant\ &\|k\|\left[\|x-x^*\|+|x^*(\varphi_1(t))-x^*(\varphi_1^*(t))|\right]\\
&+\|L\|\|x-x^*\|\|w\|[\|\theta\|\|x\|+U]\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\tau}\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}ds d\tau\\
&+\|L\|\|x-x^*\|\ F\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\ (\|L\|\|x\|+G)\ \|w\|\ \|x-x^*\|\ \|\theta\|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\tau}\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}ds\ d\tau.
\end{aligned}
$$

Taking supremum over t,

$$
\begin{aligned}
\|x\ -x^*\|\leqslant\ &\|k\|\left[\|x-x^*\|+|x^*(\varphi_1(t))-x^*(\varphi_1^*(t))|\right]+\|L\|\|x-x^*\|\left[\frac{\|w\|[\|\theta\|\|x\|+U]\ T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{F\ T^{\alpha}}{\Gamma(\alpha+1)}\right]\\
&+(\|L\|\|x\|+G)\|w\|\|\theta\|\|x-x^*\|\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},\\
\|x\ -x^*\|\leqslant\ &\frac{\|k\||x^*(\varphi_1(t))-x^*(\varphi_1^*(t))|}{1-\left(\|k\|+\|L\|\left[\frac{\|w\|[\|\theta\|\|x\|+U]\ T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{F\ T^{\alpha}}{\Gamma(\alpha+1)}\right]+[\|L\|\|x\|+G]\|w\|\|\theta\|\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)}.
\end{aligned}
$$

But from the continuity of solution x^*, we have

$$|\varphi_1(t)-\varphi_1^*(t)|\leqslant\delta\quad\Longrightarrow\quad|x^*(\varphi_1(t))-x^*(\varphi_1^*(t))|\leqslant\epsilon_1.$$

Then

$$\|x\ -x^*\|\leqslant\frac{\|k\|\epsilon_1}{1-\left(\|k\|+\|L\|\left[\frac{\|w\|[\|\theta\|\|x\|+U]\ T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{F\ T^{\alpha}}{\Gamma(\alpha+1)}\right]+[\|L\|\|x\|+G]\|w\|\|\theta\|\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)}\leqslant\epsilon.$$

This means that the solution of Eq. (1.1) depends continuously on delay function φ_1. This completes the proof.

By a similar way as done above, the continuous dependence of the solution of Eq. (1.1) on delay functions φ_2 and φ_3 can be studied. □

5. Fractional hybrid differential equation

Here, we show that Theorem 3.1 could also be used to discuss existence result for initial value problem of fractional hybrid differential equation (FHDE):

$$
\begin{cases}
D^{\alpha}\left(\frac{x(t)-k(t,x(\varphi_1(t)))}{g(t,x(\varphi_2(t)))}\right)=f(t,I^{\beta}\ u(t,x(\varphi_3(t)))),\quad t\in I=[0,T],\\
x(0)=k(0,x(0)),
\end{cases}
\tag{5.1}
$$

where D^{α} denotes the Riemann-Liouville fractional derivative of order α, $0<\alpha<1$, I^{β} is the Riemann-Liouville fractional integral of order β, $0<\beta<1$, where $g(t,x(t))\in C(I\times R, R\setminus\{0\})$, $f(t,x(t))$, $u(t,x(t))$ and $k(t,x(t))\in C(I\times R, R)$.

By a solution of the FHDE (5.1) we mean a function $x\in C(I,R)$ such that

(i) the function $t\to\frac{x(t)-k(t,x(\varphi_1(t)))}{g(t,x(\varphi_2(t)))}$ is continuous for each $x\in C(J,R)$; and
(ii) x satisfies the equations in (5.1).

Theorem 5.1. *Assume that the hypotheses* (A_1)-(A_4) *of Thorem* 3.1 *hold. Then the FHDE* (5.1) *has at least one solution defined on* I.

