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# Generalized Bernstein-Chlodowsky-Kantorovich type operators involving Gould-Hopper polynomials

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*Dedicated to the great mathematician*
*Prof. Themistocles M. Rassias on his 70-th birth anniversary*

## Abstract

In the present article, we establish a link between the theory of positive linear operators and the orthogonal polynomials by defining Bernstein-Chlodowsky-Kantorovich operators based on Gould-Hopper polynomials (orthogonal polynomials) and investigate the degree of convergence of these operators for unbounded continuous functions having a polynomial growth. In this connection, the moments of the operators are derived first, and then the approximation degree of the considered operators is established by means of the complete and the partial moduli of continuity. Next, we focus on the rate of convergence of these operators for functions in a weighted space. The associated Generalized Boolean Sum (GBS) operator of the operators under study is defined, and the degree of approximation is studied with the aid of the mixed modulus of smoothness and the Lipschitz class of Bögel continuous functions.

**Keywords:** Gould-Hopper polynomials, modulus of continuity, Peetre's K-functional, Bögel continuous functions, mixed modulus of smoothness.

**2020 MSC:** 41A10, 41A25, 41A36, 41A63.



## 1. Introduction

Chen et al. [12] introduced a new family of generalized Bernstein operators based on a parameter $\beta \in [0, 1]$ and established a new proof of the Weierstrass approximation theorem besides discovering the shape preserving properties such as convexity and monotonicity of these operators. Kajla and Acar [18] constructed a Durrmeyer modification of the operators defined by Chen et al. [12] and obtained some local and global approximation results. Baxhaku and Berisha [8] introduced the Szász-Chlodowsky operators involving Gould-Hopper polynomials and established simultaneous approximation properties of these operators besides the results on weighted approximation and statistical convergence. Ana et al. [1] presented a new class of bivariate operators and also defined the associated generalized Boolean sum operators. Recently, Ana et al. [2] gave the representations for the inverses of certain positive linear

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operators and also found the Voronovskaja type formulas for the inverses of these operators. Agrawal and Ispir [4] considered a linking of the Bernstein-Chlodowsky polynomials and the generalized Szász-Charlier type operators. For some other related papers in this direction we refer to [3, 19] etc. Inspired by this work, in the present paper we consider generalized Bernstein-Chlodowsky-Kantorovich operators blended with Gould-Hopper polynomials as follows. Let $\{a_{m_1}\}_{m_1=1}^{\infty}$, $\{b_{m_2}\}_{m_2=1}^{\infty}$, and $\{c_{m_2}\}_{m_2=1}^{\infty}$ be unbounded sequences of positive real numbers such that

$$\lim_{m_1\to\infty}\frac{a_{m_1}}{m_1}=0, \quad \text{and} \quad \frac{b_{m_2}}{c_{m_2}}=1+O\left(\frac{1}{c_{m_2}}\right), \text{ as } m_2\to\infty.$$

Then for $0\leqslant\beta\leqslant1$, $\phi\in C(I_{a_{m_1}}):=\left\{\phi: I_{a_{m_1}}\to\mathbb{R}\;\middle|\;\phi \text{ is continuous}\right\}$, $I_{a_{m_1}}=\{(y,z): 0\leqslant y\leqslant a_{m_1},\ z\geqslant 0\}$, we propose

$$\begin{aligned} R_{m_1,m_2}(\phi(t_1,t_2);y,z) &= \frac{(m_1+1)}{a_{m_1}}c_{m_2}\sum_{j=0}^{\infty}\sum_{k=0}^{m_1}p_{m_1,k}^{(\beta)}\left(\frac{y}{a_{m_1}}\right)\\ &\quad\times e^{-b_{m_2}z-h}\,\frac{g_j{}^{d+1}(b_{m_2}z,h)}{j!}\int_{\frac{j}{c_{m_2}}}^{\frac{(j+1)}{c_{m_2}}}\int_{\frac{ka_{m_1}}{m_1+1}}^{\frac{(k+1)a_{m_1}}{m_1+1}}\phi(t_1,t_2)dt_1dt_2, \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} p_{m_1,k}^{(\beta)}\left(\frac{y}{a_{m_1}}\right) &= (1-\beta)\left[\binom{m_1-2}{k}\left(\frac{y}{a_{m_1}}\right)^k\left(1-\frac{y}{a_{m_1}}\right)^{m_1-k-1}+\binom{m_1-2}{k-2}\left(\frac{y}{a_{m_1}}\right)^{k-1}\left(1-\frac{y}{a_{m_1}}\right)^{m_1-k}\right]\\ &\quad+\binom{m_1}{k}\beta\left(\frac{y}{a_{m_1}}\right)^k\left(1-\frac{y}{a_{m_1}}\right)^{m_1-k}, m_1\geqslant 2, \end{aligned}$$

and the generating function for Gould-Hopper polynomials is defined as

$$e^{ht^{d+1}}e^{yt}=\sum_{k=0}^{\infty}g_k^{d+1}(y,h)\frac{t^k}{k!}, \quad h\geqslant 0,$$

The present article deals with the approximation degree for the operators given by Eq. (1.1) using the classical approach of modulus of continuity and the Peetre's K-functional. Weighted approximation and the approximation of functions in a Bögel space are also investigated.

## 2. Preliminaries

**Lemma 2.1.** *Let $e_{p,q}(t_1,t_2)=t_1{}^pt_2{}^q$, $p,q=0,1,2,\ldots$. The raw moments of the operators $R_{m_1,m_2}(.;y,z)$ are given by the following identities:*

$$R_{m_1,m_2}(e_{0,0};y,z)=1, \tag{i}$$

$$R_{m_1,m_2}(e_{1,0};y,z)=\frac{m_1}{m_1+1}y+\frac{a_{m_1}}{2(m_1+1)}, \tag{ii}$$

$$\begin{aligned} R_{m_1,m_2}(e_{2,0};y,z) &= \left(\frac{m_1}{m_1+1}\right)^2y^2+\frac{m_1+2-2\beta}{m_1+1}\left(\frac{a_{m_1}}{m_1+1}\right)y\left(1-\frac{y}{a_{m_1}}\right)+\frac{m_1}{m_1+1}\left(\frac{a_{m_1}}{m_1+1}\right)y\\ &\quad+\frac{1}{3}\left(\frac{a_{m_1}}{m_1+1}\right)^2, \end{aligned} \tag{iii}$$

$$R_{m_1,m_2}(e_{3,0};y,z)=\left(\frac{1}{4}\right)\left(\frac{a_{m_1}}{m_1+1}\right)^3\left(1+4y\frac{m_1}{a_{m_1}}+6m_1{}^2\left\{\left(\frac{y}{a_{m_1}}\right)^2+\frac{m_1+2-2\beta}{m_1{}^2}\frac{y}{a_{m_1}}\left(1-\frac{y}{a_{m_1}}\right)\right\}\right.$$

$$+4{m_1}^3\left\{\left(\frac{y}{a_{m_1}}\right)^3+3\left(\frac{m_1+2-2\beta}{{m_1}^2}\right)\left(\frac{y}{a_{m_1}}\right)^2\left(1-\frac{y}{a_{m_1}}\right)\right. \tag{iv}$$
$$\left.+\left(\frac{m_1+6-6\beta}{{m_1}^3}\right)\left(\frac{y}{a_{m_1}}\right)\left(1-\frac{y}{a_{m_1}}\right)\left(1-\frac{2y}{a_{m_1}}\right)\right\}\Bigg),$$

$$R_{m_1,m_2}(e_{4,0};y,z)=\frac{1}{5}\left(\frac{a_{m_1}}{m_1+1}\right)^4\Bigg(1+\frac{5m_1y}{a_{m_1}}+10{m_1}^2\left\{\left(\frac{y}{a_{m_1}}\right)^2+\frac{m_1+2-2\beta}{{m_1}^2}\left(\frac{y}{a_{m_1}}\right)\left(1-\frac{y}{a_{m_1}}\right)\right\}$$
$$+10{m_1}^3\left\{\left(\frac{y}{a_{m_1}}\right)^3+\frac{3(m_1+2-2\beta)}{{m_1}^2}\left(\frac{y}{a_{m_1}}\right)^2\left(1-\frac{y}{a_{m_1}}\right)\right.$$
$$\left.+\frac{m_1+6-6\beta}{{m_1}^3}\left(\frac{y}{a_{m_1}}\right)\left(1-\frac{y}{a_{m_1}}\right)\left(1-\frac{2y}{a_{m_1}}\right)\right\} \tag{v}$$
$$+5{m_1}^4\left\{\left(\frac{y}{a_{m_1}}\right)^4+\frac{6m_1+12-12\beta}{{m_1}^2}\left(\frac{y}{a_{m_1}}\right)^3\left(1-\frac{y}{a_{m_1}}\right)\right.$$
$$+\frac{4m_1+24(1-\beta)}{{m_1}^3}\left(\frac{y}{a_{m_1}}\right)^2\left(1-\frac{y}{a_{m_1}}\right)\left(1-\frac{2y}{a_{m_1}}\right)+\frac{1}{{m_1}^4}\left\{\Big[3({m_1}^2-2m_1)\right.$$
$$+12(m_1-6)(1-\beta)\Big]\left(\frac{y}{a_{m_1}}\right)\left(1-\frac{y}{a_{m_1}}\right)$$
$$\left.\left.+(m_1+14-14\beta)\right\}\left(\frac{y}{a_{m_1}}\right)\left(1-\frac{y}{a_{m_1}}\right)\right\}\Bigg),$$

