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Approximation by a new generalization of Szász-Mirakjan operators via (p,q)-calculus

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Abstract

In this work, we obtain the approximation properties of a new generalization of Szász-Mirakjan operators based on post-quantum calculus. Firstly, for these operators, a recurrence formulation for the moments is obtained, and up to the fourth degree, the central moments are examined. Then, a local approximation result is attained. Furthermore, the degree of approximation in respect of the modulus of continuity on a finite closed set and the class of Lipschitz are computed. Next, the weighted uniform approximation on an unbounded interval is showed, and by the modulus of continuity, the order of convergence is estimated. Lastly, we proved the Voronovskaya type theorem and gave some illustrations to compare the related operators' convergence to a certain function.

Keywords: Weighted approximation, Szász-Mirakjan operators, modulus of continuity, (p,q)-calculus.

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1. Introduction

In [23, 36], the Szász-Mirakjan operators on $[0,\infty)$, which are related to the Poisson distribution, were defined as

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{(nx)^k}{k!}. \tag{1.1}$$

In recent years, many modifications and generalizations of the operators (1.1) were considered by some authors. Aral et al. [6] introduced the new generalization of Szász-Mirakyan operators. Some approximation results for the operators of Szász-Mirakjan-Durrmeyer type are obtained by Krech [19]. Çekim et al. [9] considered the Dunkl generalization of Szász beta-type operators. A modification of the Szász-Mirakjan-Kantorovich operators which preserving linear functions introduced by Duman et al. [12]. In [32], a new generalization of Szász-Mirakjan operators on a closed subintervals of $[0,\infty)$ were defined by Ousman and Izgi as follows:

$$N_n(f;x) = e^{-nx}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\frac{n+a}{n+b}\right)\frac{(nx)^k}{k!}, \qquad 0 \leqslant x < \infty, \tag{1.2}$$

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where $a, b \in \mathbb{N}$ and $0 \leqslant a \leqslant b$.

They estimated for the operators (1.2) the rate of convergence and proved the Voronovskaya type result theorem and also obtained the order of approximation of functions on the class of Lipschitz. We refer to the readers some similar type operators in [7, 16, 37].

In the last three decades, the quantum calculus, briefly q-calculus, which has a lot of application fields in mathematics, has played a serious role on the approximation theory. The first experimentation via the implementation of q-calculus to linear positive operators is done by Lupaş [20]. He investigated the q-Bernstein polynomials and examined approximation and shape-preserving properties. Later, Phillips [33] achieved several convergence results and proved the Voronovskaya-type results for the generalizations of the q-Bernstein operators. Next [20, 33] investigations, the implementing of q-calculus on the approximation theory become very popular and motivated many authors to introduce this technic to some various famous operators. In [31], Örkcü and Doğru introduced the weighted statistical approximation by kantorovich type q-Szász-Mirakjan operators. Ahasan and Mursaleen [2] obtained some approximation results of the generalized Szász-Mirakjan type operators via q-calculus. Mahmudov and Gupta [22] proposed a q-analogue of Szász Kantorovich operators. Mursaleen and Rahman [30] studied a Dunkl generalization of q-Szász-Mirakjan operators which preserve x^2. Also, we refer to [5, 8, 21].

Recently, Mursaleen et al. [28] by using of the post-quantum calculus briefly, (p, q)-calculus, added a new process to approximation theory. Moreover, by association to (p, q)-calculus the Bernstein-Schurer type operators [26], the Kantorovich variant of Szász-Mirakjan operators [27], the king type Szász-Mirakjan-Kantorovich operators [29] and the Szász-Mirakjan [25] type operators were introduced and examined in detail. Also, Alotaibi et al. [3] obtained some approximation results of a Dunkl type generalization of Szász operators via (p, q)-calculus. Karahan and Izgi [18] investigated some approximation results of the (p, q)-Bernstein operators. Acar [1] defined a new modified operators of Szász-Mirakjan based on (p, q)-integers. The Chlodowsky variant of Szász-Mirakjan-Stancu and the Szász-Mirakjan-Baskakov-Stancu type operators on the concept of (p, q)-integers were presented by [24]. The Stancu form of operators of (p, q)-Szász-Mirakjan are explored by [10].

In this research, for the operators given by (1.2), motivated by the many studies given above, we construct a new generalization of the Szász-Mirakjan operators by using of (p, q)-calculus and investigate the approximation attributes for these operators. Now, we recollect some basic notations and definitions about the (p, q)-calculus. Suppose that $0 < q < p \leqslant 1$. For all integers n,l such that $n \geqslant l \geqslant 0$, the $[n]_{p,q}$ is given as:

$$[n]_{p,q} := \frac{p^n - q^n}{p - q}.$$

The factorial and binomial factors related to the (p, q)-integers are given respectively as:

$$[n]_{p,q}! := \begin{cases} [n]_{p,q}\,[n-1]_{p,q} \cdots 1, & n \geqslant 1, \\ 1, & n = 0, \end{cases} \quad \text{and} \quad \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[l]_{p,q}!\,[n-l]_{p,q}!}.$$

The (p, q)-binomial expansion is gives as:

$$(au + bv)^n_{p,q} := \sum_{l=0}^{n} p^{\binom{n-l}{2}} q^{\binom{l}{2}} a^{n-l} b^l u^{n-l} v^l,$$

and

$$(u - v)^n_{p,q} = (u - v)(pu - qv)(p^2u - q^2v) \cdots (p^{n-1}u - q^{n-1}v).$$

Moreover, the two analogue of (p, q)-exponential function are as follows:

$$e_{p,q}(u) := \sum_{l=0}^{\infty} \frac{p^{\frac{l(l-1)}{2}} u^l}{[l]_{p,q}!} \quad \text{and} \quad E_{p,q}(u) := \sum_{l=0}^{\infty} \frac{q^{\frac{l(l-1)}{2}} u^l}{[l]_{p,q}!},$$

which verify $e_{p,q}(u)E_{p,q}(-u)=1$. Since $p=1$, $E_{p,q}(u)$ and $e_{p,q}(u)$ turn into functions of q-exponential. More detailed information about the (p,q)-calculus can be found [14, 17, 34, 35].

Now, by utilizing of (p,q)-calculus we will construct our operators as follows:

$$R^{p,q}_{n,a,b}(f;x)=E_{p,q}(-[n]_{p,q}\,x)\sum_{k=0}^{\infty} f\left(\frac{[k]_{p,q}\,[n+a]_{p,q}}{q^{k-2}\,[n]_{p,q}\,[n+b]_{p,q}}\right) q^{\frac{k(k-1)}{2}}\frac{([n]_{p,q}\,x)^k}{[k]_{p,q}!} \tag{1.3}$$

where $a,b,n\in\mathbb{N}, 0\leqslant a\leqslant b, 0<q<p\leqslant 1,\ x\in[0,\infty)$. It is clear to see $E_{p,q}(-[n]_{p,q}\,x)\sum\limits_{k=0}^{\infty} q^{\frac{k(k-1)}{2}}\frac{([n]_{p,q}x)^k}{[k]_{p,q}!}$ $=1$. The operators given by (1.3) are positive and linear. In special case, for $a=b$ the operators (1.3) reduce to operators given by [1] and also for $p=1$ and $a=b$ the operators (1.3) reduce to operators given by [21].

