J. Nonlinear Sci. Appl., 14 (2021), 423–439

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

Approximation degree of bivariate Kantorovich Stancu operators

P. N. Agrawal[a,*], Neha Bhardwaj[b], Jitendra Kumar Singh[a]

[a]*Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee, India.*

[b]*Department of Applied Mathematics, Amity Institute of Applied Sciences, Amity University Uttar Pradesh, Noida, India.*

Dedicated to the great mathematician
Prof. Themistocles M. Rassias, on his 70th birth anniversary

Abstract

Abel et al. [U. Abel, M. Ivan, R. Păltănea, Appl. Math. Comput., **259** (2015), 116–123] introduced a Durrmeyer type integral variant of the Bernstein type operators based on two parameters defined by Stancu [D. D. Stancu, Calcolo, **35** (1998), 53–62]. Kajla [A. Kajla, Appl. Math. Comput., **316** (2018), 400–408] considered a Kantorovich modification of the Stancu operators wherein he studied some basic convergence theorems and also the rate of A-statistical convergence. In the present paper, we define a bivariate case of the operators proposed in [A. Kajla, Appl. Math. Comput., **316** (2018), 400–408] to study the degree of approximation for functions of two variables. We obtain the rate of convergence of these bivariate operators by means of the complete modulus of continuity, the partial moduli of continuity and the Peetre's K-functional. Voronovskaya and Grüss Voronovskaya type theorems are also established. We introduce the associated GBS (Generalized Boolean Sum) operators of the bivariate operators and discuss the approximation degree of these operators with the aid of the mixed modulus of smoothness for Bögel continuous and Bögel differentiable functions.

Keywords: Modulus of continuity, Peetre's K-functional, GBS operator, B-continuous function, mixed modulus of smoothness.

2020 MSC: 41A10, 41A25, 41A30, 41A63, 26A15.

1. Introduction

For $z \in C(J)$, the Banach space of continuous functions on $J = [0,1]$, Stancu [16] defined Bernstein type operators, based on two parameters $i, j \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$, as follows:

$$S_{n,i,j}(z;x) = \sum_{\alpha=0}^{n-ij} b_{n-ij,\alpha}(x) \sum_{\beta=0}^{j} b_{j,\beta}(x)\, z\left(\frac{\alpha+\beta i}{n}\right), \tag{1.1}$$

*Corresponding author
Email addresses: pna_iitr@yahoo.co.in (P. N. Agrawal), nbhardwaj1@amity.edu (Neha Bhardwaj), jitendrasingh00004@gmail.com (Jitendra Kumar Singh)

doi: 10.22436/jnsa.014.06.05

Received: 2020-11-24 *Revised:* 2020-12-22 *Accepted:* 2021-04-07

where $b_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}, 0 \leqslant x \leqslant 1$ is the Bernstein basis. Evidently for $i = j = 0$, the operators (1.1) include the Bernstein operators $B_n(z;x)$ given by

$$B_n(z;x) = \sum_{k=0}^{n} b_{n,k}(x)\, z\left(\frac{k}{n}\right).$$

Abel et al. [1] gave the Durrmeyer type modification of the operators (1.1) defined by

$$D_{n,i,j}(z;x) = \sum_{\alpha=0}^{n-ij} b_{n-ij,\alpha}(x) \sum_{\beta=0}^{j} b_{j,\beta}(x)\,(n+1)\int_0^1 b_{n,\alpha+\beta i}(r) z(r)\,dr$$

and derived a complete asymptotic expansion for these operators besides some approximation properties. Kajla [13] considered the Kantorovich type modification of the Stancu operators defined by (1.1) as:

$$\mathfrak{V}_{n,i,j}(z;x) = \sum_{\alpha=0}^{n-ij} b_{n-ij,\alpha}(x) \sum_{\beta=0}^{j} b_{j,\beta}(x) \int_0^1 z\left(\frac{\alpha+\beta i+r}{n}\right) dr \tag{1.2}$$