$$R_{m_1,m_2}(e_{0,1};y,z)=z\frac{b_{m_2}}{c_{m_2}}+\frac{h(d+1)}{c_{m_2}}+\frac{1}{2c_{m_2}}, \tag{vi}$$

$$R_{m_1,m_2}(e_{0,2};y,z)=\frac{1}{3c_{m_2}^2}\Bigg(1+3\Big\{b_{m_2}z+h(d+1)\Big\} \tag{vii}$$
$$+3\Big\{b_{m_2}^2z^2+b_{m_2}z(2h(d+1)+1)+h(h+1)(d+1)^2\Big\}\Bigg),$$

$$R_{m_1,m_2}(e_{0,3};y,z)=\frac{1}{4c_{m_2}^3}\Big\{1+4(b_{m_2}z+h(d+1))+6(b_{m_2}z+h(d+1))^2+4(b_{m_2}z+h(d+1))^3 \tag{viii}$$
$$+6(b_{m_2}y+h(d+1)^2)(1+2(b_{m_2}z+h(d+1)))+4(b_{m_2}z+h(d+1)^3)\Big\},$$

*(ix)* $$R_{m_1,m_2}(e_{0,4};y,z)=\frac{1}{c_{m_2}^4}\Big\{\frac{1}{5}+(b_{m_2}z+h(d+1))+2(b_{m_2}z+h(d+1))^2+2(b_{m_2}z+h(d+1)^2)$$
$$+2(b_{m_2}z+h(d+1))^3+6(b_{m_2}z+h(d+1))(b_{m_2}z+h(d+1)^2)$$
$$+2(b_{m_2}z+h(d+1)^3)+(b_{m_2}z+h(d+1))^4 \tag{ix}$$
$$+6(b_{m_2}z+h(d+1)^2)(b_{m_2}z+h(d+1))^2$$
$$+4(b_{m_2}z+h(d+1))(b_{m_2}z+h(d+1)^3)+3(b_{m_2}z+h(d+1)^2)^2$$
$$+(b_{m_2}z+h(d+1)^4)\Big\}.$$

By simple computations, we obtain the identities from (i) to (ix) hence the details are omitted.
As a consequence of Lemma 2.1, we have the following.

**Lemma 2.2.** *The central moments of the operators $R_{m_1,m_2}$ are given by Eq.* (1.1) *verify the following equalities:*

$$R_{m_1,m_2}((t_1-y);y,z)=\frac{1}{2}\frac{a_{m_1}}{m_1+1}-\frac{y}{m_1+1}, \tag{1}$$
$$R_{m_1,m_2}((t_2-z);y,z)=\frac{1}{2c_{m_2}}+z\left(\frac{b_{m_2}}{c_{m_2}}-1\right)+\frac{h(d+1)}{c_{m_2}}, \tag{2}$$

$$R_{m_1,m_2}((t_1-y)^2;y,z)=\frac{1}{3}\left(\frac{a_{m_1}}{m_1+1}\right)^2+\frac{a_{m_1}}{m_1+1}\left(\frac{m_1+2-2\beta}{m_1+1}\left(1-\frac{y}{a_{m_1}}\right)-\frac{1}{m_1+1}\right)y,+\frac{y^2}{(m_1+1)^2}, \tag{3}$$

$$\begin{aligned}R_{m_1,m_2}((t_2-z)^2;y,z)&=z^2\left\{1-\frac{2b_{m_2}}{c_{m_2}}+\left(\frac{b_{m_2}}{c_{m_2}}\right)^2\right\}+z\left\{\frac{2b_{m_2}(1+h(d+1))}{c_{m_2}^2}-\frac{1+2h(d+1)}{c_{m_2}}\right\}\\&\quad+\frac{1}{c_{m_2}^2}\left(h(d+1)((h+1)(d+1)+1)+\frac{1}{3}\right),\end{aligned} \tag{4}$$

$$\begin{aligned}R_{m_1,m_2}((t_1-y)^3;y,z)&=\frac{1}{4}\left(\frac{a_{m_1}}{m_1+1}\right)^3+\frac{1}{m_1+1}\left(\frac{a_{m_1}}{m_1+1}\right)^2y-\frac{3}{2}\left(\frac{a_{m_1}}{m_1+1}\right)\frac{1}{(m_1+1)^2}y^2\\&\quad-\frac{3}{m_1+1}\left(\frac{m_1+2-2\beta}{m_1+1}\right)\frac{a_{m_1}}{m_1+1}\left(1-\frac{y}{a_{m_1}}\right)y^2\\&\quad+\frac{3}{2}\left(\frac{a_{m_1}}{m_1+1}\right)^2\left(\frac{m_1+2-2\beta}{m_1+1}\right)y\left(1-\frac{y}{a_{m_1}}\right)\\&\quad+\frac{m_1+6-6\beta}{m_1+1}\left(\frac{a_{m_1}}{m_1+1}\right)^2y\left(1-\frac{y}{a_{m_1}}\right)\left(1-\frac{2y}{a_{m_1}}\right)-\frac{y^3}{(m_1+1)^3},\end{aligned} \tag{5}$$

$$\begin{aligned}R_{m_1,m_2}((t_2-z)^3;y,z)&=\left(\frac{b_{m_2}}{c_{m_2}}-1\right)^3z^3+\left(\frac{3h(d+1)}{c_{m_2}}\left(\frac{b_{m_2}}{c_{m_2}}-1\right)^2+\frac{9}{2c_{m_2}}\left(\frac{b_{m_2}}{c_{m_2}}-1\right)\left(\frac{b_{m_2}}{c_{m_2}}-\frac{1}{3}\right)\right)z^2\\&\quad+b_{m_2}\left(\frac{14+24h(d+1)+12h(h+1)(d+1)^2}{4{c_{m_2}}^3}\right)z\\&\quad-\left(\frac{1+3h(d+1)((h+1)(d+1)+1)}{{c_{m_2}}^2}\right)z\\&\quad+\left(\frac{1+4h(d+1)+6(h^2+h)(d+1)^2+4h(d+1)^3(1+3h+h^2)}{4{c_{m_2}}^3}\right),\end{aligned} \tag{6}$$

$$\begin{aligned}R_{m_1,m_2}((t_1-y)^4;y,z)&=\left(\frac{1-20m_1-60(1-\alpha)+3m_1^2+12(m_1-6)(1-\alpha)}{(m_1+1)^4}\right)y^4\\&\quad+\left(\frac{a_{m_1}}{m_1+1}\right)\left(\frac{-1+40m_1+120(1-\beta)-24(m_1-6)(1-\beta)}{(m_1+1)^3}\right)y^3\\&\quad+\left(\frac{a_{m_1}}{m_1+1}\right)^2\left(\frac{9{m_1}^2+33m_1+90(1-\beta)-24m_1\beta}{(m_1+1)^2}\right)y^2\\&\quad+\left(\frac{a_{m_1}}{m_1+1}\right)^3\left(6\frac{m_1+5(1-\beta)}{m_1+1}-1\right)y+\frac{1}{5}\left(\frac{a_{m_1}}{m_1+1}\right)^4,\end{aligned} \tag{7}$$