2. Main results

Lemma 2.1. *Let $R^{p,q}_{n,a,b}(f;x)$ operators are given by (1.3). Then, we attain the following relation:*

$$R^{p,q}_{n,a,b}(t^{s+1};x)=\sum_{u=0}^{s}\binom{s}{u}\frac{xp^u\,[n+a]^{s+1-u}_{p,q}}{q^{2u-s-1}\,[n]^{s-u}_{p,q}\,[n+b]^{s+1-u}_{p,q}}R^{p,q}_{n,a,b}(t^u;x). \tag{2.1}$$

Proof. Using the equation below

$$[n]_{p,q}=q^{n-1}+p\,[n-1]_{p,q}\,,$$

then, we may write

$$\begin{aligned}
R^{p,q}_{n,a,b}(t^{s+1};x)&=E_{p,q}(-[n]_{p,q}\,x)\sum_{k=0}^{\infty}\frac{[k]^{s+1}_{p,q}\,[n+a]^{s+1}_{p,q}}{q^{(k-2)(s+1)}\,[n]^{s+1}_{p,q}\,[n+b]^{s+1}_{p,q}}q^{\frac{k(k-1)}{2}}\frac{([n]_{p,q}\,x)^k}{[k]_{p,q}!}\\
&=E_{p,q}(-[n]_{p,q}\,x)\sum_{k=1}^{\infty}\frac{[k]^{s}_{p,q}\,[n+a]^{s+1}_{p,q}}{q^{(k-2)s}\,[n]^{s}_{p,q}\,[n+b]^{s+1}_{p,q}}q^{\frac{k(k-1)}{2}-k+2}\frac{([n]_{p,q}\,x)^{k-1}x}{[k-1]_{p,q}!}\\
&=E_{p,q}(-[n]_{p,q}\,x)\sum_{k=1}^{\infty}\frac{\left(p\,[k-1]_{p,q}+q^{k-1}\right)^s\,[n+a]^{s+1}_{p,q}}{q^{(k-2)s}\,[n]^{s}_{p,q}\,[n+b]^{s+1}_{p,q}}q^{\frac{k(k-1)}{2}-k+2}\frac{([n]_{p,q}\,x)^{k-1}x}{[k-1]_{p,q}!}\\
&=E_{p,q}(-[n]_{p,q}\,x)\sum_{k=1}^{\infty}\sum_{u=0}^{s}\binom{s}{u}q^{(k-1)(s-u)}p^u\,[k-1]^u_{p,q}!\frac{q^{\frac{(k-1)(k-2)}{2}+1}}{q^{(k-2)s}\,[n]^s_{p,q}}\\
&\quad\times\frac{[n+a]^{s+1}_{p,q}}{[n+b]^{s+1}_{p,q}}\frac{([n]_{p,q}\,x)^{k-1}x}{[k-1]_{p,q}!}\\
&=E_{p,q}(-[n]_{p,q}\,x)\sum_{u=0}^{s}\binom{s}{u}\frac{xp^u\,[n+a]^{s+1-u}_{p,q}}{q^{2u-s}\,[n]^{s-u}_{p,q}\,[n+b]^{s+1-u}_{p,q}}\sum_{k=1}^{\infty}\frac{[k-1]^u_{p,q}\,[n+a]^u_{p,q}}{q^{(k-2)u}\,[n]^u_{p,q}\,[n+b]^u_{p,q}}\\
&\quad\times q^{\frac{(k-1)(k-2)}{2}+1}\frac{([n]_{p,q}\,x)^{k-1}}{[k-1]_{p,q}!}\\
&=E_{p,q}(-[n]_{p,q}\,x)\sum_{u=0}^{s}\binom{s}{u}\frac{xp^u\,[n+a]^{s+1-u}_{p,q}}{q^{2u-s-1}\,[n]^{s-u}_{p,q}\,[n+b]^{s+1-u}_{p,q}}\sum_{k=0}^{\infty}\frac{[k]^u_{p,q}\,[n+a]^u_{p,q}}{q^{(k-2)u}\,[n]^u_{p,q}\,[n+b]^u_{p,q}}\\
&\quad\times q^{\frac{k(k-1)}{2}}\frac{([n]_{p,q}\,x)^{k}}{[k]_{p,q}!}\\
&=\sum_{u=0}^{s}\binom{s}{u}\frac{xp^u\,[n+a]^{s+1-u}_{p,q}}{q^{2u-s-1}\,[n]^{s-u}_{p,q}\,[n+b]^{s+1-u}_{p,q}}R^{p,q}_{n,a,b}(t^u;x),
\end{aligned}$$

which ends the proof. □

Lemma 2.2. *Let the* $R^{p,q}_{n,a,b}(f;x)$ *operators are given by* (1.3)*. Then, the following identities*

$$R^{p,q}_{n,a,b}(1;x)=1,$$

$$R^{p,q}_{n,a,b}(t;x)=qx\frac{[n+a]_{p,q}}{[n+b]_{p,q}}, \tag{2.2}$$

$$R^{p,q}_{n,a,b}(t^2;x)=\left(pqx^2+\frac{q^2}{[n]_{p,q}}x\right)\frac{[n+a]^2_{p,q}}{[n+b]^2_{p,q}}, \tag{2.3}$$

$$R^{p,q}_{n,a,b}(t^3;x)=\left(p^3x^3+\frac{2pq^2+p^2q}{[n]_{p,q}}x^2+\frac{q^3}{[n]^2_{p,q}}x\right)\frac{[n+a]^3_{p,q}}{[n+b]^3_{p,q}},$$

$$R^{p,q}_{n,a,b}(t^4;x)=\left(\frac{p^6}{q^2}x^4+\frac{p^5q+2p^3q^2+3p^3q^3}{q^2\,[n]_{p,q}}x^3+\frac{p^3q+3p^2q^2+3pq^3}{[n]^2_{p,q}}x^2+\frac{q^4}{[n]^3_{p,q}}x\right)\frac{[n+a]^4_{p,q}}{[n+b]^4_{p,q}},$$

are satisfies.