and studied the Korovkin type theorems for these operators (1.2) and also their A-statistical convergence. Stancu [15] introduced two dimensional Bernstein polynomials $B_n^*(z;x,y)$ on the triangle $S := \{(x,y) : x+y \leqslant 1, 0 \leqslant x,y \leqslant 1\}$ as follows:

$$B_n^*(z;x,y) = \sum_{k=0}^{n}\sum_{l=0}^{n-k} p_{n,k,l}(x,y)\, z\left(\frac{k}{n},\frac{l}{n}\right), \quad (x,y) \in S, \tag{1.3}$$

where $p_{n,k,l}(x,y) = \binom{n}{k}\binom{n-k}{l} x^k y^l (1-x-y)^{n-k-l}$ and $z(x,y) \in C(S)$, the space of continuous functions on S. Deo and Bhardwaj [11], characterized the rate of approximation by means of K-functional and estimated the order of convergence for the operators (1.3) and its Durrmeyer variant proposed by Zhou [18]. For q-Bernstein-Schurer-Durrmeyer type operators for functions of one and two variables, Kajla et al. [14] studied some approximation properties.

Let $C(J^2)$, the space of continuous functions on $J^2 = J \times J$, be endowed with the norm given by $\|z\| = \sup_{(x,y)\in J^2} |z(x,y)|$, for $z \in C(J^2)$.

Motivated by the above work, for $z \in C(J^2)$, we define a bivariate case of the operators (1.2) as follows:

$$\begin{aligned}\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y) = {} & \sum_{\alpha_1=0}^{m-i_1j_1} b_{m-i_1j_1,\alpha_1}(x) \sum_{\beta_1=0}^{i_1} b_{i_1,\beta_1}(x) \sum_{\alpha_2=0}^{n-i_2j_2} b_{n-i_2j_2,\alpha_2}(y) \\ & \times \sum_{\beta_2=0}^{i_2} b_{i_2,\beta_2}(y) \int_0^1\int_0^1 z\left(\frac{\alpha_1+\beta_1 i_1+r}{m}, \frac{\alpha_2+\beta_2 i_2+s}{n}\right) dr\,ds, \ (x,y) \in J^2.\end{aligned} \tag{1.4}$$

The purpose of the present paper is to study some approximation properties of the bivariate operators (1.4) such as uniform convergence theorem, rate of convergence in terms of modulus of continuity, Voronovskaja type asymptotic theorem and Grüss-Voronovskaja type theorem. We also construct the associated GBS operator and obtain its convergence estimates using mixed modulus of smoothness.

Throughout the paper, M denotes a positive constant which may have different values for different cases.

2. Auxiliary results

We observe that the operators (1.4), is the tensorial product of the operators $\mathfrak{V}^{x}_{m,i_1,j_1}$ and $\mathfrak{V}^{y}_{n,i_2,j_2}$, i.e., $\mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n} = \mathfrak{V}^{x}_{m,i_1,j_1} \circ \mathfrak{V}^{y}_{n,i_2,j_2}$, where

$$\mathfrak{V}^{x}_{m,i_1,j_1}(z;x,y) = \sum_{\alpha_1=0}^{m-i_1j_1} b_{m-i_1j_1,\alpha_1}(x) \sum_{\beta_1=0}^{j_1} b_{j_1,\beta_1}(x) \int_0^1 z\left(\frac{\alpha_1+\beta_1 i_1 + r}{m}, s\right) dr$$

and

$$\mathfrak{V}^{y}_{n,i_2,j_2}(z;x,y) = \sum_{\alpha_2=0}^{n-i_2j_2} b_{n-i_2j_2,\alpha_2}(y) \sum_{\beta_2=0}^{j_2} b_{j_2,\beta_2}(y) \int_0^1 z\left(t, \frac{\alpha_2+\beta_2 i_2 + s}{n}\right) ds.$$

We present some lemmas which will be useful in the sequel. Let $e_h(r) = r^h, h \in \{0,1,2,3,4\}$.