$$\begin{aligned}R_{m_1,m_2}((t_2-z)^4;y,z)&=\left(\frac{b_{m_2}}{c_{m_2}}-1\right)^4z^4+\frac{z^3}{c_{m_2}}\left\{(4h(d+1)-2)\left(\frac{b_{m_2}}{c_{m_2}}-1\right)^3+6\frac{b_{m_2}}{c_{m_2}}\left(\frac{b_{m_2}}{c_{m_2}}-1\right)^2\right\}\\&\quad+\frac{z^2}{{c_{m_2}}^2}\times\left\{\left(\frac{b_{m_2}}{c_{m_2}}\right)^2(27+6h(d+1)((h+1)(d+1)+1))\right.\\&\quad\left.-\left(\frac{b_{m_2}}{c_{m_2}}\right)(14+12h(d+1)(2+(h+1)(d+1)))\right\}\\&\quad+\frac{b_{m_2}}{{c_{m_2}}^4}(6+14h(d+1)+12h^2(d+2)(d+1)^2\\&\quad+4h^2(h+1)(d+1)^3+12h(d+1)^2)z\\&\quad-\frac{1}{{c_{m_2}}^3}\left(1+4h(d+1)+6h(h+1)(d+1)^2+4h(d+1)^3(1+3h+h^2)\right)y\\&\quad+\frac{1}{{c_{m_2}}^4}\left(\frac{1}{5}+h(d+1)+2(h+1)h(d+1)^2+2(d+1)^3(h^3+3h^2+h)\right.\end{aligned} \tag{8}$$

$$+(d+1)^4(h+7h^2+6h^3+h^4)\Big).$$

*Remark* 2.3. Using Lemma 2.2, for all $(y,z)\in I_{a_{m_1}}$ and sufficiently large $m_1$, $m_2$, we get

$$\begin{aligned}
R_{m_1,m_2}((t_1-y);y,z) &= O\left(\frac{a_{m_1}}{m_1}\right)(1+y),\\
R_{m_1,m_2}((t_2-z);y,z) &= O\left(\frac{1}{c_{m_2}}\right)(1+z),\\
R_{m_1,m_2}((t_1-y)^2;y,z) &= O\left(\frac{a_{m_1}}{m_1}\right)(1+y+y^2),\\
R_{m_1,m_2}((t_2-z)^2;y,z) &= O\left(\frac{1}{c_{m_2}}\right)(1+z+z^2),\\
R_{m_1,m_2}((t_1-y)^3;y,z) &= O\left(\left(\frac{a_{m_1}}{m_1}\right)^2\right)(1+y+y^2+y^3),\\
R_{m_1,m_2}((t_2-z)^3;y,z) &= O\left(\frac{1}{c^2_{m_2}}\right)(1+z+z^2+z^3),\\
R_{m_1,m_2}((t_1-y)^4;y,z) &= O\left(\left(\frac{a_{m_1}}{m_1}\right)^2\right)(1+y+y^2+y^3+y^4),\\
R_{m_1,m_2}((t_2-z)^4;y,z) &= O\left(\frac{1}{c^2_{m_2}}\right)(1+z+z^2+z^3+z^4).
\end{aligned}$$

## 3. Main results

Let us define the spaces: $C_B(I_{a_{m_1}}):=\left\{\phi\in C(I_{a_{m_1}})\;\middle|\;\phi\text{ is bounded}\right\}$ and $\widetilde{C_B}(I_{a_{m_1}}):=\left\{C_B(I_{a_{m_1}})\;\middle|\;\phi\text{ is uniformly continuous}\right\}$. For $\phi\in C_B(I_{a_{m_1}})$, let the norm be given by

$$\|\phi\|=\sup_{(y,z)\in I_{a_{m_1}}}|\phi(y,z)|.$$

Further for $\phi\in\widetilde{C_B}(I_{a_{m_1}})$ and any $\delta>0$, the total modulus of continuity is defined as

$$\hat{\omega}(\phi;\delta)=\sup\left\{|\phi(\nu,u)-\phi(y,z)|:(\nu,u),(y,z)\in I_{a_{m_1}}\quad\text{and}\quad\sqrt{(\nu-y)^2+(u-z)^2}\leqslant\delta\right\},$$

and the two partial moduli of continuity are given by

$$\omega^{(1)}(\phi;\delta)=\sup\left\{|\phi(\nu,z)-f(y,z)|;z\in[0,\infty)\ \text{ and }\ |\nu-y|\leqslant\delta\right\},$$

and

$$\omega^{(2)}(\phi;\delta)=\sup\left\{|f(y,u)-f(y,z)|;y\in[0,a_{m_1}]\ \text{ and }\ |u-z|\leqslant\delta\right\}.$$

It is well known that the above moduli of continuity possess the properties similar to the usual modulus of continuity. In our further study, we assume $\delta_{m_1,r}(y)=R_{m_1,m_2}((\nu-y)^r;y,z)$, $\mu_{m_2,r}(z)=R_{m_1,m_2}((\nu-z)^r;y,z), r=1,2,3,\dots,$ and

$$\nu_{m_1,m_2}(y,z)\;=\;\{R_{m_1,m_2}((\nu-y)^2;y,z)+R_{m_1,m_2}((u-z)^2;y,z)\}^{\frac{1}{2}}=\{\delta_{m_1,2}(y)+\mu_{m_2,2}(z)\}^{\frac{1}{2}}.$$

**Theorem 3.1.** *For all* $(y,z) \in I_{a_{m_1}}$ *and* $\phi \in \widetilde{C_B}(I_{a_{m_1}})$, *there follows the inequality*

$$|R_{m_1,m_2}(\phi;y,z) - \phi(y,z)| \leqslant 2\hat{\omega}(\phi;\nu_{m_1,m_2}(y,z)).$$

*Proof.* Applying the definition of total modulus of continuity, Lemma 2.1 and the Cauchy-Schwarz inequality, for any $\delta > 0$, we obtain

$$\begin{aligned}
|R_{m_1,m_2}(\phi;y,z) - \phi(y,z)| &\leqslant R_{m_1,m_2}(|\phi(\nu,u) - \phi(y,z)|;y,z)\\
&\leqslant R_{m_1,m_2}(\hat{\omega}(\phi;\sqrt{(\nu-y)^2+(u-z)^2});y,z)\\
&\leqslant \hat{\omega}(\phi;\delta)R_{m_1,m_2}\left(1+\frac{1}{\delta}\sqrt{(\nu-y)^2+(u-z)^2};y,z\right)\\
&\leqslant \hat{\omega}(\phi;\delta)\left(1+\frac{1}{\delta}\{R_{m_1,m_2}((\nu-y)^2+(u-z)^2;y,z)^{\frac{1}{2}}\}\right).
\end{aligned}$$

Now, choosing $\delta = \nu_{m_1,m_2}(y,z)$, we get the desired result. □

**Theorem 3.2.** *For all* $(y,z) \in I_{a_{m_1}}$ *and* $\phi \in \widetilde{C_B}(I_{a_{m_1}})$, *there holds the following result:*

$$|R_{m_1,m_2}(\phi;y,z) - \phi(y,z)| \leqslant 2(w^{(1)}(\phi;\delta^{\frac{1}{2}}_{m_1,2}(y)) + w^{(2)}(\phi;\mu^{\frac{1}{2}}_{m_2,2}(z))).$$

*Proof.* Using the definition of partial moduli of continuity, Lemma 2.1 and the Cauchy Schwarz inequality, we are led to

$$\begin{aligned}
|R_{m_1,m_2}(\phi;y,z) - \phi(y,z)| &\leqslant R_{m_1,m_2}(|\phi(\nu,u) - \phi(y,z)|;y,z)\\
&\leqslant R_{m_1,m_2}(|\phi(\nu,u) - \phi(y,u)|;y,z) + R_{m_1,m_2}(|\phi(y,u) - \phi(y,z)|;y,z)\\
&\leqslant \omega^{(1)}(\phi;\delta_1)R_{m_1,m_2}\left(1+\frac{|\nu-y|}{\delta_1};y,z\right) + \omega^{(2)}(\phi;\delta_2)R_{m_1,m_2}\left(1+\frac{|u-z|}{\delta_2};y,z\right)\\
&\leqslant \omega^{(1)}(\phi;\delta_1)\left[1+\frac{1}{\delta_1}\sqrt{R_{m_1,m_2}((\nu-y)^2;y,z)}\right]\\
&\quad + \omega^{(2)}(\phi;\delta_2)\left[1+\frac{1}{\delta_2}\sqrt{R_{m_1,m_2}((u-z)^2;y,z)}\right].
\end{aligned}$$

Choosing $\delta_1 = \delta^{\frac{1}{2}}_{m_1,2}(y)$ and $\delta_2 = \mu^{\frac{1}{2}}_{m_2,2}(z)$, we reach the desired result. □

**Lipschitz Class**: For $0 < \xi_1, \xi_2 \leqslant 1$, the Lipschitz class $\text{Lip}_M(\xi_1,\xi_2)$ for a function of two variables is defined as

$$\text{Lip}_M(\xi_1,\xi_2) = \{\phi \in C_B(I_{a_{m_1}}) : |\phi(\nu,u) - \phi(y,z)| \leqslant M|\nu-y|^{\xi_1}|u-z|^{\xi_2}\},$$

where $(\nu,u),(y,z) \in I_{a_{m_1}}$ and $M$ is a positive constant.

