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Generalized statistically pre-Cauchy triple sequences via Orlicz functions

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Abstract

In this study, we tried to construct $\mathcal{I}$-statistically pre-Cauchy on triple sequences via Orlicz functions $\widetilde{\phi}$. We prove that for triple sequences, $\mathcal{I}$-statistical $\widetilde{\phi}$-convergence implies $\mathcal{I}$-statistical pre-Cauchy condition and examine some main properties of these concepts.

Keywords: $\mathcal{I}$-statistical convergence, $\mathcal{I}$-statistical pre-Cauchy, triple sequences, Orlicz function.

2020 MSC: 40A35, 40D25.

1. Introduction

The idea of statistical convergence, which is a generalization of common, or, in other words, usual convergence was first showed up under the name of almost convergence in the primary release of the well-known monograph of Zygmund [44]. Later, this convergence has been studied by Fast [8] in the year 1951 and its basic properties studied by Šalát [34] and Schoenberg [40]. Later, the notion related to this concept was presented by Fridy [10], namely statistically Cauchy sequence; in addition, he also examined that both concepts are equivalent. Since then, applications of statistical convergence in different fields have been studied (see [16, 29, 36, 38, 39]).

In particular in [1] a very interesting notion was introduced, that of statistically pre-Cauchy sequences. And then, Connor et al. [1] proved that statistical convergent sequences are statistically pre-Cauchy. After that Gürdal [11] presented statistically pre-Cauchy sequences and bounded moduli.

The concept of ideal convergence was introduced as a generalization of statistical convergence. At the initial stage it was studied by Kostyrko et al. [24]. Later on it was studied by Šalát et al. [35] and Demirci [5] and others [2, 12, 13, 15, 17, 23, 32].

These concepts are also studied of double and triple sequences. The different notions on the statistical convergence of triple sequences due to Şahiner et al. [30] and Şahiner and Tripathy [33] and others (see [6, 7, 18–22, 27]).

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2. Preliminaries

We look back some definitions and notations that will be required throughout the paper.

The concept of statistical convergence can also be given asymptotic density of subsets of the set $\mathbb{N}$ of positive integers. So, firstly, we can see the notion of asymptotic density.

Definition 2.1 ([41]). A subset A of the set of positive integer $\mathbb{N}$, then A_n denotes the set $\{k \in A : k \leqslant n\}$ and $|A_n|$ denotes the number of elements in A_n, then $d(A)$ denotes the asymptotic density if

$$d(A) = \lim_{n\to\infty} \frac{1}{n} |A_n| = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k),$$

where χ_A is the characteristic function of the set A and $d(A)$ lie in the unit interval $[0,1]$.

Definition 2.2 ([8]). A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$ if for any $\varepsilon > 0$, we have $d(A(\varepsilon)) = 0$, where

$$A(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\}.$$

Definition 2.3 ([1]). A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be statistically pre-Cauchy if for any $\varepsilon > 0$,

$$\lim_{n} \frac{1}{n^2} \left|\{j, k \leqslant n : \left|x_k - x_j\right|\} \geqslant \varepsilon\right| = 0.$$

Next we recall the following definitions, where X represents an arbitrary set, from [24].

Definition 2.4. Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

(a) $\emptyset \in \mathcal{I}$;
(b) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$;
(c) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$;

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

Definition 2.5. Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:

(a) $\emptyset \notin F$;
(b) $A, B \in F$ implies $A \cap B \in F$;
(c) $A \in F, A \subset B$ implies $B \in F$.

A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

For more details about $\mathcal{I}$-convergence and $\mathcal{I}$-statistically convergence and $\mathcal{I}$-statistically convergence of sequences see [4, 13, 24–26, 31, 42, 43].

Definition 2.6 ([24]). Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$. A sequence $\{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be convergence to $L \in \mathbb{R}$ with respect to the ideal $\mathcal{I}$, if for every $\varepsilon > 0$

$$A(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\} \in \mathcal{I}.$$

In this case we write $\mathcal{I} - \lim_{k\to\infty} x_k = L$.