Proof. In view of (2.1), it is obvious that $R^{p,q}_{n,a,b}(1;x)=1$. Then,

$$R^{p,q}_{n,a,b}(t;x)=qx\frac{[n+a]_{p,q}}{[n+b]_{p,q}}R^{p,q}_{n,a,b}(1;x)=qx\frac{[n+a]_{p,q}}{[n+b]_{p,q}},$$

$$\begin{aligned}R^{p,q}_{n,a,b}(t^2;x)&=px\frac{[n+a]_{p,q}}{[n+b]_{p,q}}R^{p,q}_{n,a,b}(t;x)+\frac{q^2}{[n]_{p,q}}x\frac{[n+a]^2_{p,q}}{[n+b]^2_{p,q}}R^{p,q}_{n,a,b}(1;x)\\&=px\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\left(qx\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right)+\frac{q^2}{[n]_{p,q}}x\frac{[n+a]^2_{p,q}}{[n+b]^2_{p,q}}=\left(pqx^2+\frac{q^2}{[n]_{p,q}}x\right)\frac{[n+a]^2_{p,q}}{[n+b]^2_{p,q}}\end{aligned}$$

and

$$\begin{aligned}R^{p,q}_{n,a,b}(t^3;x)&=\frac{p^2}{q}x\frac{[n+a]_{p,q}}{[n+b]_{p,q}}R^{p,q}_{n,a,b}(t^2;x)+\frac{2pq}{[n]_{p,q}}x\frac{[n+a]^2_{p,q}}{[n+b]^2_{p,q}}R^{p,q}_{n,a,b}(t;x)+\frac{q^3}{[n]^2_{p,q}}x\frac{[n+a]^3_{p,q}}{[n+b]^3_{p,q}}R^{p,q}_{n,a,b}(1;x)\\&=\frac{p^2}{q}x\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\left(\left(pqx^2+\frac{q^2}{[n]_{p,q}}x\right)\frac{[n+a]^2_{p,q}}{[n+b]^2_{p,q}}\right)\\&\quad+\frac{2pq}{[n]_{p,q}}x\frac{[n+a]^2_{p,q}}{[n+b]^2_{p,q}}\left(qx\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right)+\frac{q^3}{[n]^2_{p,q}}x\frac{[n+a]^2_{p,q}}{[n+b]^2_{p,q}}\\&=\left(p^3x^3+\frac{2pq^2+p^2q}{[n]_{p,q}}x^2+\frac{q^3}{[n]^2_{p,q}}x\right)\frac{[n+a]^3_{p,q}}{[n+b]^3_{p,q}},\\R^{p,q}_{n,a,b}(t^4;x)&=\frac{p^3}{q^2}x\frac{[n+a]_{p,q}}{[n+b]_{p,q}}R^{p,q}_{n,a,b}(t^3;x)+\frac{3p^2}{[n]_{p,q}}x\frac{[n+a]^2_{p,q}}{[n+b]^2_{p,q}}R^{p,q}_{n,a,b}(t^2;x)\\&\quad+\frac{3pq^2}{[n]^2_{p,q}}x\frac{[n+a]^3_{p,q}}{[n+b]^3_{p,q}}R^{p,q}_{n,a,b}(t;x)+\frac{q^4}{[n]^3_{p,q}}x\frac{[n+a]^4_{p,q}}{[n+b]^4_{p,q}}R^{p,q}_{n,a,b}(1;x)\\&=\frac{p^3}{q^2}x\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\left\{\left(p^3x^3+\frac{2pq^2+p^2q}{[n]_{p,q}}x^2+\frac{q^3}{[n]^2_{p,q}}x\right)\frac{[n+a]^3_{p,q}}{[n+b]^3_{p,q}}\right\}\\&\quad+\frac{3p^2}{[n]_{p,q}}x\frac{[n+a]^2_{p,q}}{[n+b]^2_{p,q}}\left\{\left(pqx^2+\frac{q^2}{[n]_{p,q}}x\right)\frac{[n+a]^2_{p,q}}{[n+b]^2_{p,q}}\right\}\end{aligned}$$

$$+\frac{3pq^2}{[n]^2_{p,q}}x\frac{[n+a]^3_{p,q}}{[n+b]^3_{p,q}}\left\{qx\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right\}+\frac{q^4}{[n]^3_{p,q}}x\frac{[n+a]^4_{p,q}}{[n+b]^4_{p,q}}$$
$$=\left(\frac{p^6}{q^2}x^4+\frac{p^5q+2p^3q^2+3p^3q^3}{q^2[n]_{p,q}}x^3+\frac{p^3q+3p^2q^2+3pq^3}{[n]^2_{p,q}}x^2+\frac{q^4}{[n]^3_{p,q}}x\right)\frac{[n+a]^4_{p,q}}{[n+b]^4_{p,q}},$$

this ends the proof. □

Corollary 2.3. *Taking into account Lemma 2.2, the following central moments*

$$R^{p,q}_{n,a,b}(t-x;x)=\{q-1\}\frac{[n+a]_{p,q}}{[n+b]_{p,q}}x, \tag{2.4}$$

$$R^{p,q}_{n,a,b}((t-x)^2;x)=\left\{pq\frac{[n+a]^2_{p,q}}{[n+b]^2_{p,q}}-2q\frac{[n+a]_{p,q}}{[n+b]_{p,q}}+1\right\}x^2+q^2\frac{[n+a]^2_{p,q}}{[n+b]^2_{p,q}}\frac{x}{[n]_{p,q}}, \tag{2.5}$$

$$\begin{aligned}R^{p,q}_{n,a,b}((t-x)^4;x)=&\left\{\frac{p^6}{q^2}\frac{[n+a]^4_{p,q}}{[n+b]^4_{p,q}}-4p^3\frac{[n+a]^3_{p,q}}{[n+b]^3_{p,q}}+6pq\frac{[n+a]^2_{p,q}}{[n+b]^2_{p,q}}-4q\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right\}x^4\\&\times\left\{\frac{p^5q+2p^3q^2+3p^3q^3}{q^2}\frac{[n+a]^4_{p,q}}{[n+b]^4_{p,q}}-4(2pq^2+p^2q)\frac{[n+a]^3_{p,q}}{[n+b]^3_{p,q}}\right.\\&\left.+6q^2\frac{[n+a]^2_{p,q}}{[n+b]^2_{p,q}}\frac{x^3}{[n]_{p,q}}p^3q+3p^2q^2+3pq^3\frac{[n+a]^4_{p,q}}{[n+b]^4_{p,q}}-4q^3\frac{[n+a]^3_{p,q}}{[n+b]^3_{p,q}}\right\}\frac{x^2}{[n]^2_{p,q}}\\&+q^4\frac{[n+a]^4_{p,q}}{[n+b]^4_{p,q}}\frac{x}{[n]^3_{p,q}},\end{aligned} \tag{2.6}$$

are satisfies.