**Theorem 3.3.** *For all* $(y,z) \in I_{a_{m_1}}$ *and* $\phi \in \text{Lip}_M(\xi_1,\xi_2)$, *we have*

$$|R_{m_1,m_2}(\phi;y,z) - \phi(y,z)| \leqslant M\,\delta^{\frac{\xi_1}{2}}_{m_1,2}(y)\,\mu^{\frac{\xi_2}{2}}_{m_2,2}(z),\ \xi_1,\xi_2 \in (0,1].$$

*Proof.* By the hypothesis

$$|R_{m_1,m_2}(\phi;y,z) - \phi(y,z)| \leqslant R_{m_1,m_2}(|\phi(\nu,u) - \phi(y,z)|;y,z) \leqslant MR_{m_1,m_2}(|\nu-y|^{\xi_1}|u-z|^{\xi_2};y,z).$$

Applying Hölder's inequality with $q_1 = \frac{2}{\xi_1}$, $r_1 = \frac{2}{2-\xi_1}$ and $q_2 = \frac{2}{\xi_2}$, $r_2 = \frac{2}{2-\xi_2}$, in view of Lemma 2.1, we get

$$\begin{aligned}
|R_{m_1,m_2}(\phi;y,z) - \phi(y,z)| &\leqslant M(R_{m_1,m_2}(\nu-y)^2;y,z)^{\frac{\xi_1}{2}}(R_{m_1,m_2}(1;y,z))^{\frac{2-\xi_1}{2}}\\
&\quad\times R_{m_1,m_2}((u-z)^2;y,z)^{\frac{\xi_2}{2}}R_{m_1,m_2}(1;y,z)^{\frac{2-\xi_2}{2}} \leqslant M\,\delta^{\frac{\xi_1}{2}}_{m_1,2}(y)\,\mu^{\frac{\xi_2}{2}}_{m_2,2}(z).
\end{aligned}$$

This completes the proof. □

In the following theorem, we obtain the degree of approximation for $\phi \in C_B^1(I_{a_{m_1}})$, the space of bounded and continuous functions in $I_{a_{m_1}}$ whose first order partial derivatives are bounded and continuous in $I_{a_{m_1}}$, i.e., $C_B^1(I_{a_{m_1}}) := \{\phi \in C_B(I_{a_{m_1}}) : \phi'_y, \phi'_z \in C_B(I_{a_{m_1}})\}$.

**Theorem 3.4.** *Let $\phi \in C_B^1(I_{a_{m_1}})$. Then for each $(y,z) \in I_{a_{m_1}}$, we have*

$$|R_{m_1,m_2}(\phi; y,z) - \phi(y,z)| \leqslant \|\phi'_y\|\delta^{\frac{1}{2}}_{m_1,2}(y) + \|\phi'_z\|\mu^{\frac{1}{2}}_{m_2,2}(z).$$

*Proof.* Since $\phi \in C_B^1(I_{a_{m_1}})$, we get

$$\phi(\nu, t_2) - \phi(y,z) = \int_y^\nu \phi'_{t_1}(t_1,u)dt_1 + \int_z^u \phi'_{t_2}(y,t_2)dt_2.$$

Hence, applying Cauchy Schwarz inequality and Lemma 2.1, we obtain

$$\begin{aligned}|R_{m_1,m_2}(\phi(\nu,t_2); y,z) - \phi(y,z)| &\leqslant R_{m_1,m_2}\left(\left|\int_y^\nu |\phi'_{t_1}(t_1,u)|dt_1\right| + \left|\int_z^u |\phi'_{t_2}(y,t_2)|dt_2\right|; y,z\right)\\ &\leqslant \|\phi'_y\|R_{m_1,m_2}\left(\left|\int_y^\nu dt\right|; y,z\right) + \|\phi'_z\|R_{m_1,m_2}\left(\left|\int_z^u dt_2\right|; y,z\right)\\ &\leqslant \|\phi'_y\|\sqrt{R_{m_1,m_2}((\nu-y)^2; y,z)} + \|\phi'_z\|\sqrt{R_{m_1,m_2}((u-z)^2; y,z)},\end{aligned}$$

which leads us to the desired result. □

**Peetre's K-functional:** Let us define $C_B^2(I_{a_{m_1}}) := \left\{\phi \in C_B(I_{a_{m_1}}) : \frac{\partial^{i+j}\phi}{\partial y^i \partial z^j} \in C_B(I_{a_{m_1}});\ 1 \leqslant i+j \leqslant 2\right\}$, with the norm given by

$$\|\phi\|_{C_B^2(I_{a_{m_1}})} = \|\phi\| + \sum_{i=1}^{2}\left(\left\|\frac{\partial^i \phi}{\partial y^i}\right\| + \left\|\frac{\partial^i \phi}{\partial z^i}\right\|\right) + \left\|\frac{\partial^2 \phi}{\partial y \partial z}\right\|.$$

For $\phi \in \widetilde{C_B}(I_{a_{m_1}})$, the Peetre's K-functional is defined as

$$K(\phi;\delta) = \inf_{\varphi \in C_B^2(I_{a_{m_1}})} \left\{\|\phi - \varphi\| + \delta\|\varphi\|_{C_B^2(I_{a_{m_1}})},\ \delta > 0\right\}.$$

From [11] it is known that

$$K(\phi;\delta) \leqslant M_1\{\hat{\omega}_2(\phi;\sqrt{\delta}) + \min(1,\delta)\|\phi\|\}, \tag{3.1}$$

holds true $\forall\ \delta > 0$, where $M_1$ is a positive constant that does not depend on $\delta$ and $\phi$, and $\hat{\omega}_2(\phi;\delta)$ is the second order modulus of continuity for a function of two variables.

**Theorem 3.5.** *For $\phi \in \widetilde{C_B}(I_{a_{m_1}})$, the operator which is given by Eq. (1.1) satisfies the following inequality*

$$|R_{m_1,m_2}(\phi; y,z) - \phi(y,z)| \leqslant M\left\{\hat{\omega}_2(\phi;\sqrt{\zeta_{m_1,m_2}(y,z)}) + \min(1,\zeta_{m_1,m_2}(y,z))\|\phi\|\right\} + \hat{\omega}(\phi;\tau_{m_1,m_2}(y,z)),$$

*where*

$$\zeta_{m_1,m_2}(y,z) = \left\{\delta_{m_1,2}(y) + \delta^2_{m_1,1}(y) + \mu_{m_2,2}(z) + \mu^2_{m_2,1}(z) + \sqrt{\delta_{m_1,2}(y)}\sqrt{\mu_{m_2,2}(z)} + |\delta_{m_1,1}(y)|\,|\mu_{m_2,1}(z)|\right\},$$

*and*

$$\tau_{m_1,m_2}(y,z) = \sqrt{\delta^2_{m_1,1}(y) + \mu^2_{m_2,1}(z)}.$$

*Proof.* Let us describe an auxiliary operator as follows:

$$R^{\star}_{m_1,m_2}(\phi;y,z)=R_{m_1,m_2}(\phi;y,z)-\phi\left(\frac{m_1y}{m_1+1}+\frac{a_{m_1}}{2(m_1+1)},\ \frac{1}{2c_{m_2}}+z\frac{b_{m_2}}{c_{m_2}}+\frac{h(d+1)}{c_{m_2}}\right)+\phi(y,z). \tag{3.2}$$

Clearly $R^{\star}_{m_1,m_2}$ is a linear operator and in view of Lemma 2.1,

$$R^{\star}_{m_1,m_2}(1;y,z)=1,\ \ R^{\star}_{m_1,m_2}((t_1-y);y,z)=0,\ \ R^{\star}_{m_1,m_2}((t_2-z);y,z)=0. \tag{3.3}$$