Lemma 2.1 ([13]). *For the operators $\mathfrak{V}_{n,i,j}(f;x)$, we have*

(i) $\mathfrak{V}_{n,i,j}(e_0;x) = 1$;
(ii) $\mathfrak{V}_{n,i,j}(e_1;x) = x + \frac{1}{2n}$;
(iii) $\mathfrak{V}_{n,i,j}(e_2;x) = x^2 + \frac{1}{n}(2x - x^2) + \frac{1}{n^2}(i^2jx - ijx - i^2jx^2 + ijx^2 + \frac{1}{3})$;
(iv)

$$\begin{aligned}\mathfrak{V}_{n,i,j}(e_3;x) = x^3 &+ \frac{1}{2n}3x(3x-2x^2) + \frac{1}{2n^2}[6ijx^3 - 6ijx^2 + 7x - 7x^2 + 4x^3 + 6i^2jx^2 - 3i^2jx^3] \\ &+ \frac{1}{4n^3}[-8ijx^3 + 18ijx^2 - 10ijx - 12i^2jx^2 + 4i^3jx + 6i^2jx + 1];\end{aligned}$$

(v)

$$\begin{aligned}\mathfrak{V}_{n,i,j}(e_4;x) = x^4 &+ \frac{1}{n}[8x^3 - 6x^4] + \frac{1}{n^2}[11x^4 - 24x^3 + 15x^2 + 6i^2j^2x^4 + 18ijx^4 - 18ijx^3] \\ &+ \frac{1}{n^3}[-4i^3j^3x^4 - 18i^2j^2x^4 - 22ijx^4 - 6x^4 + 42ijx^3 + 18i^2j^2x^3 + 16x^3 - 20ijx^2 \\ &- 15x^2 + 6i^2jx^2 - 6i^2jx^3 + 6x] + \frac{1}{n^4}[i^4j^4x^4 - 6i^3j^3x^4 + 11i^2j^2x^4 + 6ijx^4 + 6i^3j^3x^3 \\ &- 18i^2j^2x^3 - 16ijx^3 + 7i^2j^2x^2 + 15ijx^2 - 8i^2jx^2 - 5ijx + 2i^2jx + \frac{1}{5}].\end{aligned}$$

Consequently, let $e^x_h(r) = (r-x)^h, h \in \{1,2,3,4\}$.

Lemma 2.2 ([13]). *For the operators $\mathfrak{V}_{n,i,j}(f;x)$, we get*

(i) $\mathfrak{V}_{n,i,j}\left(e^x_1(r);x\right) = \dfrac{1}{2n}$;
(ii) $\mathfrak{V}_{n,i,j}\left(e^x_2(r);x\right) = \dfrac{1}{n}(x - x^2) + \dfrac{1}{n^2}(i^2jx - ijx - i^2jx^2 + ijx^2 + \dfrac{1}{3})$;
(iii) $\mathfrak{V}_{n,i,j}\left(e^x_3(r);x\right) = \frac{1}{2n^2}[2 + 7x - 9x^2 + 4x^3] + \frac{1}{2n^3}[-4ijx^3 + 9ijx^2 - 5ijx - 6i^2jx^2 + 2i^3jx + 3i^2jx]$;
(iv) $\mathfrak{V}_{n,i,j}\left(e^x_4(r);x\right) = \dfrac{1}{n^2}[3x^4 - 6x^3 + 3x^2 + 6i^2j^2x^4 + 12ijx^4 - 12ijx^3 - 6i^2jx^3 - 6i^2jx^4]$.