Definition 2.7 ([37]). A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is said to be $\mathcal{I}$-statistically convergent to L if for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} |\{k \leqslant n : |x_k - L| \geqslant \varepsilon\}| \geqslant \delta\right\} \in \mathcal{I}$$

or equivalently if for each $\varepsilon > 0$

$$\delta_{\mathcal{I}}(A(\varepsilon)) = \mathcal{I} - \lim \delta_n(A(\varepsilon)) = 0,$$

where $A(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\}$ and $\delta_n(A(\varepsilon)) = \frac{|A(\varepsilon)|}{n}$.

Definition 2.8 ([3]). A sequence $(x_k)_{k\in\mathbb{N}}$ is said to $\mathcal{I}$-statistically pre-Cauchy if, for any $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n^2} \left| \left\{ (j,k) : \left| x_k - x_j \right| \geqslant \varepsilon,\ j,k \leqslant n \right\} \right| \geqslant \delta \right\} \in \mathcal{I}.$$

Definition 2.9 ([30]). A triple sequence $x = \{x_{jkl}\}$ is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| x_{jkl} - L \right| < \varepsilon \text{ whenever } j,k,l \geqslant N.$$

A real triple sequence $x = \{x_{jkl}\}$ can be defined as a function $X : \mathbb{N}^3 = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}$.

Definition 2.10 ([30]). A subset A of $\mathbb{N}^3$ is said to have natural density $d_3(A)$ if

$$d_3(A) = \lim_{p,q,r\to\infty} \frac{|A(p,q,r)|}{pqr}$$

exists where the vertical bars denote the numbers of (j,k,l) in A such that $j \leqslant p$, $k \leqslant q$, $l \leqslant r$.

Definition 2.11 ([30]). A triple sequence $x = \{x_{jkl}\}$ is said to be statistically convergent to $L \in \mathbb{R}$ if for every $\varepsilon > 0$, we have $d_3(A(\varepsilon)) = 0$, where

$$d_3\left(\left\{ j,k,l \in \mathbb{N}^3 : \left| x_{jkl} - L \right| \geqslant \varepsilon \right\} \right) = 0.$$

We denote this by $st_3 - \lim_{jkl} = L$.

Throughout the paper we take $\mathcal{I}_3$ as a nontrivial admissible ideal in $\mathbb{N}^3$. We shall denote the ideals of $\mathcal{P}(\mathbb{N})$ by $\mathcal{I}$ and $\mathcal{P}(\mathbb{N}^3)$ by $\mathcal{I}_3$.

Definition 2.12. Let $\mathcal{I}_3 \subset \mathcal{P}(\mathbb{N}^3)$ be an ideal. Then a triple sequence $x = \{x_{jkl}\}$ is said to be $\mathcal{I}_3$-convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\left\{ (j,k,l) \in \mathbb{N}^3 : \left| x_{jkl} - L \right| \geqslant \varepsilon \right\} \in \mathcal{I}_3.$$

If $\{x_{jkl}\}$ is $\mathcal{I}_3$-convergent to L, we write $\mathcal{I}_3 - \lim x_{jkl} = L$.

Statistical convergence of any real sequence is identified relatively to absolute value. While we have known that the absolute value of real numbers is special case of an Orlicz function [28], that is, a function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ in such a way that it is even, non-decreasing on $\mathbb{R}^+$, continuous on $\mathbb{R}$, and satisfying

$$\widetilde{\phi}(x) = 0 \text{ if and only if } x = 0 \text{ and } \widetilde{\phi}(x) \to \infty \text{ as } x \to \infty.$$

Further, an Orlicz function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ is said to satisfy the $\triangle_2$ condition, if there exists an positive real number M such that $\widetilde{\phi}(2x) \leqslant M.\widetilde{\phi}(x)$ for every $x \in \mathbb{R}^+$.

The notion of $\mathcal{I}$-statistically pre-Cauchy of triple sequences via Orlicz function $\widetilde{\phi}$ has not been studied previously. Motivated by this fact, in this paper, we are concerned with $\mathcal{I}_3$-statistically pre-Cauchy triple sequences Orlicz function $\widetilde{\phi}$ and some important results are established.

3. Main results

As it is shown here, we present our results for ideal statistical pre-Cauchy and ideal statistical convergence with triple sequences via Orlicz functions $\widetilde{\phi}$.