Let $\varphi\in C^2_B(I_{a_{m_1}})$ and $(y,z)\in I_{a_{m_1}}$ be arbitrary. Then by Taylor's formula for two variables,

$$\begin{aligned}\varphi(t_1,t_2)-\varphi(y,z)&=\varphi(t_1,z)-\varphi(y,z)+\varphi(t_1,t_2)-\varphi(t_1,z)\\&=(t_1-y)\frac{\partial\varphi(y,z)}{\partial y}+\int_y^{t_1}(t_1-v)\frac{\partial^2\varphi(v,z)}{\partial v^2}dv+(t_2-z)\frac{\partial\varphi(y,z)}{\partial z}\\&\quad+\int_z^{t_2}(t_2-u)\frac{\partial^2\varphi(y,u)}{\partial u^2}du+\int_y^{t_1}\int_z^{t_2}\frac{\partial^2\varphi(v,u)}{\partial v\partial u}dvdu.\end{aligned} \tag{3.4}$$

Let

$$\alpha_{m_1}(y)=\frac{m_1y}{m_1+1}+\frac{a_{m_1}}{2(m_1+1)}\qquad\text{and}\qquad\beta_{m_2}(z)=\frac{1}{2c_{m_2}}+z\frac{b_{m_2}}{c_{m_2}}+\frac{h(d+1)}{c_{m_2}}.$$

Now applying the linear operator $R^{\star}_{m_1,m_2}(;x,y)$ on both sides of Eq. (3.4) and using Eq. (3.3), we have

$$\begin{aligned}R^{\star}_{m_1,m_2}(\varphi(t_1,t_2);y,z)-\varphi(y,z)&=R^{\star}_{m_1,m_2}\left(\int_y^{t_1}(t_1-v)\frac{\partial^2\varphi(v,z)}{\partial v^2}dv;y,z\right)\\&\quad+R^{\star}_{m_1,m_2}\left(\int_z^{t_2}(t_2-u)\frac{\partial^2\varphi(y,u)}{\partial u^2}du;y,z\right)\\&\quad+R^{\star}_{m_1,m_2}\left(\int_y^{t_1}\int_z^{t_2}\frac{\partial^2\varphi(v,u)}{\partial v\partial u}dvdu;y,z\right).\end{aligned}$$

Then using Eq. (3.3), we obtain

$$\begin{aligned}&R^{\star}_{m_1,m_2}(\varphi(t_1,t_2);y,z)-\varphi(y,z)\\&=R_{m_1,m_2}\left(\int_y^{t_1}(t_1-v)\frac{\partial^2\varphi(v,z)}{\partial v^2}dv;y,z\right)-\int_y^{\alpha_{m_1}(y)}\left(\alpha_{m_1}(y)-v\right)\frac{\partial^2\varphi(v,z)}{\partial v^2}dv\\&\quad+R_{m_1,m_2}\left(\int_z^{t_2}(t_2-u)\frac{\partial^2\varphi(y,u)}{\partial u^2}du;y,z\right)-\int_z^{\beta_{m_2}(z)}\left(\beta_{m_2}(z)-u\right)\frac{\partial^2\varphi(y,u)}{\partial u^2}du\\&\quad+R_{m_1,m_2}\left(\int_y^{t_1}\int_z^{t_2}\frac{\partial^2\varphi(v,u)}{\partial v\partial u}dvdu;y,z\right)-\int_y^{\alpha_{m_1}(y)}\int_z^{\beta_{m_2}(z)}\frac{\partial^2\varphi(v,u)}{\partial v\partial u}dvdu.\end{aligned}$$

Hence,

$$\begin{aligned}&|R^{\star}_{m_1,m_2}(\varphi(t_1,t_2);y,z)-\varphi(y,z)|\\&\leqslant R_{m_1,m_2}\left(\left|\int_y^{t_1}|t_1-v|\left|\frac{\partial^2\varphi(v,z)}{\partial v^2}\right|dv\right|;y,z\right)+\left|\int_y^{\alpha_{m_1}(y)}|\alpha_{m_1}(y)-v|\left|\frac{\partial^2\varphi(v,z)}{\partial v^2}\right|dv\right|\\&\quad+R_{m_1,m_2}\left(\left|\int_z^{t_2}|t_2-u|\left|\frac{\partial^2\varphi(y,u)}{\partial u^2}\right|du\right|;y,z\right)+\left|\int_z^{\beta_{m_2}(z)}|\beta_{m_2}(z)-u|\left|\frac{\partial^2\varphi(y,u)}{\partial u^2}\right|du\right|\\&\quad+R_{m_1,m_2}\left(\left|\int_y^{t_2}\int_z^{t_2}\left|\frac{\partial^2\varphi(v,u)}{\partial v\partial u}\right|dvdu;y,z\right|\right)+\left|\int_y^{\alpha_{m_1}(y)}\int_z^{\beta_{m_2}(z)}\left|\frac{\partial^2\varphi(v,u)}{\partial v\partial u}\right|dvdu\right|\end{aligned}$$

$$\leqslant \Big\{\delta_{m_1,2}(y)+\delta^2_{m_1,1}+\mu_{m_2,2}(z)+\mu^2_{m_2,1}(z)+\sqrt{\delta_{m_1,2}(y)}\sqrt{\mu_{m_2,2}(z)}$$
$$+|\delta_{m_1,1}(y)|\,|\mu_{m_2,1}(z)|\Big\}\|\varphi\|_{C^2_B(I_{a_{m_1}})}$$
$$=\zeta_{m_1,m_2}(y,z)\|\varphi\|_{C^2_B(I_{a_{m_1}})}.$$

From Eq. (3.2) and using Lemma 2.1, we get

$$|R^{\star}_{m_1,m_2}(\phi(t_1,t_2);y,z)| \leqslant 3\|\phi\|. \tag{3.5}$$

For $\phi \in \widetilde{C_B}(I_{a_{m_1}})$ and any $\varphi \in C^2_B(I_{a_{m_1}})$ using equations Eq. (3.1), Eq. (3.4), and Eq. (3.5), we get

$$\begin{aligned}|R_{m_1,m_2}(\phi;y,z)-\phi(y,z)| &= \Big||R^{\star}_{m_1,m_2}(\phi;y,z)-\phi(y,z)+\phi(\alpha_{m_1}(y),\beta_{m_2}(z))-\phi(y,z)\Big|\\ &\leqslant \Big|R^{\star}_{m_1,m_2}((\phi-\varphi);y,z)\Big|+\Big|R^{\star}_{m_1,m_2}(\varphi;y,z)-\varphi(y,z)\Big|+\Big|\varphi(y,z)-\phi(y,z)\Big|\\ &\quad+\Big|\phi(\alpha_{m_1}(y),\beta_{m_2}(z))-\phi(y,z)\Big|\\ &\leqslant 4\|\phi-\varphi\|+\zeta_{m_1,m_2}(y,z)\|\varphi\|_{C^2_B(I_{a_{m_1}})}+\hat{\omega}\Big(\phi;\sqrt{\delta^2_{m_1,1}(y)+\mu^2_{m_2,1}(z)}\Big).\end{aligned}$$

Now taking the infimum on the right hand side over all $\varphi \in C^2_B(I_{a_{m_1}})$, we obtain

$$|R_{m_1,m_2}(\phi;y,z)-\phi(y,z)| \leqslant 4K(\phi;\zeta_{m_1,m_2}(y,z))+\hat{\omega}\Big(\phi;\tau_{m_1,m_2}(y,z)\Big).$$

Finally, using the relation between the Peetre's K-functional and the second order modulus of continuity given by Eq. (3.1), we arrive at

$$|R_{m_1,m_2}(\phi;y,z)-\phi(y,z)| \leqslant M\Big\{\hat{\omega}_2(\phi;\sqrt{\zeta_{m_1,m_2}(y,z)})+\min\{1,\zeta_{m_1,m_2}(y,z)\}\|\phi\|\Big\}+\hat{\omega}\Big(\phi;\tau_{m_1,m_2}(y,z)\Big),$$

which completes the proof of the theorem. □

## 4. Weighted approximation

In this section we consider the approximation of functions in a weighted space by the operators defined by Eq. (1.1). Let $B_\rho := \{\phi : \mathbb{R}^+_0 \times \mathbb{R}^+_0 \to \mathbb{R} \mid |\phi(y,z)| \leqslant M_\phi\, \rho(y,z)\}$, where $\mathbb{R}^+_0 = [0,\infty)$, $M_\phi$ is a positive constant depending only on $\phi$, and $\rho(y,z) = 1+y^2+z^2$, with the norm

$$\|\phi\|_\rho = \sup_{(y,z)\in R^+_0\times R^+_0}\frac{|f(y,z)|}{\rho(y,z)}.$$

Let $C_\rho = \{\phi \in B_\rho : \phi \text{ is continuous}\}$ and

$$C^0_\rho := \Big\{\phi \in C_\rho : \lim_{\sqrt{(y^2+z^2)}\to\infty}\frac{|f(y,z)|}{\rho(y,z)} \text{ exists finitely}\Big\}.$$