Remark 2.4. It can be clearly seen that $\lim_{n\to\infty}[n]_{p,q}=0$ or $\frac{1}{p-q}$ for $0<q<p\leqslant 1$. For the purpose of to provide the results, we get the sequences $q=(q_n)\in(0,1)$ and $p=(p_n)\in(q_n,1]$ so that $\lim_{n\to\infty}p_n=1$, $\lim_{n\to\infty}q_n=1$ and $\lim_{n\to\infty}p_n^n=c$, $\lim_{n\to\infty}q_n^n=d$, by $0<c,d\leqslant 1$. Then, we obtain $\lim_{n\to\infty}[n]_{p_n,q_n}=\infty$. Furthermore, following relations hold

$$\lim_{n\to\infty}[n]_{p_n,q_n}\left\{(q_n-1)\frac{[n+a]_{p_n,q_n}}{[n+b]_{p_n,q_n}}\right\}=\alpha,$$
$$\lim_{n\to\infty}[n]_{p_n,q_n}\left\{p_nq_n\frac{[n+a]^2_{p_n,q_n}}{[n+b]^2_{p_n,q_n}}-2q_n\frac{[n+a]_{p_n,q_n}}{[n+b]_{p_n,q_n}}+1\right\}=\beta$$
$$\lim_{n\to\infty}[n]_{p_n,q_n}\left\{\frac{p_n^6}{q_n^2}\frac{[n+a]^4_{p_n,q_n}}{[n+b]^4_{p_n,q_n}}-4p_n^3\frac{[n+a]^3_{p_n,q_n}}{[n+b]^3_{p_n,q_n}}+6p_nq_n\frac{[n+a]^2_{p_n,q_n}}{[n+b]^2_{p_n,q_n}}-4q_n\frac{[n+a]_{p_n,q_n}}{[n+b]_{p_n,q_n}}+1\right\}=\gamma.$$

Let us give an example for the sequences $(p_n),(q_n)$ given by Remark 2.4. Taking $(p_n)=1-\frac{2}{n+2}$, $(q_n)=1-\frac{3}{n+3}$, so it is clear to see that $\lim_{n\to\infty}p_n=1$, $\lim_{n\to\infty}q_n=1$, $\lim_{n\to\infty}p_n^n=\frac{1}{e^2}$, $\lim_{n\to\infty}q_n^n=\frac{1}{e^3}$ $\lim_{n\to\infty}\frac{1}{[n]_{p_n,q_n}}=0$ as $n\to\infty$. Further, we obtain $\alpha=c(e^{-3}-e^{-2})$, $\beta=e^{-2}-e^{-3}$, $\gamma=0$.

Remark 2.5. Let the sequences $(p_n),(q_n)$ are given by Remark 2.4. Then, we obtain

$$\lim_{n\to\infty}[n]_{p_n,q_n}R^{p_n,q_n}_{n,a,b}(t-x;x)=\alpha x, \tag{2.7}$$
$$\lim_{n\to\infty}[n]_{p_n,q_n}R^{p_n,q_n}_{n,a,b}((t-x)^2;x)=\beta x^2+x, \tag{2.8}$$
$$\lim_{n\to\infty}[n]_{p_n,q_n}R^{p_n,q_n}_{n,a,b}((t-x)^4;x)=\gamma x^4. \tag{2.9}$$

3. Local approximation results for $R^{p,q}_{n,a,b}(f;x)$ operators

Suppose that the space $C_B[0,\infty)$ indicates for all real-valued continuous and bounded functions g. On $C_B[0,\infty)$ the norm and K-functional of Peetre's are given respectively as

$$\|g\| = \sup_{x\in[0,\infty)} |g(x)| \quad \text{and} \quad K_2(g,\eta) = \inf_{h\in C_B^2} \left\{\|g-h\| + \eta \left\|h''\right\|\right\},$$

where $\eta > 0$ and $C_B^2 = \{h \in C_B[0,\infty) : h', h'' \in C_B[0,\infty)\}$. Taking into account [11], we attain

$$K_2(g;\eta) \leqslant C\omega_2(g;\sqrt{\eta}), \qquad \eta > 0, \tag{3.1}$$

where

$$\omega_2(g;\eta) = \sup_{0<h\leqslant\sqrt{\eta}} \sup_{x\in[0,\infty)} |g(x+2h) - 2g(x+h) + g(x)|$$

is the second order of modulus of smoothness of function $g \in C_B[0,\infty)$. Further,

$$\omega(g;\eta) := \sup_{0<h\leqslant\eta} \sup_{x\in[0,\infty)} |g(x+h) - g(x)|$$

is the ordinary modulus of continuity of $g \in C_B[0,\infty)$. More details for $\omega(g,\eta)$ can be found by [4].

Theorem 3.1. *Suppose that* $f \in C_B[0,\infty)$ *and* $0 < q < p \leqslant 1$. *Then, for all* $x \in [0,\infty)$, *we obtain*

$$\left|R^{p,q}_{n,a,b}(f;x) - f(x)\right| \leqslant C\omega_2(f;\sqrt{\eta_n(x)}) + \omega(f;\vartheta_n(x))$$

where a constant $C > 0$, $\eta_n(x) = R^{p,q}_{n,a,b}((t-x)^2;x) + \vartheta(x)^2$, *and* $\vartheta_n(x) = \left((q-1)\frac{[n+a]_{p,q}}{[n+b]_{p,q}}x\right)$.

Proof. Firstly, we give the following auxiliary operators

$$\widetilde{R}^{p,q}_{n,a,b}(f;x) = R^{p,q}_{n,a,b}(f;x) - f\left(q\frac{[n+a]_{p,q}}{[n+b]_{p,q}}x\right) + f(x), \tag{3.2}$$

where $x \in [0,\infty)$. From (2.2),

$$\widetilde{R}^{p,q}_{n,a,b}(t-x;x) = 0.$$

For $g \in C_B^2$, making use of Taylor's expansion,

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du. \tag{3.3}$$

Operating of $\widetilde{R}^{p,q}_{n,a,b}(.;x)$ operators to (3.3), we have

$$\begin{aligned}
\widetilde{R}^{p,q}_{n,a,b}(g;x) - g(x) &= \widetilde{R}^{p,q}_{n,a,b}((t-x)g'(x);x) + \widetilde{R}^{p,q}_{n,a,b}(\int_x^t (t-u)g''(u)du;x) \\
&= g'(x)\widetilde{R}^{p,q}_{n,a,b}(t-x;x) + R^{p,q}_{n,a,b}(\int_x^t (t-u)g''(u)du;x) \\
&\quad - \int_x^{q\frac{[n+a]_{p,q}}{[n+b]_{p,q}}x} (q\frac{[n+a]_{p,q}}{[n+b]_{p,q}}x - u)g''(u)du
\end{aligned}$$

$$= R^{p,q}_{n,a,b}\Big(\int_x^t (t-u)g''(u)du;x\Big) - \int_x^{q\frac{[n+a]_{p,q}}{[n+b]_{p,q}}x} \Big(q\frac{[n+a]_{p,q}}{[n+b]_{p,q}}x-u\Big)g''(u)du.$$

In view of Lemma 2.2 and (3.2),

$$\left|\widetilde{R}^{p,q}_{n,a,b}(g;x)-g(x)\right| \leqslant \left|R^{p,q}_{n,a,b}\Big(\int_x^t (t-u)g''(u)du;x\Big)\right| + \left|\int_x^{q\frac{[n+a]_{p,q}}{[n+b]_{p,q}}x} \Big(q\frac{[n+a]_{p,q}}{[n+b]_{p,q}}x-u\Big)g''(u)du\right|$$

$$\leqslant R^{p,q}_{n,a,b}\Big(\left|\int_x^t (t-u)\right|\left|g''(u)\right||du|;x\Big) + \int_x^{q\frac{[n+a]_{p,q}}{[n+b]_{p,q}}x} \left|q\frac{[n+a]_{p,q}}{[n+b]_{p,q}}x-u\right|\left|g''(u)\right||du|$$

$$\leqslant \left\|g''(u)\right\|\left\{R^{p,q}_{n,a,b}((t-x)^2;x) + \left(q\frac{[n+a]_{p,q}}{[n+b]_{p,q}}x-x\right)^2\right\}.$$