Definition 3.1. A triple sequence $x = \{x_{jkl}\}$ is said to be $\mathcal{I}_3$-statistically pre-Cauchy if, for any $\varepsilon > 0$ and $\delta > 0$,

$$\left\{(m,n,o) \in \mathbb{N}^3 : \frac{1}{m^2n^2o^2} \left|\{(j,k,l) : j,p \leqslant m,\ k,q \leqslant n,\ l,r \leqslant o,\ \left|x_{jkl} - x_{pqr}\right| \geqslant \varepsilon\}\right| \geqslant \delta\right\} \in \mathcal{I}_3.$$

Definition 3.2. Let $\mathcal{I}_3$ be an ideal of $\mathcal{P}\left(\mathbb{N}^3\right)$ and $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. Then a triple sequence $x = \{x_{jkl}\}$ is said to be triple $\mathcal{I}_3^{\widetilde{\phi}}$-convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\left\{(j,k,l) \in \mathbb{N}^3 : \widetilde{\phi}\left(x_{jkl} - L\right) \geqslant \varepsilon\right\} \in \mathcal{I}_3.$$

In this case, we write $\mathcal{I}_3^{\widetilde{\phi}}\text{-}\lim x_{jkl} = L$.

Remark 3.3. If we take $\widetilde{\phi}(x) = |x|$, then triple $\mathcal{I}_3^{\widetilde{\phi}}$-convergent concepts coincide with ideal convergent.

Definition 3.4. Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $x = \{x_{jkl}\}$ be a triple sequence. Then x is said to be $\mathcal{I}_3$-statistically $\widetilde{\phi}$-convergent to L if for every $\varepsilon > 0$ and $\delta > 0$,

$$\left\{(m,n,o) \in \mathbb{N}^3 : \frac{1}{mno} \left|\left\{(j,k,l) : j \leqslant m,\ k \leqslant n,\ l \leqslant o,\ \widetilde{\phi}\left(x_{jkl} - L\right) \geqslant \varepsilon\right\}\right| \geqslant \delta\right\} \in \mathcal{I}_3.$$

Definition 3.5. Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A triple sequence $x = \{x_{jkl}\}$ is said to be $\mathcal{I}_3^{\widetilde{\phi}}$-statistically pre-Cauchy if for every $\varepsilon > 0$ and $\delta > 0$,

$$\left\{(m,n,o) \in \mathbb{N}^3 : \frac{1}{m^2n^2o^2} \left|\left\{(j,k,l) :\ j,p \leqslant m,\ k,q \leqslant n,\ l,r \leqslant o,\ \widetilde{\phi}\left(x_{jkl} - x_{pqr}\right) \geqslant \varepsilon,\right\}\right| \geqslant \delta\right\} \in \mathcal{I}_3.$$

Theorem 3.6. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. An $\mathcal{I}_3$-statistically $\widetilde{\phi}$-convergent triple sequence is $\mathcal{I}_3^{\widetilde{\phi}}$-statistically pre-Cauchy.*

Proof. Let $\{x_{jkl}\}$ be $\mathcal{I}_3$-statistically $\widetilde{\phi}$-convergent to L. Choose $\delta > 0$ such that $1 - (1-\delta)^2 < \delta_1$. Let

$$H = \left\{(m,n,o) \in \mathbb{N}^3 : \frac{1}{mno} \left|\left\{j \leqslant m,\ k \leqslant n,\ l \leqslant o,\ \widetilde{\phi}\left(x_{jkl} - L\right) \geqslant \frac{\varepsilon}{2}\right\}\right| \geqslant \delta\right\} \in \mathcal{I}_3. \tag{3.1}$$

Then for all $(m,n,o) \in H^c$,

$$\frac{1}{mno} \left|\left\{j \leqslant m\ , k \leqslant n\ , l \leqslant o,\ \widetilde{\phi}\left(x_{jkl} - L\right) \geqslant \frac{\varepsilon}{2}\right\}\right| < \delta,$$

that is,

$$\frac{1}{mno} \left|\left\{j \leqslant m,\ k \leqslant n,\ l \leqslant o,\ \widetilde{\phi}\left(x_{jkl} - L\right) < \frac{\varepsilon}{2}\right\}\right| > 1 - \delta,$$

where c means complement. Let

$$B_{mno} = \left\{(j,k,l) : j \leqslant m,\ k \leqslant n,\ l \leqslant o,\ \widetilde{\phi}\left(x_{jkl} - L\right) < \frac{\varepsilon}{2}\right\}.$$