Proof. Let $x(t)$ be a solution of Eq. (5.1). Applying Riemann-Liouville fractional integral of order α on both sides of (5.1), we obtain

$$I^\alpha D^\alpha\left(\frac{x(t)-k(t,x(\varphi_1(t)))}{g(t,x(\varphi_2(t)))}\right)=I^\alpha f(t,I^\beta\ u(t,x(\varphi_3(t)))),$$

so, from Lemma 2.5 we conclude that

$$\frac{x(t)-k(t,x(\varphi_1(t)))}{g(t,x(\varphi_2(t)))}-\frac{I^{1-\alpha}[\frac{x(t)-k(t,x(\varphi_1(t)))}{g(t,x(\varphi_2(t)))}]|_{t=0}}{\Gamma(\alpha)}t^{\alpha-1}=I^\alpha f(t,I^\beta u(t,x(\varphi_3(t))),\ t\in I.$$

Since $\frac{x(t)-k(t,x(\varphi_1(t)))}{g(t,x(\varphi_2(t)))}|_{t=0}=\frac{x(0)-k(0,x(0))}{g(0,x(0))}=\frac{0}{g(0,x(0))}=0,$ (given $g(0,x(0))\neq 0$), hence

$$\frac{x(t)-k(t,x(\varphi_1(t)))}{g(t,x(\varphi_2(t)))}=I^\alpha f(t,I^\beta u(t,x(\varphi_3(t))),$$

i.e.,

$$x(t)=g(t,x(\varphi_2(t)))I^\alpha f_1(t,I^\beta f_2(t,x(\varphi_3(t)))+k(t,x(\varphi_1(t)))\quad t\in I.$$

Thus, Eq. (1.1) holds.

Conversely, assume that $x(t)$ satisfies Eq. (1.1), with this form

$$x(t)-k(t,x(\varphi_1(t)))=g(t,x(\varphi_2(t)))I^\alpha f_1(t,I^\beta f_2(t,x(\varphi_3(t)))\quad t\in I. \tag{5.2}$$

So dividing (5.2) by $g(t,x(\varphi_2(t)))$ and applying D^α for both sides of (5.2), we obtain FHDE (5.1),

$$D^\alpha\left(\frac{x(t)-k(t,x(\varphi_1(t)))}{g(t,x(\varphi_2(t)))}\right)=I^\alpha f_1(t,I^\beta f_2(t,x(\phi(t))).$$

Again, substituting $t=0$ in Eq. (1.1) (due to the fact that $g(0,x(0))\neq 0$ and $\varphi_i(0)=0,\ i=1,2,3.$), then

$$\frac{x(0)-k(0,x(0))}{g(0,x(0))}\to 0\ \text{ as }\ t\to 0.$$

This yield that

$$x(0)=k(0,x(0)).$$

Now an application of Theorem 3.1 , we deduce that the FHDE (5.1) has at least one solution defined on I. This completes the proof □

6. General quadratic integral equation of fractional order

In this section, we study the general quadratic integral equation of fractional order

$$x=\sum_{i=1}^{m}g_i(t,x(t))\ I^{\alpha_i}\ f_i(t,I^{\beta_i}\ u_i(t,x(t)),\quad \alpha_i,\ \beta_i\in(0,1), \tag{6.1}$$

by applying the following fixed point theorem point theorem due to Dhage.

Lemma 6.1 ([19]). *Let* m *be a positive integer, and* S^* *be a nonempty, closed, convex and bounded subset of a Banach algebra* X. *Assume that the operators* $A_i : X\to X$ *and* $B_i : S^*\to X$, $i=1,2,\dots,m$, *satisfy*

(1) *for each* $i\in\{1,2,\dots,m\}$, A_i *are Lipschitzian with Lipschitz constant* δ_i;

(2) *for each* $i \in \{1, 2, \ldots, m\}$, B_i *are continuous and* $B_i(S^*)$ *are precompact;*
(3) *for each* $y \in S^*$, $x = \sum_{i=1}^{m} A_i x \cdot B_i y$ *implies that* $x \in S^*$.

Then, the operator equation $x = \sum_{i=1}^{m} A_i x \cdot B_i x$ *has a solution provided that*

$$\sum_{i=1}^{m} F_i \delta_i < r, \quad \forall r > 0,$$

where $F_i = \sup_{x \in S^*} \|B_i x\|$, $i = 1, 2, \ldots, m$.

Eq. (6.1) is investigated under following assumptions.