Further, by (2.2), (2.3), and (3.2), we obtain

$$\left|\widetilde{R}^{p,q}_{n,a,b}(f;x)\right| \leqslant \left|R^{p,q}_{n,a,b}(f;x)\right| + 2\left\|f\right\| \leqslant \left\|f\right\| R^{p,q}_{n,a,b}(1;x) + 2\left\|f\right\| \leqslant 3\left\|f\right\|.$$

Then,

$$\left|R^{p,q}_{n,a,b}(f;x)-f(x)\right| \leqslant \left|\widetilde{R}^{p,q}_{n,a,b}(f-g;x)-(f-g)(x)\right| + \left|\widetilde{R}^{p,q}_{n,a,b}(g;x)-g(x)\right| + \left|f(x)-f\Big(q\frac{[n+a]_{p,q}}{[n+b]_{p,q}}x\Big)\right|$$

$$\leqslant 4\left\|f-g\right\| + \left\{R^{p,q}_{n,a,b}((t-x)^2;x) + \left((q-1)\frac{[n+a]_{p,q}}{[n+b]_{p,q}}x\right)^2\right\}\left\|g''\right\|$$

$$+\,\omega\left(f;(q-1)\frac{[n+a]_{p,q}}{[n+b]_{p,q}}x\right).$$

Over all $g \in C^2_B[0,\infty)$ utilize the infimum on the right hand side by make use of (3.1) and for a constant $C>0$, $\vartheta_n(x) = \left((q-1)\frac{[n+a]_{p,q}}{[n+b]_{p,q}}x\right)$, $\eta_n(x) = R^{p,q}_{n,a,b}((t-x)^2;x)+\vartheta(x)^2$, then

$$\left|R^{p,q}_{n,a,b}(f;x)-f(x)\right| \leqslant 4K_2(f;\eta_n(x)(x)) + \omega(f;\vartheta_n(x)) \leqslant C\omega_2(f;\sqrt{\eta_n(x)}) + \omega(f;\vartheta_n(x)),$$

which ends the proof. □

4. Order of convergence of $R^{p,q}_{n,a,b}(f;x)$ operators

In this section, by utilizing the ordinary modulus of continuity, we computed the order of convergence. Also, to see the smoothness of approximation for a function g on Lipschitz class $Lip_M(\zeta)$, where $M>0$ and $0<\zeta\leqslant 1$, we established the degree of convergence of the operators (1.3). Since

$$|g(t)-g(x)| \leqslant M\,|t-x|^{\zeta}, \quad (t,x\in\mathbb{R}),$$

holds, then a function g belongs to $Lip_M(\zeta)$. Let $C_{x^2}[0,\infty) := \{h : |h(x)| \leqslant M_h(1+x^2),\ h$ is continuous, $M_h>0\}$ and $C^*_{x^2}[0,\infty) := \{h :\ h\in C_{x^2}[0,\infty),\ \lim\limits_{x\to\infty}\frac{|h(x)|}{1+x^2} < \infty\}$. On $C^*_{x^2}[0,\infty)$, the norm and ordinary modulus of continuity of h on $[0,\delta]$ are given respectively as follows:

$$\|h\|_{x^2} = \sup_{x\in[0,\infty)}\frac{|h(x)|}{1+x^2} \quad\text{and}\quad \omega_\delta(h;\eta) = \sup_{|t-x|\leqslant\eta}\ \sup_{x,t\in[0,\delta]} |h(t)-h(x)| \qquad \delta>0.$$

Theorem 4.1. *Assume that* $f \in C_B[0,\infty)$, *the sequences* (p_n), (q_n) *are given as in Remark 2.4 and* $\omega_{\delta+1}(f;\eta)$ *be its modulus of continuity on* $[0,\delta+1] \subset [0,\infty)$. *Then, for all* $x \in [0,\infty)$ *the following relation*

$$\left\| R^{p_n,q_n}_{n,a,b}(f;x) - f(x) \right\|_{C_{[0,\delta]}} \leqslant 6M_f(1+\delta^2)\mu_n(x) + 2\omega_{\delta+1}(f;\sqrt{\mu_n(x)})$$

holds, where $\mu_n(x) = (1-p_nq_n)\,\delta^2 + \frac{\delta}{[n]_{p_n,q_n}}$.

Proof. Suppose that $x \in [0,\delta]$ and $t > \delta+1$, so $t-x > 1$, then

$$|f(t)-f(x)| \leqslant M_f(x^2+t^2+2) \leqslant M_f(3x^2+2(t-x)^2+2) \leqslant 6M_f(1+\delta^2)(t-x)^2. \tag{4.1}$$

Also, by $x \in [0,\delta]$ and $t \leqslant \delta+1$,

$$|f(t)-f(x)| \leqslant \left(1+\frac{|t-x|}{\eta}\right)\omega_{\delta+1}(f;\eta), \quad \eta > 0. \tag{4.2}$$

Combining (4.1), (4.2), and for $t > 0$, $x \in [0,\delta]$, we obtain

$$|f(t)-f(x)| \leqslant 6M_f(1+\delta^2)(t-x)^2 + \left(1+\frac{|t-x|}{\eta}\right)\omega_{\delta+1}(f;\eta). \tag{4.3}$$

Operating $R^{p_n,q_n}_{n,a,b}(.;x)$ operators to both sides of (4.3),

$$\begin{aligned}\left| R^{p_n,q_n}_{n,a,b}(f;x) - f(x) \right| &\leqslant R^{p_n,q_n}_{n,a,b}(|f(t)-f(x)|\,;x) \\ &\leqslant 6M_f(1+\delta^2)R^{p_n,q_n}_{n,a,b}((t-x)^2;x) + \left(1+\frac{1}{\eta^2}R^{p_n,q_n}_{n,a,b}((t-x)^2;x)\right)^{\frac{1}{2}}\omega_{\delta+1}(f;\eta).\end{aligned}$$

For $x \in [0,\delta]$ and by (2.5), since $a,b,n \in \mathbb{N}$, $0 \leqslant a \leqslant b$, then $\frac{[n+a]_{p_n,q_n}}{[n+b]_{p_n,q_n}} \leqslant 1$, thus we obtain

$$\begin{aligned}R^{p_n,q_n}_{n,a,b}((t-x)^2;x) &= \left(p_nq_n\frac{[n+a]^2_{p_n,q_n}}{[n+b]^2_{p_n,q_n}} - 2q_n\frac{[n+a]_{p_n,q_n}}{[n+b]_{p_n,q_n}} + 1\right)x^2 + q_n^2\frac{[n+a]^2_{p_n,q_n}}{[n+b]^2_{p_n,q_n}}\frac{x}{[n]_{p_n,q_n}} \\ &\leqslant (p_nq_n - 2q_n + 1)\,\delta^2 + q_n^2\frac{\delta}{[n]_{p_n,q_n}} \\ &\leqslant (1-p_nq_n)\,\delta^2 + \frac{\delta}{[n]_{p_n,q_n}} \\ &= \mu_n(x).\end{aligned}$$