Then for $(j,k,l), (p,q,r) \in B_{mno}$,

$$\widetilde{\phi}\left(x_{jkl} - x_{pqr}\right) = \widetilde{\phi}\left(x_{jkl} - L + L - x_{pqr}\right) \leqslant \widetilde{\phi}\left(x_{jkl} - L\right) + \widetilde{\phi}\left(x_{pqr} - L\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This implies

$$\left[\frac{|B_{mno}|^2}{m^2n^2o^2}\right] \leqslant \frac{1}{m^2n^2o^2}\left|\left\{(j,k,l):\ j,p\leqslant m,\ k,q\leqslant n,\ l,r\leqslant o,\ \widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)<\varepsilon\right\}\right|.$$

Thus for all $(m,n,o)\in H^c$, we have

$$(1-\delta)^2 < \left[\frac{|B_{mno}|^2}{m^2n^2o^2}\right] \leqslant \frac{1}{m^2n^2o^2}\left|\left\{(j,k,l):j,p\leqslant m,\ k,q\leqslant n,\ l,r\leqslant o,\ \widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)<\varepsilon\right\}\right|,$$

that is

$$\frac{1}{m^2n^2o^2}\left|\left\{(j,k,l):j,p\leqslant m,\ k,q\leqslant n,\ l,r\leqslant o,\ \widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)\geqslant\varepsilon\right\}\right| < 1-(1-\delta)^2 < \delta_1$$

for all $(m,n,o)\in H^c$ and so

$$\left\{(m,n,o)\in\mathbb{N}^3:\frac{1}{m^2n^2o^2}\left|\left\{(j,k,l):j,p\leqslant m,\ k,q\leqslant n,\ l,r\leqslant o,\ \widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)\geqslant\varepsilon\right\}\right|\geqslant\delta_1\right\}\subset H.$$

By (3.1),

$$\left\{(m,n,o)\in\mathbb{N}^3:\frac{1}{m^2n^2o^2}\left|\left\{(j,k,l):j,p\leqslant m,\ k,q\leqslant n,\ l,r\leqslant o,\ \widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)\geqslant\varepsilon\right\}\right|\geqslant\delta_1\right\}\in\mathcal{I}_3.$$

Hence x is $\mathcal{I}_3^{\widetilde{\phi}}$-statistically pre-Cauchy. This completes the proof of the theorem. □

Now this theorem highlights that a necessary and sufficient condition for a triple sequence to be $\mathcal{I}_3$-statistically pre-Cauchy via Orlicz function $\widetilde{\phi}$.

Theorem 3.7. *Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. A triple sequence $x=\left\{x_{jkl}\right\}$ is $\mathcal{I}_3^{\widetilde{\phi}}$-statistically pre-Cauchy if and only if $\mathcal{I}_3-\lim_{m,n,o}\frac{1}{m^2n^2o^2}\sum\limits_{j,p\leqslant m}\sum\limits_{k,q\leqslant n}\sum\limits_{l,r\leqslant o}\widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)=0$.*

Proof. Suppose that

$$\mathcal{I}_3-\lim_{m,n,o}\frac{1}{m^2n^2o^2}\sum_{j,p\leqslant m}\sum_{k,q\leqslant n}\sum_{l,r\leqslant o}\widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)=0.$$

For every $\varepsilon>0$ and $(m,n,o)\in\mathbb{N}^3$ we have

$$\begin{aligned}&\frac{1}{m^2n^2o^2}\sum_{j,p\leqslant m}\sum_{k,q\leqslant n}\sum_{l,r\leqslant o}\widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)\\&\qquad\geqslant\varepsilon\left(\frac{1}{m^2n^2o^2}\left|\left\{(j,k,l):j,p\leqslant m,\ k,q\leqslant n,\ l,r\leqslant o,\ \widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)\geqslant\varepsilon\right\}\right|\right).\end{aligned}$$

Hence for any $\delta>0$,

$$\begin{aligned}H&=\left\{(m,n,o)\in\mathbb{N}^3:\frac{1}{m^2n^2o^2}\left|\left\{(j,k,l):j,p\leqslant m,\ k,q\leqslant n,\ l,r\leqslant o,\widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)\geqslant\varepsilon\right\}\right|\geqslant\delta\right\}\\&\subset\left\{(m,n,o)\in\mathbb{N}^3:\frac{1}{m^2n^2o^2}\sum_{j,p\leqslant m}\sum_{k,q\leqslant n}\sum_{l,r\leqslant o}\widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)\geqslant\delta\varepsilon\right\}.\end{aligned}\tag{3.2}$$