Let $e_{h,k}(r,s) = r^h s^k,\ h,k \in \{0,1,2,3,4\}$ and $(r,s) \in J^2$. From Lemma 2.1 and the definition (1.4), by simple calculations, we obtain:

Lemma 2.3. *The bivariate operators defined by (1.4) satisfy the following equalities:*

(i) $\mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}(e_{0,0};x,y) = 1$;

(ii) $\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(e_{1,0};x,y) = x + \frac{1}{2m}$;

(iii) $\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(e_{0,1};x,y) = y + \frac{1}{2n}$;

(iv) $\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(e_{1,1};x,y) = \left(x + \frac{1}{2m}\right)\left(y + \frac{1}{2n}\right)$;

(v) $\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(e_{2,0};x,y) = x^2 + \dfrac{1}{m}(2x - x^2) + \dfrac{1}{m^2}(i_1^2 j_1 x - i_1 j_1 x - i_1^2 j_1 x^2 + i_1 j_1 x^2 + \frac{1}{3})$;

(vi) $\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(e_{0,2};x,y) = y^2 + \dfrac{1}{n}(2y - y^2) + \frac{1}{n^2}(i_2^2 j_2 y - i_2 j_2 y - i_2^2 j_2 y^2 + i_2 j_2 y^2 + \frac{1}{3})$;

(vii)

$$\begin{aligned}\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(e_{3,0};x,y) = x^3 + \frac{1}{2m}3x\left(3x - 2x^2\right) + \frac{1}{2m^2}[6i_1j_1x^3 - 6i_1j_1x^2 + 7x - 7x^2 \\ + 4x^3 + 6i_1^2j_1x^2 - 3i_1^2j_1x^3] + \frac{1}{4m^3}[-8i_1j_1x^3 + 18i_1j_1x^2 - 10i_1j_1x \\ - 12i_1^2j_1x^2 + 4i_1^3j_1x + 6i_1^2j_1x + 1];\end{aligned}$$

(viii)

$$\begin{aligned}\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(e_{0,3};x,y) = y^3 + \frac{1}{2n}3y\left(3y - 2y^2\right) + \frac{1}{2n^2}[6i_2j_2y^3 - 6i_2j_2y^2 + 7y - 7y^2 + 4y^3 \\ + 6i_2^2j_2y^2 - 3i_2^2j_2y^3] + \frac{1}{4n^3}[-8i_2j_2y^3 + 18i_2j_2y^2 - 10i_2j_2y - 12i_2^2j_2y^2 \\ + 4i_2^3j_2y + 6i_2^2j_2y + 1];\end{aligned}$$

(ix)

$$\begin{aligned}\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(e_{4,0};x,y) = x^4 + \frac{1}{m}[8x^3 - 6x^4] + \frac{1}{m^2}[11x^4 - 24x^3 + 15x^2 + 6i_1^2j_1^2x^4 + 18i_1j_1x^4 \\ - 18i_1j_1x^3] + \frac{1}{m^3}[-4i_1^3j_1^3x^4 - 18i_1^2j_1^2x^4 - 22i_1j_1x^4 - 6x^4 + 42i_1j_1x^3 \\ + 18i_1^2j_1^2x^3 + 16x^3 - 20i_1j_1x^2 - 15x^2 + 6i_1^2j_1x^2 - 6i_1^2j_1x^3 + 6x] \\ + \frac{1}{m^4}[i_1^4j_1^4x^4 - 6i_1^3j_1^3x^4 + 11i_1^2j_1^2x^4 + 6i_1j_1x^4 + 6i_1^3j_1^3x^3 - 18i_1^2j_1^2x^3 \\ - 16i_1j_1x^3 + 7i_1^2j_1^2x^2 + 15i_1j_1x^2 - 8i_1^2j_1x^2 - 5i_1j_1x + 2i_1^2j_1x + \frac{1}{5}];\end{aligned}$$