For all $\phi \in C^0_\rho$ and $\delta_1,\delta_2 > 0$, the weighted modulus of continuity [17] is defined by

$$\Omega_\rho(\phi;\delta_1,\delta_2) = \sup_{(y,z)\in R^+_0\times R^+_0}\ \sup_{0<|k_1|\leqslant\delta_1, 0<|k_2|\leqslant\delta_2}\frac{|f(y+k_1,z+k_2)-f(y,z)|}{\rho(y,z)\rho(k_1,k_2)}.$$

From [17], it is known that for any $\delta_1, \delta_2 > 0$

$$|\phi(t_1,t_2)-\phi(y,z)| \leqslant 4(1+y^2+z^2)(1+\delta_1^2)(1+\delta_2^2)\left(1+(t_1-y)^2\right)\left(1+\frac{|t_1-y|}{\delta_1}\right)$$
$$\times\left(1+(t_2-z)^2\right)\left(1+\frac{|t_2-z|}{\delta_2}\right)\Omega_\rho(\phi;\delta_1,\delta_2). \tag{4.1}$$

Let us consider the positive linear operators $C_{m_1,m_2}$, defined by

$$C_{m_1,m_2}(\phi;y,z)=\begin{cases} R_{m_1,m_2}(\phi;y,z), & \text{if } (y,z)\in I_{a_{m_1},d_{m_2}},\\ \phi(y,z), & \text{if } (y,z)\in R_0^+\times R_0^+\setminus I_{a_{m_1},d_{m_2}},\end{cases} \tag{4.2}$$

where $I_{a_{m_1},d_{m_2}}=\{(y,z); 0\leqslant y\leqslant a_{m_1},\ 0\leqslant z\leqslant d_{m_2}\}$, and $(d_{m_2})$ is a sequence such that

$$\lim_{m_2\to\infty} d_{m_2}=\infty.$$

Then applying the Korovkin type theorems given in [14, 15] we get following theorem.

**Theorem 4.1.** *For the sequence of operators given by Eq. (4.2) and $\phi\in C_\rho^0$, we have*

$$\lim_{m_1,m_2\to\infty}\|C_{m_1,m_2}\phi-\phi\|_\rho=0.$$

**Theorem 4.2.** *If $\phi\in C_\rho^0$, then for sufficiently large $m_1, m_2$, the inequality holds:*

$$\sup_{(y,z)\in R_0^+\times R_0^+}\frac{|R_{m_1,m_2}(\phi;y,z)-\phi(y,z)|}{(\rho(y,z))^{\frac{5}{2}}}\leqslant C\ \Omega_\rho\left(\phi;\sqrt{\frac{a_{m_1}}{m_1}},\sqrt{\frac{1}{c_{m_2}}}\right),$$

*where C is a positive constant independent of $m_1, m_2$ and $\phi$.*

*Proof.* From the inequality (4.1), for $\delta_1,\delta_2>0$, we have

$$|\phi(t_1,t_2)-\phi(y,z)|\leqslant 4(1+y^2+z^2)\left(1+(t_1-y)^2+(t_2-z)^2+(t_1-y)^2(t_2-z)^2\right)\left(1+\frac{|t_1-y|}{\delta_1}\right)$$
$$\times\left(1+\frac{|t_2-z|}{\delta_2}\right)(1+\delta_1^2)(1+\delta_2^2)\Omega_\rho(\phi;\delta_1,\delta_2).$$

Now, applying the linear operator $R_{m_1,m_2}(.;y,z)$ on both sides of inequality, in view of the positivity of this operator we get

$$\begin{aligned}|R_{m_1,m_2}(\phi(t_1,t_2;y,z)-\phi(y,z))|\leqslant 4(1+y^2+z^2)&\left[1+\frac{1}{\delta_1}R_{m_1,m_2}(|t_1-y|;y,z)\right.\\ &\left.+\frac{1}{\delta_1}R_{m_1,m_2}\left(|t_1-y|(t_1-y)^2;y,z\right)+R_{m_1,m_2}((t_1-y)^2;y,z)\right]\\ &\times\left[1+\frac{1}{\delta_2}R_{m_1,m_2}(|t_2-z|;y,z)+\frac{1}{\delta_2}R_{m_1,m_2}\left(|t_2-z|(t_2-z)^2;y,z\right)\right.\\ &\left.+R_{m_1,m_2}\left((t_2-z)^2;y,z\right)\right]\Omega_\rho(\phi;\delta_1,\delta_2)(1+\delta_1^2)(1+\delta_2^2).\end{aligned}$$

Now, using the Cauchy-Schwarz inequality, we get

$$|R_{m_1,m_2}(\phi(t_1,t_2;y,z)-\phi(y,z))|\leqslant 4(1+y^2+z^2)\left[1+\frac{1}{\delta_1}\sqrt{R_{m_1,m_2}((t_1-y)^2;y,z)}+R_{m_1,m_2}((t_1-y)^2;y,z)\right.$$

$$
\begin{aligned}
&+\frac{1}{\delta_1}\sqrt{R_{m_1,m_2}((t_1-y)^2;y,z)}\sqrt{R_{m_1,m_2}(t_1-y)^4;y,z}\Bigg]\\
&\times\left[1+\frac{1}{\delta_2}\sqrt{R_{m_1,m_2}((t_2-z)^2;y,z)}+R_{m_1,m_2}((t_2-z)^2;y,z)\right.\\
&\left.+\frac{1}{\delta_2}\sqrt{R_{m_1,m_2}((t_2-z)^2;y,z)}\sqrt{R_{m_1,m_2}(t_2-z)^4;y,z}\right]\\
&\times\Omega_\rho(\phi;\delta_1,\delta_2)(1+\delta_1^2)(1+\delta_2^2).
\end{aligned}
$$

In view of the Remark 2.3, we are led to

$$
\begin{aligned}
|R_{m_1,m_2}(\phi(t_1,t_2;y,z)-\phi(y,z))| &\leqslant C(1+y^2+z^2)(1+\delta_1^2)(1+\delta_2^2)\\
&\times\left[1+\frac{1}{\delta_1}\sqrt{\left(\frac{a_{m_1}}{m_1}\right)(1+y+y^2)}+\left(\frac{a_{m_1}}{m_1}\right)(1+y+y^2)\right.\\
&\left.+\frac{1}{\delta_1}\sqrt{\left(\frac{a_{m_1}}{m_1}\right)(1+y+y^2)}\sqrt{\left(\frac{a_{m_1}}{m_1}\right)(1+y+y^2+y^3+y^4)}\right]\\
&\times\left[1+\frac{1}{\delta_2}\sqrt{\left(\frac{1}{c_{m_2}}\right)(1+z+z^2)}+\left(\frac{1}{c_{m_2}}\right)(1+z+z^2)\right.\\
&\left.+\frac{1}{\delta_2}\sqrt{\left(\frac{1}{c_{m_2}}\right)(1+z+z^2)}\sqrt{\left(\frac{1}{c_{m_2}}\right)(1+z+z^2+z^3+z^4)}\right]\Omega_\rho(\phi;\delta_1,\delta_2).
\end{aligned}
$$

Now, taking $\delta_1=\left(\frac{a_{m_1}}{m_1}\right)^{\frac{1}{2}}$ and $\delta_2=\left(\frac{1}{c_{m_2}}\right)^{\frac{1}{2}}$, we obtain,

$$
|R_{m_1,m_2}(\phi(t_1,t_2;y,z)-\phi(y,z))|\leqslant C(1+y^2+z^2)^{\frac{5}{2}}\,\Omega_\rho(\phi;\delta_1,\delta_2),
$$

which leads us to the desired result. □

## 5. Approximation by associated GBS operators

The concepts of Bögel continuity and Bögel differentiability were introduced by Bögel ([9, 10]). Dobrescu and Matei [13] established the convergence of the Generalized Boolean Sum (GBS) of the Bivariate generalization of Bernstein polynomials for the Bögel continuous functions. Badea et al. [6] gave the "test function theorem" for Bögel continuous functions. A Shisha-Mond type theorem for Bögel continuous functions using GBS operators was derived by Badea et al. [5]. Badea and Cottin [7] established Korovkin type theorems for GBS operators. For a detailed account of the research work in this direction, we refer the readers to the book [16] and the references therein.