Since $\mathcal{I}_3 - \lim_{m,n,o} \frac{1}{m^2n^2o^2} \sum_{j,p\leqslant m} \sum_{k,q\leqslant n} \sum_{l,r\leqslant o} \widetilde{\phi}\left(x_{jkl}-x_{pqr}\right) = 0$, thus the set in (3.2) belongs to $\mathcal{I}_3$ which implies that

$$\left\{(m,n,o)\in\mathbb{N}^3 : \frac{1}{m^2n^2o^2}\left|\left\{(j,k,l) : j,p\leqslant m,\ k,q\leqslant n,\ l,r\leqslant o,\ \widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)\geqslant\varepsilon\right\}\right|\geqslant\delta\right\}\in\mathcal{I}_3.$$

So, x is $\mathcal{I}_3^{\widetilde{\phi}}$-statistically pre-Cauchy.

Conversely suppose that $x=\{x_{jkl}\}$ is $\mathcal{I}_3^{\widetilde{\phi}}$-statistically pre-Cauchy and let $\varepsilon>0$. Choose $\varepsilon>0$ and $\delta>0$ such that $\frac{\varepsilon}{2}+2M\delta<\delta_1$. Since x is bounded, there exists an integer M such that $\widetilde{\phi}\left(x_{jkl}\right)\leqslant M$ for each $j,k,l\in\mathbb{N}$. Then for each $m,n,o\in\mathbb{N}$,

$$\begin{aligned}
&\frac{1}{m^2n^2o^2}\sum_{j,p\leqslant m}\sum_{k,q\leqslant n}\sum_{l,r\leqslant o}\widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)\\
&\quad=\frac{1}{m^2n^2o^2}\sum_{\substack{\widetilde{\phi}(x_{jkl}-x_{pqr})<\frac{\varepsilon}{2}\\ j,p\leqslant m,\ k,q\leqslant n,\ l,r\leqslant o}}\widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)+\frac{1}{m^2n^2o^2}\sum_{\substack{\widetilde{\phi}(x_{jkl}-x_{pqr})\geqslant\frac{\varepsilon}{2}\\ j,p\leqslant m,\ k,q\leqslant n,\ l,r\leqslant o}}\widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)\\
&\quad\leqslant\frac{\varepsilon}{2}+2M\left(\frac{1}{m^2n^2o^2}\left|\left\{(j,k,l) : j,p\leqslant m,\ k,q\leqslant n,\ l,r\leqslant o,\widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)\geqslant\varepsilon\right\}\right|\right).
\end{aligned}$$

Since x is $\mathcal{I}_3^{\widetilde{\phi}}$-statistically pre-Cauchy for that $\delta>0$,

$$H=\left\{(m,n,o)\in\mathbb{N}^3 : \frac{1}{m^2n^2o^2}\left|\left\{(j,k,l) : j,p\leqslant m,\ k,q\leqslant n,\ l,r\leqslant o,\ \widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)\geqslant\frac{\varepsilon}{2}\right\}\right|\geqslant\delta\right\}\in\mathcal{I}_3.$$

Then for $(m,n,o)\in H^c$,

$$\frac{1}{m^2n^2o^2}\left|\left\{(j,k,l) : j,p\leqslant m,\ k,q\leqslant n,\ l,r\leqslant o,\ \widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)\geqslant\frac{\varepsilon}{2}\right\}\right|<\delta.$$

Hence

$$\frac{1}{m^2n^2o^2}\sum_{j,p\leqslant m}\sum_{k,q\leqslant n}\sum_{l,r\leqslant o}\widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)\leqslant\frac{\varepsilon}{2}+2M\delta<\delta_1.$$

For every $(m,n,o)\in H^c$ we have

$$\frac{1}{m^2n^2o^2}\sum_{j,p\leqslant m}\sum_{k,q\leqslant n}\sum_{l,r\leqslant o}\widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)<\delta_1.$$