(x)

$$\begin{aligned}\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(e_{0,4};x,y) = y^4 + \frac{1}{n}[8y^3 - 6y^4] + \frac{1}{n^2}[11y^4 - 24y^3 + 15y^2 + 6i_2^2j_2^2y^4 + 18i_2j_2y^4 \\ - 18i_2j_2y^3] + \frac{1}{n^3}[-4i_2^3j_2^3y^4 - 18i_2^2j_2^2y^4 - 22i_2j_2y^4 - 6y^4 + 42i_2j_2y^3 \\ + 18i_2^2j_2^2y^3 + 16y^3 - 20i_2j_2y^2 - 15y^2 + 6i_2^2j_2y^2 - 6i_2^2j_2y^3 + 6y] \\ + \frac{1}{n^4}[i_2^4j_2^4y^4 - 6i_2^3j_2^3y^4 + 11i_2^2j_2^2y^4 + 6i_2j_2y^4 + 6i_2^3j_2^3y^3 - 18i_2^2j_2^2y^3 \\ - 16i_2j_2y^3 + 7i_2^2j_2^2y^2 + 15i_2j_2y^2 - 8i_2^2j_2y^2 - 5i_2j_2y + 2i_2^2j_2y + \frac{1}{5}].\end{aligned}$$

Let $\phi_{m,n,h,k}^{i_1,j_1,i_2,j_2}(x,y) = \mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\left((r-x)^h(s-y)^k;x,y\right)$, $h,k \in \{0,1,2,3,4\}$ and $(r,s),(x,y) \in J^2$. As a consequence of the proceeding lemma, we have following.

Lemma 2.4. *The central moments for the operators (1.4) are obtained as follows:*

(i) $\phi_{m,n,1,0}^{i_1,j_1,i_2,j_2}(x,y) = \dfrac{1}{2m}$;

(ii) $\phi^{i_1,j_1,i_2,j_2}_{m,n,0,1}(x,y) = \dfrac{1}{2n}$;

(iii) $\phi^{i_1,j_1,i_2,j_2}_{m,n,1,1}(x,y) = \dfrac{1}{4mn}$;

(iv) $\phi^{i_1,j_1,i_2,j_2}_{m,n,2,0}(x,y) = \dfrac{1}{m}\left(x-x^2\right) + \dfrac{1}{m^2}\left\{i_1^2 j_1 x - i_1 j_1 x - i_1^2 j_1 x^2 + i_1 j_1 x^2 + \dfrac{1}{3}\right\}$;

(v) $\phi^{i_1,j_1,i_2,j_2}_{m,n,0,2}(x,y) = \dfrac{1}{n}\left(y-y^2\right) + \dfrac{1}{n^2}\left\{i_2^2 j_2 y - i_2 j_2 y - i_2^2 j_2 y^2 + i_2 j_2 y^2 + \dfrac{1}{3}\right\}$;

(vi)

$$\begin{aligned}\phi^{i_1,j_1,i_2,j_2}_{m,n,2,2}(x,y) = {} & \frac{1}{mn}(xy - xy^2 - x^2y + x^2y^2) + \frac{1}{mn^2}(x-x^2)(i_2^2 j_2 y - i_2 j_2 y - i_2^2 j_2 y^2 \\ & + i_2 j_2 y^2 + \frac{1}{3}) + \frac{1}{m^2 n}(y-y^2)(i_1^2 j_1 x - i_1 j_1 x - i_1^2 j_1 x^2 + i_1 j_1 x^2 + \frac{1}{3}) \\ & + \frac{1}{m^2 n^2}\{(i_1^2 j_1 x - i_1 j_1 x - i_1^2 j_1 x^2 + i_1 j_1 x^2 + \frac{1}{3})(i_2^2 j_2 y - i_2 j_2 y - i_2^2 j_2 y^2 \\ & + i_2 j_2 y^2 + \frac{1}{3})\};\end{aligned}$$