Choosing $\eta = \sqrt{\mu_n(x)}$, gives the proof. □

Theorem 4.2. *Assume that the sequences* (p_n) *and* (q_n) *are given as in Remark 2.4. Then, for all* $f \in \mathrm{Lip}_M(\zeta)$, $M > 0$, $0 < \zeta \leqslant 1$, *the following inequality*

$$\left| R^{p_n,q_n}_{n,a,b}(f;x) - f(x) \right| \leqslant M(\mu_n(x))^{\frac{\zeta}{2}},$$

holds, where $\mu_n(x)$ *is given as Theorem 4.1.*

Proof. Let $f \in \mathrm{Lip}_M(\zeta)$ and $0 < \zeta \leqslant 1$. Since $R^{p_n,q_n}_{n,a,b}(f;x)$, utilizing the linearity and monotononicity, then we get

$$\left| R^{p_n,q_n}_{n,a,b}(f;x) - f(x) \right| \leqslant E_{p_n,q_n}(-[n]_{p_n,q_n}x)\sum_{k=0}^{\infty}\frac{([n]_{p_n,q_n}x)^k}{[k]_{p_n,q_n}!}q_n^{\frac{k(k-1)}{2}}$$

$$\times\left|f\left(\frac{[k]_{p_n,q_n}[n+a]_{p_n,q_n}}{q_n^{k-2}[n]_{p_n,q_n}[n+b]_{p_n,q_n}}\right)-f(x)\right|$$

$$\leqslant ME_{p_n,q_n}(-[n]_{p_n,q_n}x)\sum_{k=0}^{\infty}\frac{([n]_{p_n,q_n}x)^k}{[k]_{p_n,q_n}!}q_n^{\frac{k(k-1)}{2}}$$

$$\times\left|f\left(\frac{[k]_{p_n,q_n}[n+a]_{p_n,q_n}}{q_n^{k-2}[n]_{p_n,q_n}[n+b]_{p_n,q_n}}\right)-f(x)\right|^{\zeta}.$$

Utilizing Hölder's inequality, then

$$\left|R^{p_n,q_n}_{n,a,b}(f;x)-f(x)\right|$$

$$\leqslant ME_{p_n,q_n}(-[n]_{p_n,q_n}x)\sum_{k=0}^{\infty}\left\{\frac{([n]_{p_n,q_n}x)^k}{[k]_{p_n,q_n}!}q_n^{\frac{k(k-1)}{2}}\left(\frac{[k]_{p_n,q_n}[n+a]_{p_n,q_n}}{q_n^{k-2}[n]_{p_n,q_n}[n+b]_{p_n,q_n}}-x\right)^2\right\}^{\frac{\zeta}{2}}$$

$$\times\left\{\frac{([n]_{p_n,q_n}x)^k}{[k]_{p_n,q_n}!}q_n^{\frac{k(k-1)}{2}}\right\}^{\frac{2-\zeta}{2}}\leqslant M(R^{p_n,q_n}_{n,a,b}((t-x)^2;x))^{\frac{\zeta}{2}}.$$

Taking $\mu_n(x)=R^{p_n,q_n}_{n,a,b}((t-x)^2;x)$, gives the proof. □

5. Approximation in weighted spaces

In this section, we determined the approximation features of the operators (1.3) on the weighted spaces of continuous functions on $[0,\infty)$.

Theorem 5.1. *Assume that the sequences* (p_n) *and* (q_n) *are given as in Remark 2.4. For all* $x\in[0,\infty)$ *and* $f\in C^*_{x^2}[0,\infty)$, *we have the following relation:*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{\left|R^{p_n,q_n}_{n,a,b}(f;x)-f(x)\right|}{1+x^2}=0.$$

Proof. Considering to the Korovkin's theorem which is given by [13], we have to show that operators (1.3) fulfill the following three conditions:

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{\left|R^{p_n,q_n}_{n,a,b}(t^s;x)-x^s\right|}{1+x^2}=0,\qquad s=0,1,2.\tag{5.1}$$

By (2.1), for $s=0$ the first condition in (5.1) is trivial. Also, from (2.3),

$$\sup_{x\in[0,\infty)}\frac{\left|R^{p_n,q_n}_{n,a,b}(t;x)-x\right|}{1+x^2}\leqslant\left|q_n\frac{[n+a]_{p_n,q_n}}{[n+b]_{p_n,q_n}}-1\right|\sup_{x\in[0,\infty)}\frac{x}{1+x^2}\leqslant\left|q_n\frac{[n+a]_{p_n,q_n}}{[n+b]_{p_n,q_n}}-1\right|\leqslant|q_n-1|.$$

Then,

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{\left|R^{p_n,q_n}_{n,a,b}(t;x)-x\right|}{1+x^2}=0.$$

Analogously, from (2.4), we get

$$\sup_{x\in[0,\infty)}\frac{\left|R^{p_n,q_n}_{n,a,b}(t^2;x)-x^2\right|}{1+x^2}\leqslant\left|p_nq_n\frac{[n+a]^2_{p_n,q_n}}{[n+b]^2_{p_n,q_n}}-1\right|\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}+\left|q_n^2\frac{[n+a]^2_{p_n,q_n}}{[n]_{p_n,q_n}[n+b]^2_{p_n,q_n}}\right|\sup_{x\in[0,\infty)}\frac{x}{1+x^2}$$

$$\leqslant |1-p_n q_n| + \left|\frac{1}{[n]_{q_n}}\right|.$$

Hence,

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{\left|R_{n,a,b}^{p_n,q_n}(t^2;x)-x^2\right|}{1+x^2} = 0,$$

which ends the proof. □

Now, we take the modulus of continuity $\Omega(g;\eta)$ on weighted spaces for every $g \in C^*_{x^2}[0,\infty)$ as below

$$\Omega(g;\eta) = \sup_{0<h\leqslant\eta, x\geqslant 0} \frac{|g(x+h)-g(x)|}{1+(x+h)^2}.$$

Lemma 5.2 ([15])**.** *Let $g \in C^*_{x^2}[0,\infty)$. The following relations are fulfilled:*

i) $\Omega(g;\eta)$ *is a monotone increasing function of* g;

ii) $\lim_{\eta\to 0^+} \Omega(g;\eta) = 0$;

iii) *for* $\lambda > 0$, $\Omega(g;\lambda\eta) \leqslant 2(1+\lambda)(1+\eta^2)\Omega(g;\eta)$.

Theorem 5.3. *Suppose that the sequences (p_n) and (q_n) are given as in Remark 2.4. Then, for all $x \in [0,\infty)$, $f \in C^*_{x^2}[0,\infty)$, we get*

$$\sup_{x\in[0,\infty)} \frac{\left|R_{n,a,b}^{p_n,q_n}(f;x)-f(x)\right|}{(1+x^2)^{\frac{5}{2}}} \leqslant K\Omega(f;\frac{1}{\sqrt{\vartheta_{p_n,q_n}(n)}})$$

where $K > 0$ is a constant and $\vartheta_{p_n,q_n}(n) = \max\left\{1-p_n q_n, \frac{1}{[n]_{p_n,q_n}}\right\}$.