That is

$$\left\{(m,n,o)\in\mathbb{N}^3 : \frac{1}{m^2n^2o^2}\sum_{j,p\leqslant m}\sum_{k,q\leqslant n}\sum_{l,r\leqslant o}\widetilde{\phi}\left(x_{jkl}-x_{pqr}\right)\geqslant\delta_1\right\}\subset H\in\mathcal{I}_3.$$

Therefore $\mathcal{I}_3 - \lim_{m,n,o} \frac{1}{m^2n^2o^2} \sum_{j,p\leqslant m} \sum_{k,q\leqslant n} \sum_{l,r\leqslant o} \widetilde{\phi}\left(x_{jkl}-x_{pqr}\right) = 0$. This proves the necessity of the theorem. □

We give a sufficient condition under which an $\mathcal{I}_3^{\widetilde{\phi}}$-statistically pre-Cauchy triple sequence can be $\mathcal{I}_3$-statistically $\widetilde{\phi}$-convergent.

Before we prove the following theorem, we bear in mind a definition of $\mathcal{I}_3$-limit inferior [21].

Definition 3.8. Let $\mathcal{I}_3$ be an admissible ideal of $\mathcal{P}\left(\mathbb{N}^3\right)$ and $x = \left\{x_{jkl}\right\}$ be a real triple sequence. Let

$$A^{\alpha} = \left\{(m,n,o) \in \mathbb{N}^3 : \frac{1}{mno}\left|j \leqslant m,\ k \leqslant n,\ l \leqslant o : x_{jkl} < \alpha\right| > \gamma\right\}.$$

If there is a $\alpha \in \mathbb{R}$ such that $A^{\alpha} \notin \mathcal{I}_3$, then we put

$$\mathcal{I}_3 - \liminf x = \inf\{\alpha \in \mathbb{R} : A^{\alpha} \notin \mathcal{I}_3\}.$$

It is known ([21, Theorem 2.3]) that $\mathcal{I}_3$-$\liminf x = \alpha$ (finite) if and only if for every positive number ε,

$$\left\{(m,n,o) \in \mathbb{N}^3 : \frac{1}{mno}\left|j \leqslant m,\ k \leqslant n,\ l \leqslant o : x_{jkl} < \alpha + \varepsilon\right| > \gamma\right\} \notin \mathcal{I}_3$$

and

$$\left\{(m,n,o) \in \mathbb{N}^3 : \frac{1}{mno}\left|j \leqslant m,\ k \leqslant n,\ l \leqslant o : x_{jkl} < \alpha - \varepsilon\right| > \gamma\right\} \in \mathcal{I}_3.$$

Theorem 3.9. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $x = \left\{x_{jkl}\right\}$ be a $\mathcal{I}_3^{\widetilde{\phi}}$-statistically pre-Cauchy triple sequence. If $x = \left\{x_{jkl}\right\}$ has a subsequence $\left\{x_{t_j t_k t_l}\right\}$ which converges to L and*

$$0 < \mathcal{I}_3 - \lim_{m,n,o} \inf \frac{1}{mno}\left|\left\{\left(t_j, t_k, t_l\right) : j,k,l \in \mathbb{N},\ t_j \leqslant m, t_k \leqslant n, t_l \leqslant o \in \mathbb{N}\right\}\right| < \infty,$$

then x is $\mathcal{I}_3$-statistically $\widetilde{\phi}$-convergent to L.

Proof. We suppose that $\mathcal{I}_3 - \lim_{m,n,o} \inf \frac{1}{mno}\left|\left\{\left(t_j, t_k, t_l\right) : j,k,l \in \mathbb{N},\ t_j \leqslant m, t_k \leqslant n, t_l \leqslant o \in \mathbb{N}\right\}\right| = b$. We choose $\delta > 0$ such that $\frac{2\delta}{b} < \delta_1$. Let $\varepsilon > 0$ be given and select $n_0 \in \mathbb{N}$ such that if $t_j,\ t_k,\ t_l > n_0$ for some j,k,l, then $\widetilde{\phi}\left(x_{t_j t_k t_l} - L\right) < \frac{\varepsilon}{2}$. Let $P = \left\{\left(t_j, t_k, t_l\right) : t_j,\ t_k,\ t_l > n_0\ j,k,l \in \mathbb{N}\right\}$ and $P(\varepsilon) = \left\{(j,k,l) : \widetilde{\phi}\left(x_{jkl} - L\right) \geqslant \varepsilon\right\}$. Since