(vii)

$$\begin{aligned}\phi^{i_1,j_1,i_2,j_2}_{m,n,3,0}(x,y) = {} & \frac{1}{2m^2}[2 + 7x - 9x^2 + 4x^3] + \frac{1}{2m^3}[-4i_1 j_1 x^3 + 9 i_1 j_1 x^2 - 5 i_1 j_1 x - 6 i_1^2 j_1 x^2 \\ & + 2 i_1^3 j_1 x + 3 i_1^2 j_1 x];\end{aligned}$$

(viii)

$$\begin{aligned}\phi^{i_1,j_1,i_2,j_2}_{m,n,0,3}(x,y) = {} & \frac{1}{2n^2}[2 + 7y - 9y^2 + 4y^3] + \frac{1}{2n^3}[-4i_2 j_2 y^3 + 9 i_2 j_2 y^2 - 5 i_2 j_2 y - 6 i_2^2 j_2 y^2 \\ & + 2 i_2^3 j_2 y + 3 i_2^2 j_2 y];\end{aligned}$$

(ix)

$$\begin{aligned}\phi^{i_1,j_1,i_2,j_2}_{m,n,4,0}(x,y) = {} & \frac{1}{m^2}[3x^4 - 6x^3 + 3x^2 + 6 i_1^2 j_1^2 x^4 + 12 i_1 j_1 x^4 - 12 i_1 j_1 x^3 - 6 i_1^2 j_1 x^3 - 6 i_1^2 j_1 x^4] \\ & + \frac{1}{m^3}[-4 i_1^3 j_1^3 x^4 - 18 i_1^2 j_1^2 x^4 - 14 i_1 j_1 x^4 - 6x^4 + 24 i_1 j_1 x^3 + 18 i_1^2 j_1^2 x^3 + 16x^3 \\ & - 10 i_1 j_1 x^2 - 15x^2 + 6 i_1^2 j_1 x^3 + 6x + 4 i_1^3 j_1 x^2] + \frac{1}{m^4}[i_1^4 j_1^4 x^4 - 6 i_1^3 j_1^3 x^4 + 11 i_1^2 j_1^2 x^4 \\ & + 6 i_1 j_1 x^4 + 6 i_1^3 j_1^3 x^3 - 18 i_1^2 j_1^2 x^3 - 16 i_1 j_1 x^3 + 7 i_1^2 j_1^2 x^2 + 15 i_1 j_1 x^2 - 8 i_1^2 j_1 x^2 \\ & - 5 i_1 j_1 x + 2 i_1^2 j_1 x + \frac{1}{5}];\end{aligned}$$

(x)

$$\begin{aligned}\phi^{i_1,j_1,i_2,j_2}_{m,n,0,4}(x,y) = {} & \frac{1}{n^2}[3y^4 - 6y^3 + 3y^2 + 6 i_2^2 j_2^2 y^4 + 12 i_2 j_2 y^4 - 12 i_2 j_2 y^3 - 6 i_2^2 j_2 y^3 - 6 i_2^2 j_2 y^4] \\ & + \frac{1}{n^3}[-4 i_2^3 j_2^3 y^4 - 18 i_2^2 j_2^2 y^4 - 14 i_2 j_2 y^4 - 6y^4 + 24 i_2 j_2 y^3 + 18 i_2^2 j_2^2 y^3 + 16y^3 \\ & - 10 i_2 j_2 y^2 - 15y^2 + 6 i_2^2 j_2 y^3 + 6y + 4 i_2^3 j_2 y^2] + \frac{1}{n^4}[i_2^4 j_2^4 y^4 - 6 i_2^3 j_2^3 y^4 + 11 i_2^2 j_2^2 y^4 \\ & + 6 i_2 j_2 y^4 + 6 i_2^3 j_2^3 y^3 - 18 i_2^2 j_2^2 y^3 - 16 i_2 j_2 y^3 + 7 i_2^2 j_2^2 y^2 + 15 i_2 j_2 y^2 - 8 i_2^2 j_2 y^2 \\ & - 5 i_2 j_2 y + 2 i_2^2 j_2 y + \frac{1}{5}].\end{aligned}$$