**B-continuous function**: Let Y and Z be any two subsets of $\mathbb{R}$ then a mapping $\phi : Y\times Z\to\mathbb{R}$ is called a B-continuous, i.e., (Bögel continuous) function at $(y,z)\in Y\times Z$ if and only if

$$
\lim_{(t_1,t_2)\to(y,z)}\triangle f[(t_1,t_2);(y,z)]=0,
$$

where

$$
\triangle\phi[(t_1,t_2);(y,z)]=\phi(t_1,t_2)-\phi(y,t_2)-\phi(t_1,z)+\phi(y,z)
$$

is known as "mixed difference" of $\phi$. Moreover, $\phi$ is called a B-continuous function in $Y\times Z$ if it is B-continuous at each point of $Y\times Z$. The space of B-continuous functions is denoted by $C_b(Y\times Z)$.

**B-bounded function**: A function $\phi : Y \times Z \to \mathbb{R}$ is called Bögel-bounded on $Y \times Z$ if $\exists$ $M > 0$ such that

$$|\triangle \phi[(t_1, t_2); (y, z)]| \leqslant M, \quad \forall (y, z), (t_1, t_2) \in Y \times Z.$$

Let $B(Y \times Z) = \{\phi : Y \times Z \to \mathbb{R} \mid \phi \text{ is bounded}\}$ and $C(Y \times Z) = \{\phi : Y \times Z \to \mathbb{R} \mid \phi \text{ is continuous}\}$ and the norm in $B(Y \times Z)$ be defined by

$$\|\phi\|_\infty = \sup_{(y,z)\in Y\times Z} |\phi(Y, Z)|.$$

**Uniformly B-continuous function**: A function $\phi : Y \times Z \to \mathbb{R}$ is said to be uniformly B-continuous in $Y \times Z$ if for any $\epsilon > 0$ $\exists$ $\delta = \delta(\epsilon) > 0$ such that $|\triangle\phi[(t_1, t_2); (y, z)]| < \epsilon$ whenever $|t_1 - y| < \delta$, $|t_2 - z| < \delta$, $(t_1, t_2), (y, z) \in Y \times Z$.

Let $\tilde{C_b}(Y \times Z)$ denote the space of uniformly B-continuous functions on $Y \times Z$.

**B-differentiable function**: A function $\phi : Y \times Z \to \mathbb{R}$ is called a Bögel differentiable function at $(y, z) \in Y \times Z$ if

$$\lim_{(t_1,t_2)\to(y,z)} \frac{\triangle\phi[(t_1, t_2); (y, z)]}{(t_1 - y)(t_2 - z)}$$

exists and is finite.

The limit is said to be the B-differential of $\phi$ at the point $(y, z)$ and is denoted by $D_B(\phi; y, z)$ and the set of all Bögel-differentiable functions is denoted by $D_b(Y \times Z)$.

**Mixed modulus of smoothness**: The mixed modulus of smoothness of $\phi \in C_b(Y \times Z)$ is defined as

$$\omega_{\text{mixed}}(\phi; \delta_1, \delta_2) = \sup \Big[ |\triangle \phi[(t_1, t_2); (y, z)]| : |y - t_1| \leqslant \delta_1, |z - t_2| \leqslant \delta_2 \Big],$$

for all $(y, z)$, $(t_1, t_2) \in Y \times Z$ and for $(\delta_1, \delta_2) \in (0, \infty) \times (0, \infty)$. It is known that the mixed modulus of smoothness $\omega_{\text{mixed}}(\phi; \delta_1, \delta_2)$ satisfies the following property,

$$\omega_{\text{mixed}}(\phi; \lambda_1\delta_1, \lambda_2\delta_2) \leqslant (1 + \lambda_1)(1 + \lambda_2)\omega_{\text{mixed}}(\phi; \delta_1, \delta_2); \quad \lambda_1,\ \lambda_2 > 0. \tag{5.1}$$

For any $\phi \in C_b(I_{a_{m_1}})$, we define the GBS operators $R^\star_{m_1,m_2}$ associated with this operators $R_{m_1,m_2}$ as follows:

$$\begin{aligned} R^\star_{m_1,m_2}(\phi(t_1, t_2); y, z) = {} & \frac{m_1 + 1}{a_{m_1}} c_{m_2} \sum_{j=0}^{\infty} \sum_{k=0}^{m_1} p^{(\alpha)}_{m_1,k}\left(\frac{y}{a_{m_1}}\right) e^{-b_{m_2}z-h} \frac{g_j^{d+1}(b_{m_2}z, h)}{j!} \\ & \times \int_{\frac{(k)a_{m_1}}{m_1+1}}^{\frac{(k+1)a_{m_1}}{m_1+1}} \int_{\frac{j}{c_{m_2}}}^{\frac{(j+1)}{c_{m_2}}} \Big( \phi(t_1, z) - \phi(t_1, t_2) + \phi(y, t_2) \Big) dt_1 dt_2, \end{aligned}$$

$0 \leqslant y \leqslant a_{m_1}$ and $z \geqslant 0$. It is clear that $R^\star_{m_1,m_2}$ is a linear operator.

**Lipschitz class**: For $0 < \xi_1, \xi_2 \leqslant 1$, the Lipschitz class $\text{Lip}^\star_M(\xi_1, \xi_2)$ of B-continuous functions is defined as $\text{Lip}^\star_M(\xi_1, \xi_2) = \Big\{ \phi \in C_b(I_{a_{m_1}}) : |\triangle\phi[(t_1, t_2); (y, z)]| \leqslant M|t_1 - y|^{\xi_1} |t_2 - z|^{\xi_2} \Big\}$, for some $M > 0$ and $(t_1, t_2), (y, z) \in I_{a_{m_1}}$.

The following theorem provides the degree of approximation by the operator $R^*_{m_1,m_2}(\phi(t_1, t_2); y, z)$ for the Lipschitz class of Bögel continuous functions.

**Theorem 5.1.** *For $\phi \in \text{Lip}^\star_M(\xi_1, \xi_2)$, there holds the following inequality:*

$$|R^\star_{m_1,m_2}(\phi(t_1, t_2); y, z) - \phi(y, z)| \leqslant M\delta^{\frac{\xi_1}{2}}_{m_1,2}(y)\mu^{\frac{\xi_2}{2}}_{m_2,2}(z), \qquad \xi_1, \xi_2 \in (0, 1].$$

*Proof.* From the definition of the operator $R_{m_1,m_2}(\phi(t_1,t_2);y,z)$, for any $(y,z) \in I_{a_{m_1}}$, we have

$$R^{\star}_{m_1,m_2}(\phi(t_1,t_2);y,z) = R_{m_1,m_2}\Big(\phi(t_1,z)+\phi(y,t_2)-f(t_1,t_2);y,z\Big).$$

By the definition of mixed difference of $\phi$,

$$\begin{aligned}\triangle\phi[(t_1,t_2);(y,z)] &= \phi(t_1,t_2)-\phi(y,t_2)-\phi(t_1,z)+\phi(y,z), \quad \text{or}\\ \phi(y,t_2)+\phi(t_1,z)-\phi(t_1,t_2) &= \phi(y,z)-\triangle\phi[(t_1,t_2);(y,z)].\end{aligned}$$

Applying the linear operator $R_{m_1,m_2}$ on both sides of the above equation, in view of $R_{m_1,m_2}(1;y,z)=1$, we obtain

$$R_{m_1,m_2}\Big(\phi(t_1,z)+\phi(y,t_2)-\phi(t_1,t_2);y,z\Big) = \phi(y,z)-R_{m_1,m_2}\Big(\triangle\phi[(t_1,t_2);(y,z)];y,z\Big)$$

or,

$$R^{\star}_{m_1,m_2}(\phi(t_1,t_2);y,z) = \phi(y,z)-R_{m_1,m_2}\Big(\triangle\phi[(t_1,t_2);(y,z)];y,z\Big),$$

which implies that

$$|R^{\star}_{m_1,m_2}(\phi(t_1,t_2);y,z)-\phi(y,z)| \leqslant R_{m_1,m_2}(|\triangle\phi[(t_1,t_2);(y,z)]|;y,z). \tag{5.2}$$

Since $\phi \in Lip^{\star}_{M}(\xi_1,\xi_2)$, we get

$$\begin{aligned}|R^{\star}_{m_1,m_2}(\phi(t_1,t_2);y,z)-\phi(y,z)| &\leqslant M\, R_{m_1,m_2}\Big(|t_1-y|^{\xi_1}|t_2-z|^{\xi_2};y,z\Big)\\ &\leqslant M\, R_{m_1,m_2}\Big(|t_1-y|^{\xi_1};y,z\Big)R_{m_1,m_2}\Big(|t_2-z|^{\xi_2};y,z\Big).\end{aligned}$$