(B_1) $f_i : I \times R \to R$, and $u_i : I \times R \to R$, $i = 1, 2, \ldots, m$, satisfy Caratheodory condition, i.e., f_i and u_i are measurables in t for any $x \in R$ and continuous in x for almost all $t \in I$. There exists functions $t \to a_i(t)$, $t \to b_i(t)$, $t \to m_i(t)$, such that

$$|f_i(t, x)| \leqslant \; a_i(t) + \; b_i(t)|x|, \quad i = 1, 2, \ldots, m, \quad \forall \,(t, x) \in I \times R,$$
$$|u_i(t, x)| \leqslant \; m_i(t), \quad i = 1, 2, \ldots, m, \quad \forall \,(t, x) \in I \times R,$$

where $a_i(.)$, $m_i(.) \in L^1$ and $b_i(.)$ are measurable and bounded. And $I_c^{\gamma}\, a_i(.) \leqslant M_{1i}$ and $I_c^{\gamma}\, c_i(.) \leqslant M_{2i}, \;\; \forall \;\; \gamma_i \;\leqslant \alpha_i, \; c \geqslant 0$.

(B_2) $g_i : I \times R \to R \backslash \{0\}$, $\;i = 1, 2, \ldots, m$ are continuous and there exists a positive function $L_i(t)$, with norm $\|L_i\|$ satisfying

$$|g_i(t, x) - g_i(t, y)| \;\leqslant L_i(t)|x - y|, \quad i = 1, 2, \ldots, m$$

for all $t \in I$ and $x, y \in R$.

(B_3) There exists a number $r^* > 0$ such that

$$r^* \geqslant \frac{\sum_{i=1}^{m} \left[\|G_i\| + \frac{G_i \; M_{1i} \; T^{\alpha_i - \gamma_i}}{\Gamma(\alpha_i - \gamma_i + 1)} + \frac{G_i \;\; \|b_i\| M_{2i} \; T^{\alpha_i + \beta_i - \gamma}}{\Gamma(\alpha_i + \beta_i - \gamma_i + 1)}\right]}{1 - \sum_{i=1}^{m} \left[\|L_i\| + \frac{\|L_i\| \; M_{1i} T^{\alpha_i - \gamma_i}}{\Gamma(\alpha_i - \gamma_i + 1)} + \frac{+\|L_i\| \;\; \|b_i\| M_{2i} T^{\alpha_i + \beta_i - \gamma_i}}{\Gamma(\alpha_i + \beta_i - \gamma_i + 1)}\right]},$$

where $G_i = \sup_{t \in I} |g_i(t, 0)|$, and

$$\sum_{i=1}^{m} \left(\|L_i\| + \frac{\|L_i\| \; M_{1i} T^{\alpha_i - \gamma_i}}{\Gamma(\alpha_i - \gamma_i + 1)} + \frac{\|L_i\| \; \|b_i\| M_{2i} T^{\alpha_i + \beta_i - \gamma_i}}{\Gamma(\alpha_i + \beta_i - \gamma_i + 1)} \right) < 1.$$

Theorem 6.2. *Let the assumptions* (B_1)-(B_3) *be satisfied. Then the general Eq.* (6.1) *has at least one solution defined on* I.

Proof. Let us define a subset S^* of $X = C(I, R)$ by

$$S^* := \{x \in X, \|x\| \leqslant r\}.$$

Obviously, S^* is nonempty, bounded, convex and closed subset of $C(I, R)$. Consider the operators $A_i : X \to X$, $\;B_i : S^* \to X$ defined by:

$$(A_i x)(t) = g_i(t, x(t)), \quad (B_i x)(t) = I^{\alpha_i} \; f_i(t, I^{\beta_i} \; u_i(t, x(t))).$$

Then Eq. (6.1) can be written in the form:

$$x(t) = \sum_{i=1}^{m} A_i(x) \cdot B_i(x), \;\; i = 1, 2, \ldots, m.$$

We shall show that A_i and B_i satisfy all the conditions of Theorem 6.2.

Step 1. As done before in the proof of Theorem 3.1, we can deduce that

$$\|A_i x - A_i y\| \leqslant \|L_i\| \|x - y\|,$$

for all $x, y \in S^*$. This shows that A_i are Lipschitz mappings on S^* with the Lipschitz constants $\|L_i\|$.

Step 2. we show that B_i are compact and continuous operators on S^* into X.