Proof. Let $x \in [0,\infty)$, $\eta > 0$, making use of $\Omega(f;\eta)$ and by Lemma 5.2, we attain

$$\begin{aligned} |f(t)-f(x)| &\leqslant (1+(t-x)^2)\left(1+x^2\right)\Omega(f;|t-x|) \\ &\leqslant 2(1+(t-x)^2)\left(1+x^2\right)\left(1+\frac{|t-x|}{\eta}\right)(1+\eta^2)\Omega(f;\eta). \end{aligned} \tag{5.2}$$

Consider $R_{n,a,b}^{p_n,q_n}(1;x) = 1$ and making use of monotononicity of $R_{n,a,b}^{p_n,q_n}$ operators, we get

$$\left|R_{n,a,b}^{p_n,q_n}(f;x)-f(x)\right| \leqslant R_{n,a,b}^{p_n,q_n}(|f(t)-f(x);x)|).$$

By (5.2),

$$\begin{aligned} \left|R_{n,a,b}^{p_n,q_n}(f;x)-f(x)\right| &\leqslant 2(1+\eta^2)\Omega(f;\eta)\left(1+x^2\right)\left\{R_{n,a,b}^{p_n,q_n}(\left(1+\frac{|t-x|}{\eta}\right)(1+(t-x)^2);x)\right\} \\ &\leqslant 2(1+\eta^2)\Omega(f;\eta)\left(1+x^2\right)\left\{R_{n,a,b}^{p_n,q_n}(1;x)+R_{n,a,b}^{p_n,q_n}((t-x)^2;x)\right. \\ &\quad \left.+\frac{1}{\eta}R_{n,a,b}^{p_n,q_n}(|t-x|;x)+\frac{1}{\eta}R_{n,a,b}^{p_n,q_n}(|t-x|(t-x)^2;x)\right\}. \end{aligned}$$

Next, utilizing the Cauchy-Schwarz inequality,

$$\begin{aligned} \left|R_{n,a,b}^{p_n,q_n}(f;x)-f(x)\right| &\leqslant 2(1+\eta^2)\Omega(f;\eta)\left(1+x^2\right)\left\{R_{n,a,b}^{p_n,q_n}(1;x)+R_{n,a,b}^{p_n,q_n}((t-x)^2;x)\right. \\ &\quad \left.+\frac{1}{\eta}\sqrt{R_{n,a,b}^{p_n,q_n}((t-x)^2;x)}+\frac{1}{\eta}\sqrt{R_{n,a,b}^{p_n,q_n}((t-x)^2;x)}\sqrt{R_{n,a,b}^{p_n,q_n}((t-x)^4;x)}\right\}. \end{aligned}$$

Also, from (2.5),

$$\begin{aligned} R^{p_n,q_n}_{n,a,b}((t-x)^2;x) &= \left(p_nq_n\frac{[n+a]^2_{p_n,q_n}}{[n+b]^2_{p_n,q_n}} - 2q_n\frac{[n+a]_{p_n,q_n}}{[n+b]_{p_n,q_n}}+1\right)x^2 + q_n^2\frac{[n+a]^2_{p_n,q_n}}{[n+b]^2_{p_n,q_n}}\frac{x}{[n]_{p_n,q_n}} \\ &\leqslant (1-p_nq_n)\,x^2 + \frac{x}{[n]_{p_n,q_n}} \\ &\leqslant K_1O(\vartheta_{p_n,q_n}(n))(1+x^2), \end{aligned}$$

where $K_1>0$ and $\vartheta_{p_n,q_n}(n)=\max\left\{1-p_nq_n, \frac{1}{[n]_{p_n,q_n}}\right\}$. Since $\lim\limits_{n\to\infty} p_nq_n=1$ and $\lim\limits_{n\to\infty}[n]_{p_n,q_n}=\infty$, hence there consists a constant $K_2>0$ such that

$$R^{p_n,q_n}_{n,a,b}((t-x)^2;x)\leqslant K_2\left(1+x^2\right).$$

Moreover, by (2.6),

$$\sqrt{R^{p_n,q_n}_{n,a,b}((t-x)^4;x)}\leqslant K_3\left(1+x^2\right)$$

and

$$\frac{1}{\eta}\sqrt{R^{p_n,q_n}_{n,a,b}((t-x)^2;x)}\leqslant \frac{K_4}{\eta}O\left(\sqrt{\vartheta_{p_n,q_n}(n)}\right)\sqrt{(1+x^2)}$$

for $K_3>0$ and $K_4>0$. Then, we obtain

$$\begin{aligned} \left|R^{p_n,q_n}_{n,a,b}(f;x)-f(x)\right| &\leqslant 2\Big(1+\frac{1}{\vartheta_{p_n,q_n}(n)}\Big)(1+x^2)\Omega\Big(f;\frac{1}{\sqrt{\vartheta_{p_n,q_n}(n)}}\Big)\Big\{1+K_2\left(1+x^2\right) \\ &\quad +\frac{K_4}{\eta}O\left(\sqrt{\vartheta_{p_n,q_n}(n)}\right)\sqrt{(1+x^2)}+K_3\left(1+x^2\right)\frac{K_4}{\eta}O\left(\sqrt{\vartheta_{p_n,q_n}(n)}\right)\sqrt{(1+x^2)}\Big\}. \end{aligned}$$

Taking $\eta=\sqrt{\vartheta_{p_n,q_n}(n)}$, in above equation,

$$\left|R^{p_n,q_n}_{n,a,b}(f;x)-f(x)\right|\leqslant 2\Big(1+\frac{1}{\vartheta_{p_n,q_n}(n)}\Big)(1+x^2)\Omega\Big(f;\frac{1}{\sqrt{\vartheta_{p_n,q_n}(n)}}\Big)+CK_4\left(1+x^2\right)^{\frac{1}{2}}.$$

Thus, for $\vartheta_{p_n,q_n}(n)\leqslant 1$, we may write

$$\sup_{x\in[0,\infty)}\frac{\left|R^{p_n,q_n}_{n,a,b}(f;x)-f(x)\right|}{(1+x^2)^{\frac{5}{2}}}\leqslant K\Omega\Big(f;\frac{1}{\sqrt{\vartheta_{p_n,q_n}(n)}}\Big),$$

where $K=4\left(1+K_2+CK_4+K_1K_3K_4\right)$, which ends the proof. □

6. Voronovskaya type theorem

Theorem 6.1. *Let the sequences* $(q_n)\in(0,1)$ *and* $(p_n)\in(q_n,1]$ *such that* $\lim\limits_{n\to\infty}p_n=1$, $\lim\limits_{n\to\infty}q_n=1$ *and* $\lim\limits_{n\to\infty}p_n^n=c$, $\lim\limits_{n\to\infty}q_n^n=d$, *by* $0<c,d\leqslant 1$. *For any* $f\in C^*_{x^2}[0,\infty)$ *such that* $f',f''\in C^*_{x^2}[0,\infty)$ *we get*

$$\lim_{n\to\infty}[n]_{p_n,q_n}\left(R^{p_n,q_n}_{n,a,b}(f;x)-f(x)\right)=\alpha x f'(x)+\frac{(\beta x^2+x)}{2}f''(x)$$

uniformly on the interval $[0,A]$, $A>0$.