$$\begin{aligned}
&\frac{1}{m^2n^2o^2}\left|\left\{(j,k,l) :\ j,p \leqslant m,\ k,q \leqslant n,\ l,r \leqslant o,\ \widetilde{\phi}\left(x_{jkl} - x_{pqr}\right) \geqslant \frac{\varepsilon}{2}\right\}\right| \\
&\quad \geqslant \frac{1}{mno}\left|\left\{\left(t_j, t_k, t_l\right) \in P : t_j \leqslant m, t_k \leqslant n, t_l > o\right\}\right| \times \frac{1}{mno}\left|\left\{j \leqslant m, k \leqslant n, l \leqslant o : \widetilde{\phi}\left(x_{jkl} - L\right) \geqslant \varepsilon\right\}\right|.
\end{aligned}$$

Since x is $\mathcal{I}_3$-statistically pre-Cauchy, for $\delta > 0$,

$$R = \left\{(m,n,o) \in \mathbb{N}^3 : \frac{1}{m^2n^2o^2}\left|\left\{(j,k,l) :\ j,p \leqslant m,\ k,q \leqslant n,\ l,r \leqslant o,\ \widetilde{\phi}\left(x_{jkl} - x_{pqr}\right) \geqslant \frac{\varepsilon}{2}\right\}\right| \geqslant \delta\right\} \in \mathcal{I}_3.$$

Therefore for every $(m,n,o) \in R^c$, we have

$$\frac{1}{m^2n^2o^2}\left|\left\{(j,k,l) \in \mathbb{N}^3 :\ j,p \leqslant m,\ k,q \leqslant n,\ l,r \leqslant o,\ \widetilde{\phi}\left(x_{jkl} - x_{pqr}\right) \geqslant \frac{\varepsilon}{2}\right\}\right| < \delta. \tag{3.3}$$

Again since

$$\mathcal{I}_3 - \lim_{m,n,o} \inf \frac{1}{mno}\left|\left\{\left(t_j, t_k, t_l\right) : t_j \leqslant m, t_k \leqslant n, t_l \leqslant o; j,k,l \in \mathbb{N}\right\}\right| = b > 0,$$

so

$$\left\{(m,n,o) : \frac{1}{mno}\left|\left\{\left(t_j, t_k, t_l\right) : t_j \leqslant m, t_k \leqslant n, t_l \leqslant o; j,k,l \in \mathbb{N}\right\}\right| < \frac{b}{2}\right\} = S \in \mathcal{I}_3.$$

Thus every $(m, n, o) \in S^c$, we have

$$\frac{1}{mno}\left|\left\{\left(t_j, t_k, t_l\right) : t_j \leqslant m, t_k \leqslant n, t_l \leqslant o; j, k, l \in \mathbb{N}\right\}\right| \geqslant \frac{b}{2}. \tag{3.4}$$

From (3.3) and (3.4), it follows that every $(m, n, o) \in R^c \cap S^c = (R \cup S)^c$,

$$\frac{1}{mno}\left|\left\{p \leqslant m,\ q \leqslant n,\ r \leqslant o : \widetilde{\phi}\left(x_{pqr} - L\right) \geqslant \varepsilon\right\}\right| < \frac{2\delta}{b} < \delta_1.$$

This implies that every $(m, n, o) \in (R \cup S)^c$, that is,

$$\left\{(m, n, o) \in \mathbb{N}^3 : \frac{1}{mno}\left|\left\{(j, k, l) : j \leqslant m, k \leqslant n, l \leqslant o; \widetilde{\phi}\left(x_{jkl} - L\right) \geqslant \varepsilon\right\}\right| \geqslant \delta\right\} \subseteq R \cup S.$$

Since $R, S \in \mathcal{I}_3 \Rightarrow R \cup S \in \mathcal{I}_3$, we get

$$\left\{(m, n, o) \in \mathbb{N}^3 : \frac{1}{mno}\left|\left\{(j, k, l) : j \leqslant m, k \leqslant n, l \leqslant o; \widetilde{\phi}\left(x_{jkl} - L\right) \geqslant \varepsilon\right\}\right| \geqslant \delta\right\} \in \mathcal{I}_3.$$

This shows that x is $\mathcal{I}_3$-statistically convergent to L. □

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