Now using Hölder's inequality with $q_1=\frac{2}{\xi_1}$, $r_1=\frac{2}{2-\xi_1}$, $q_2=\frac{2}{\xi_2}$, $r_2=\frac{2}{2-\xi_2}$, and Lemma 2.1 we obtain

$$\begin{aligned}|R^{\star}_{m_1,m_2}(\phi(t_1,t_2);y,z)-\phi(y,z)| &\leqslant MR_{m_1,m_2}\Big((t_1-y)^2;y,z\Big)^{\frac{\xi_1}{2}}R_{m_1,m_2}\Big(e_{0,0};y,z\Big)^{\frac{2-\xi_1}{2}}\\ &\quad\times R_{m_1,m_2}\Big((t_2-z)^2;y,z\Big)^{\frac{\xi_2}{2}}R_{m_1,m_2}\Big(e_{0,0};y,z\Big)^{\frac{2-\xi_2}{2}}\\ &\leqslant M\delta^{\frac{\xi_1}{2}}_{m_1,2}(y)\mu^{\frac{\xi_2}{2}}_{m_2,2}(z),\end{aligned}$$

completing the proof. □

**Theorem 5.2.** *For any $\phi \in \tilde{C_b}(I_{a_{m_1}})$, at every point $(y,z) \in I_{a_{m_1}}$, the operator $R^{\star}_{m_1,m_2}$ satisfies the following inequality*

$$|R^{\star}_{m_1,m_2}(\phi;y,z)-\phi(y,z)| \leqslant 4\omega_{\text{mixed}}\Big(\phi;\sqrt{\delta_{m_1,2}(y)},\sqrt{\mu_{m_2,2}(z)}\Big).$$

*Proof.* Using the property Eq. (5.1) of the mixed modulus of smoothness, we have

$$|\triangle\phi[(t_1,t_2);(y,z)]| \leqslant \omega_{\text{mixed}}(\phi;|t_1-y|,|t_2-z|) \leqslant \left(1+\frac{|t_1-y|}{\delta_1}\right)\left(1+\frac{|t_2-z|}{\delta_2}\right)\omega_{\text{mixed}}(f;\delta_1,\delta_2) \tag{5.3}$$

for any $(t_1,t_2),(y,z) \in I_{a_{m_1}}$ and for any $\delta_1,\delta_2 > 0$. From Eq. (5.2) we have

$$|R^{\star}_{m_1,m_2}(\phi;y,z)-\phi(y,z)| \leqslant R_{m_1,m_2}(|\triangle\phi[(t_1,t_2);(y,z)]|;y,z).$$

Hence using Eq. (5.3) and applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}|R^{\star}_{m_1,m_2}(\phi;y,z)-\phi(y,z)| &\leqslant \Bigg[(R_{m_1,m_2}(e_{0,0};y,z))+\frac{1}{\delta_1}\sqrt{R_{m_1,m_2}((t_1-y)^2;y,z)}+\frac{1}{\delta_2}\sqrt{R_{m_1,m_2}((t_2-z)^2;y,z)}\\ &\quad+\frac{1}{\delta_1\delta_2}\sqrt{R_{m_1,m_2}((t_1-y)^2;y,z)R_{m_1,m_2}((t_2-z)^2;y,z)}\Bigg]\omega_{\text{mixed}}\Big(\phi;\delta_1,\delta_2\Big)\\ &\leqslant 4\omega_{\text{mixed}}\Big(\phi;\delta_1,\delta_2\Big),\end{aligned}$$

which leads us to the desired result on choosing $\delta_1=\delta^{\frac{1}{2}}_{m_1,2}(y),\ \delta_2=\mu^{\frac{1}{2}}_{m_2,2}(z)$ and using Lemma 2.1. □

**Theorem 5.3.** *If the function $\phi \in D_b(I_{a_{m_1}})$ with $D_B\phi \in \tilde{C}_b(I_{a_{m_1}})\bigcap B(I_{a_{m_1}})$, then for every $(y,z) \in I_{a_{m_1}}$, we have*

$$|R^{\star}_{m_1,m_2}(\phi(t_1,t_2);y,z)-\phi(y,z)| \leqslant M\Bigg(\|D_b\phi\|_\infty+\omega_{\text{mixed}}(D_B\phi;\sqrt{\frac{a_{m_1}}{m_1}},\sqrt{\frac{1}{c_{m_2}}}\Bigg).$$

*Proof.* Since $\phi \in D_b(I_{a_{m_1}})$, we have

$$\triangle\phi[(t_1,t_2);(y,z)]=(t_1-y)(t_2-z)D_B\phi(v,u) \tag{5.4}$$

with $y \leqslant v \leqslant t_1$ and $z \leqslant u \leqslant t_2$. Since

$$D_B\phi(v,u)=\triangle D_B\phi(v,u)+D_B\phi(v,z)+D_B\phi(y,u)-D_B\phi(y,z),$$

and $D_B\phi \in B(I_{a_{m_1}})$, from (5.4), we can write

$$\begin{aligned}|R^{\star}_{m_1,m_2}(\triangle\phi[(t_1,t_2);y,z];y,z)| &= |R^{\star}_{m_1,m_2}((t_1-y)(t_2-z)D_B\phi(v,u);y,z|\\ &\leqslant R^{\star}_{m_1,m_2}(|t_1-y|\,|t_2-z|\,|\triangle D_B\phi(v,u)|;y,z)\\ &\quad+R^{\star}_{m_1,m_2}\Big(|t_1-y|\,|t_2-z|\,|(D_B\phi(v,z)\\ &\quad+D_B\phi(y,u)-D_B\phi(y,z))|;y,z\Big) \qquad (5.5)\\ &\leqslant R^{\star}_{m_1,m_2}\Big(|t_1-y||t_2-z|\omega_{\text{mixed}}(D_B\phi;|v-y|,|u-z|);y,z\Big)\\ &\quad+3\|D_B\phi\|_\infty R^{\star}_{m_1,m_2}(|t_1-y||t_2-z|;y,z).\end{aligned}$$

Since the mixed modulus of smoothness $\omega_{\text{mixed}}$ is a non decreasing function, we have

$$\omega_{\text{mixed}}(D_B\phi;|v-y|,|u-z|) \leqslant \Bigg(1+\frac{|t_1-y|}{\delta_1}\Bigg)\Bigg(1+\frac{|t_2-z|}{\delta_2}\Bigg)\omega_{\text{mixed}}(D_B\phi;\delta_1,\delta_2), \tag{5.6}$$

for any $\delta_1,\delta_2 > 0$. From Eq. (5.2), we have

$$|R^{\star}_{m_1,m_2}(\phi((t_1,t_2);y,z))-\phi(y,z)|=|R_{m_1,m_2}(\triangle\phi[(v,u);(y,z)];y,z)|,$$

hence using Eq. (5.5) and Eq. (5.6) and applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&|R^{\star}_{m_1,m_2}(\phi((t_1,t_2);y,z)) - \phi(y,z)| \\
&\leqslant C\Bigg[\|D_B\phi\|_\infty\sqrt{\frac{a_{m_1}}{m_1}}\sqrt{\frac{1}{c_{m_2}}} + \Bigg\{\sqrt{\frac{a_{m_1}}{m_1}}\sqrt{\frac{1}{c_{m_2}}} + \frac{1}{\delta_1}\left(\frac{a_{m_1}}{m_1}\right)\sqrt{\frac{1}{c_{m_2}}} \\
&\quad + \frac{1}{\delta_2}\sqrt{\frac{a_{m_1}}{m_1}}\left(\frac{1}{c_{m_2}}\right) + \frac{1}{\delta_1\delta_2}\left(\frac{a_{m_1}}{m_1}\right)\left(\frac{1}{c_{m_2}}\right)\Bigg\}\omega_{\text{mixed}}\left(D_B\phi;\sqrt{\frac{a_{m_1}}{m_1}},\sqrt{\frac{1}{c_{m_2}}}\right)\Bigg],
\end{aligned}$$

in view of the Remark 2.3. Now, choosing $\delta_1 = \sqrt{\frac{a_{m_1}}{m_1}}$ and $\delta_2 = \sqrt{\frac{1}{c_{m_2}}}$, we obtain the desired result. □

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