First we show that B_i are continuous on X. Let $\{x_n\}$ be a sequence in S^* converging to a point $x \in S^*$. Then by the Lebesgue dominated convergence theorem, let us assume that $t \in I$ and since $u_i(t, x(t))$ are continuous in X, then $u_i(t, x_n(t))$ converges to $u_i(t, x(t))$, (see assumption (B_1)) applying Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty} I^{\beta}\, u_i(s, x_n(s)) \;=\; I^{\beta}\, u_i(s, x(s)).$$

Also, since $f_i(t, x(t))$ are continuous in x, then using the properties of the fractional-order integral and applying Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty} B_i x_n(t) = \lim_{n\to\infty} \left(I^{\alpha}\, f_i(t, I^{\beta}\, u_i(t, x_n(t))\right) = I^{\alpha}\, f_i(t, I^{\beta}\, u_i(t, x(t)) = B_i x(t).$$

Thus, $B_i x_n \to B_i x$ as $n \to \infty$ uniformly on R^+, and hence B_i are continuous operator on S^* into S^*.

Now, we show that B_i are compact operator on S^*. It is enough to show that $B_i(S^*)$ are a uniformly bounded and equicontinuous set in X. On the one hand, let $x \in S^*$ be arbitrary. Then by hypothesis (B_1) and as done before in the proof of Theorem 3.1, we can get that

$$|B_i x(t)| \leqslant \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, |f_i(s, I^{\beta} u_i(s, x_n(s))| ds \leqslant M_{1i} \frac{T^{\alpha_i - \gamma_i}}{\Gamma(\alpha_i - \gamma_i + 1)} + \|b_i\| M_{2i} \frac{T^{\alpha_i + \beta_i - \gamma_i}}{\Gamma(\alpha_i + \beta_i - \gamma_i + 1)} = K_i$$

for all $t \in I$. Taking supermom over t,

$$\|B_i x(t)\| \leqslant M_{1i} \frac{T^{\alpha_i - \gamma_i}}{\Gamma(\alpha_i - \gamma_i + 1)} + \|b_i\| M_{2i} \frac{T^{\alpha_i + \beta_i - \gamma_i}}{\Gamma(\alpha_i + \beta_i - \gamma_i + 1)} = K_i.$$

Then

$$\|B_i x(t)\| \leqslant K_i,$$

for all $x \in S^*$. This shows that B_i are uniformly bounded on S^*.

Now, we proceed to show that $B_i(S^*)$ are also equicontinuous set in X. Let t_1, $t_2 \in I$, and $x \in S^*$. Without loss of generality assume that $t_1 < t_2$, then as done before in the proof of Theorem 3.1, we can get that

$$|(B_i x)(t_2) - (B_i x)(t_1)| \leqslant \frac{M_{1i}(t_2 - t_1)^{\alpha_i - \gamma_i}}{\Gamma(\alpha_i - \gamma_i + 1)} + \frac{\|b_i\| M_{2i} (t_2 - t_1)^{\alpha_i} T^{\beta_i - \gamma_i}}{\Gamma(\alpha_i + 1)\Gamma(\beta_i - \gamma_i + 1)}.$$

Hence, for $\epsilon_i > 0$, there exists a $\delta > 0$ such that

$$|t_2 - t_1| < \delta \quad \Longrightarrow \quad |(B_i x)(t_2) - (B_i x)(t_1)| < \epsilon_i,$$

for all $t_1, t_2 \in I$ and for all $x \in S^*$. This shows that $B_i(S^*)$ are an equicontinuous set in X. Now, the set $B_i(S^*)$ are a uniformly bounded and equicontinuous set in X, so it is compact by the Arzela-Ascoli theorem. As a result, B_i are complete continuous operator on S^*.

Step 3. The hypothesis (c) of Lemma 6.1 is satisfied. Let $x \in X$ and $y \in S^*$ be arbitrary elements such that $x = A_i x B_i y$. Then as done before in the proof of Theorem 3.1, we can get that

$$\|x(t)\| \leqslant \sum_{i=1}^{m} (\|L_i\| r + G_i) \left(\frac{M_{1i} T^{\alpha_i - \gamma_i}}{\Gamma(\alpha_i - \gamma_i + 1)} + \frac{\|b_i\| M_{2i} T^{\alpha_i + \beta_i - \gamma_i}}{\Gamma(\alpha_i + \beta_i - \gamma_i + 1)} \right) \leqslant r.$$

Therefore, $x \in S^*$.