Proof. Suppose that $x\in[0,\infty)$ and considering $f,f',f''\in C^*_{x^2}[0,\infty)$ and making use of Taylor's expansion formula, then

$$f(t)=f(x)+(t-x)f'(x)+\frac{1}{2}(t-x)^2f''(x)+\psi(t;x)(t-x)^2. \tag{6.1}$$

Here, $\psi(t;x)$ is a form of Peano of the rest term. Since $\psi(.;x)\in C^*_{x^2}[0,\infty)$, then $\lim\limits_{t\to x}\psi(t;x)=0$. Operating $R^{p_n,q_n}_{n,a,b}(.;x)$ to (6.1), then we get,

$$[n]_{p_n,q_n} \left(R^{p_n,q_n}_{n,a,b}(f;x) - f(x)\right) = [n]_{p_n,q_n} R^{p_n,q_n}_{n,a,b}((t-x);x)f'(x) + \frac{1}{2}[n]_{p_n,q_n} R^{p_n,q_n}_{n,a,b}((t-x)^2;x)f''(x) + [n]_{p_n,q_n} R^{p_n,q_n}_{n,a,b}(\psi(t;x)(t-x)^2;x).$$

Utilizing the Cauchy-Schwarz inequality to last part of the above equality, it pursues that

$$R^{p_n,q_n}_{n,a,b}(\psi(t;x)(t-x)^2;x) \leqslant \sqrt{R^{p_n,q_n}_{n,a,b}(\psi^2(t;x);x)}\sqrt{R^{p_n,q_n}_{n,a,b}((t-x)^4;x)}. \tag{6.2}$$

Considering $\psi(t;x) \in C^*_{x^2}[0,\infty)$ and by Theorem 5.1, we get $\lim_{t\to x}\psi(t;x) = 0$. Then,

$$\lim_{n\to\infty} R^{p_n,q_n}_{n,a,b}(\psi^2(t;x);x) = \psi^2(t;x) = 0 \tag{6.3}$$

uniformly on $x \in [0, A]$. Combining (6.2), (6.3), and by (2.9), we get

$$\lim_{n\to\infty} [n]_{p_n,q_n} R^{p_n,q_n}_{n,a,b}(\psi(t;x)(t-x)^2;x) = 0. \tag{6.4}$$

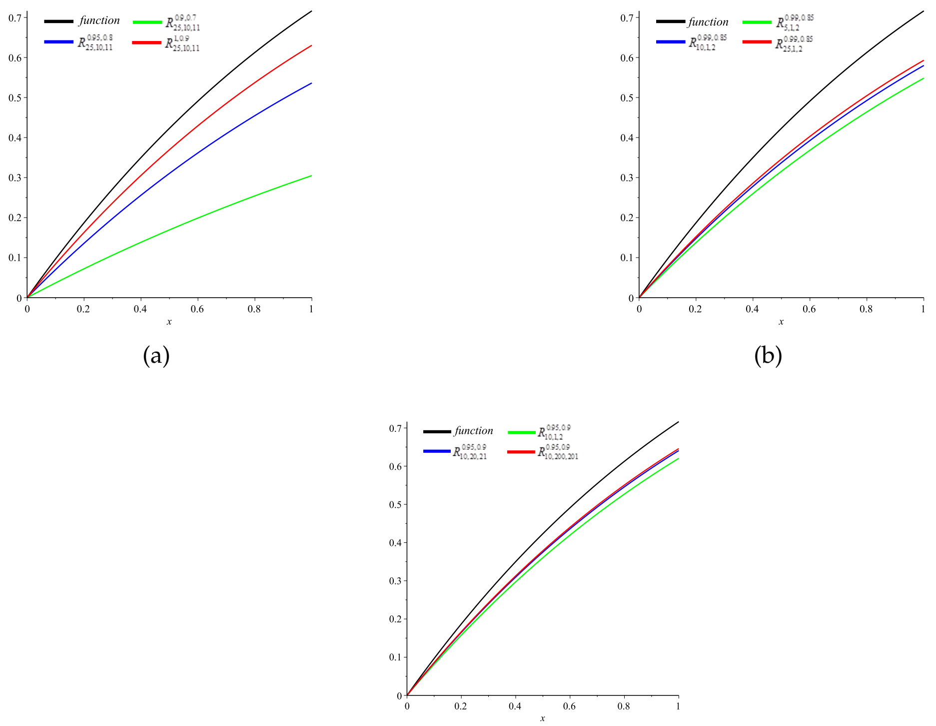
Consequently, in view of (2.7), (2.8), and (6.4),

$$\lim_{n\to\infty} [n]_{p_n,q_n} \left(R^{p_n,q_n}_{n,a,b}(f;x) - f(x)\right) = \alpha x f'(x) + \frac{(\beta x^2 + x)}{2} f''(x),$$

which gives the desired result. □

7. Some plots

In this section, we compare the convergence of $R^{p,q}_{n,a,b}(f;x)$ operators with the different parameters to a particular function.



(a) (b) (c)

Figure 1: The convergence of $R^{p,q}_{n,a,b}(f;x)$ operators to $f(x) = xe^{-\frac{x}{3}}$ under the different parameters.

In Figure 1a, we examine the convergence of $R^{p,q}_{n,a,b}(f;x)$ operators to the $f(x) = xe^{-\frac{x}{3}}$ by keeping the parameters n, a, b constant and increasing the values of p and q. It is obvious that, in view of $0 < q < p \leqslant 1$, as the values of p and q increasing then the convergence of the $R^{p,q}_{n,a,b}(f;x)$ operators to the $f(x) = xe^{-\frac{x}{3}}$ becomes better.

In Figure 1b, we examine the convergence of $R^{p,q}_{n,a,b}(f;x)$ operators to the $f(x) = xe^{-\frac{x}{3}}$ by keeping the parameters p, q, a, b constant and increasing the values of n. It is clear that since the values of n are increasing, then the convergence of the $R^{p,q}_{n,a,b}(f;x)$ operators to the $f(x) = xe^{-\frac{x}{3}}$ becomes better.

Also, in Figure 1c, we examine the convergence of $R^{p,q}_{n,a,b}(f;x)$ operators to the $f(x) = xe^{-\frac{x}{3}}$ by keeping the parameters n, p, q constant and increasing the values of a and b. We have seen that if we choose the natural numbers of a and b very close and large then the convergence of the $R^{p,q}_{n,a,b}(f;x)$ operators to the $f(x) = xe^{-\frac{x}{3}}$ becomes better.

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