Step 4. Finally we show that $\sum_i^m \delta_i F_i < 1$, that is, last condition of Lemma 6.1 holds.
Since

$$F_i = \|B_i(S^*)\| = \sup_{x\in S^*}\left\{\sup_{t\in I}|B_i x(t)|\right\} \leqslant \frac{M_{1i}\, T^{\alpha_i-\gamma_i}}{\Gamma(\alpha_i-\gamma_i+1)} + \frac{\|b_i\| M_{2i}\, T^{\alpha_i+\beta_i-\gamma_i}}{\Gamma(\alpha_i+\beta_i-\gamma_i+1)}$$

and by (B_3) we have

$$\sum_i^m \|L_i\|\; F_i \; < 1$$

with $\delta_i = \|L_i\|$. Thus all the conditions of Lemma 6.1 are satisfied and hence the operator equation $x = \sum_i^m A_i x B_i x$ has a solution in S^*. In consequence, Eq. (6.1) has a solution on I. This completes the proof. □

6.1. Some remarks and applications

As particular cases of Theorem 6.2 we can obtain theorems on the existence of solutions belonging to the space $C(I, R)$ for the following integral equations.

(i) Let $m = 1$, then we have

$$x = g_1(t, x(t)) I^{\alpha_1} f_1(t, I^{\beta_1} u_1(t, x(t)), \quad t \in I, \quad \alpha_1,\ \beta_1 \in (0, 1).$$

(ii) Let $m = 2$, $g_1(t, x(t)) = g_2(t, x(t)) = g(t, x(t))$, and $\alpha_1 = \alpha_2 = \alpha$. Then, we have

$$x(t) = g(t, x(t)) I^{\alpha}(f_1(t, I^{\beta_1}u_1(t, x(t)) + f_2(t, I^{\beta_2}u_2(t, x(t))), \quad t \in I, \quad \alpha_i,\ \beta_i \in (0, 1),\ (i = 1, 2).$$

This kind of equation had studied in [12].

(iii) Let $m = 2$, $\alpha_1, \beta_1 \to 1$, and $f_1(t, u_1(t, x(t))) = 1$, then we have

$$x(t) = t g_1(t, x(t)) + g_2(t, x(t)) I^{\alpha_2} f_2(t, I^{\beta_2} u_2(t, x(t))),\ t \in I,\ \alpha_2,\ \beta_2 \in (0, 1).$$

Taking $K(t, x(t)) = t\, g_1(t, x(t))$, $g = g_2$, $f = f_2$, $u = u_2$, $\alpha_2 = \alpha$, and $\beta_2 = \beta$, we get

$$x(t) = k(t, x(t)) + g(t, x(t)) I^{\alpha} f(t, I^{\beta} u(t, x((t))), \quad t \in I, \quad \alpha_2,\ \beta_2 \in (0, 1). \tag{6.2}$$

This equation had studied in Section 2.

- Let $K(t, x(t)) = p(t)$ and $g(t, x(t)) = 1$, in Eq. (6.2). Then we have

$$x(t) = p(t) + I^{\alpha} f(t, I^{\beta} u(t, x((t))), \quad t \in I, \quad \alpha,\ \beta \in (0, 1).$$

This kind of equation had studied in [1, 8].

- Furthermore let $I^{\beta} u(t, x((t)) = x(t)$, in Eq. (6.2). Then we have

$$x = k(t, x(t)) + g(t, x(t)) I^{\alpha} f(t, , x(t)), \quad t \in I, \quad \alpha \in (0, 1).$$

Which is a well-known results had studied in [14].

7. Conclusion

In this paper, we proved some fixed point theorems for the nonlinear operator $A \times B + C$ in a Banach algebra due to Dhage [5]. Results on the existence continuous dependence of solutions for Eq. (1.1) on delay function φ_1 also studied. It should be noted that in the same way, the reader can get the continuous dependence of solutions for Eq. (1.1) on the other delay functions. Furthermore, one of our results is applied to investigate sufficient conditions for existence of solutions to initial value problem of (FHDE) (5.1). As a second problem we have developed some adequate conditions for the existence of at least one solution to the general quadratic integral equation of fractional order Eq. (6.1) in a Banach algebra by applying the fixed point theorem due to Dhage [19]. Our results improved and generalized some interesting results in the literature.

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