Consequently,

Lemma 2.5. *We have the following results:*

(i) $\lim_{m\to\infty} m\phi^{i_1,j_1,i_2,j_2}_{m,n,1,0}(x,y) = \frac{1}{2}$;
(ii) $\lim_{m\to\infty} m\phi^{i_1,j_1,i_2,j_2}_{m,n,0,1}(x,y) = \frac{1}{2}$;
(iii) $\lim_{m\to\infty} m\phi^{i_1,j_1,i_2,j_2}_{m,n,2,0}(x,y) = x(1-x)$;
(iv) $\lim_{n\to\infty} n\phi^{i_1,j_1,i_2,j_2}_{m,n,0,2}(x,y) = y(1-y)$;
(v) $\lim_{n\to\infty} n\phi^{i_1,j_1,i_2,j_2}_{n,n,1,1}(x,y) = 0$;
(vi) $\lim_{m\to\infty} m^2\phi^{i_1,j_1,i_2,j_2}_{m,n,4,0}(x,y) = 3x^4-6x^3+3x^2+6i_1^2j_1^2x^4+12i_1j_1x^4-12i_1j_1x^3-6i_1^2j_1x^3-6i_1^2j_1x^4$;
(vii) $\lim_{n\to\infty} n^2\phi^{i_1,j_1,i_2,j_2}_{m,n,0,4}(x,y) = 3y^4-6y^3+3y^2+6i_2^2j_2^2y^4+12i_2j_2y^4-12i_2j_2y^3-6i_2^2j_2y^3-6i_2^2j_2y^4$.

Let I_1 and I_2 be compact intervals of the real line and $B(I_1\times I_2)$ be the space of functions bounded on $(I_1\times I_2)$.

Theorem 2.6 ([17])**.** *Let $D_{m,n}: C(I_1\times I_2)\to B(I_1\times I_2), (m,n)\in\mathbb{N}\times\mathbb{N}$ be linear positive operators. If*

$$\lim_{m,n\to\infty} D_{m,n}(e_{ts}) = (e_{ts}), \forall (t,s)\in\{(0,0),(1,0),(0,1)\}$$

and

$$\lim_{m,n\to\infty} D_{m,n}(e_{20}+e_{02}) = e_{20}+e_{02},$$

uniformly on $I_1\times I_2$, then the sequence $(D_{m,n}z)$ converges to z uniformly on $I_1\times I_2$, for any $z\in C(I_1\times I_2)$.

3. Approximation properties of the operator $\mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}$

First, we show that the operator $\mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}(z)$ is an approximation process for all $z\in C(J^2)$.

Theorem 3.1. *For $z\in C(J^2)$, the operators $\mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}(z)$ converge to f as $m,n\to\infty$, uniformly on J^2.*

Proof. Applying Lemma 2.3 and Theorem 2.6, the proof easily follows. Hence, we skip the details. □

Now, we proceed to determine the order of convergence of the operators (1.4). Let us recall some definitions.

For $z\in C(J^2)$, for the bivariate case, the complete modulus of continuity is defined as follows:

$$\bar{\omega}(z;\sigma_1,\sigma_2) = \sup\left\{|z(r,s)-z(x,y)| : (r,s),(x,y)\in J^2 \text{ and } |r-x|\leqslant\sigma_1, |s-y|\leqslant\sigma_2\right\},$$

where $\sigma_1>0, \sigma_2>0$. Further, $\bar{\omega}(z;\sigma_1,\sigma_2)$ satisfies following properties:

(i) $\bar{\omega}(z;\sigma_1,\sigma_2)\to 0$, if $\sigma_1\to 0$ and $\sigma_2\to 0$;
(ii) $|z(r,s)-z(x,y)|\leqslant\bar{\omega}(z;\sigma_1,\sigma_2)\left(1+\frac{|r-x|}{\sigma_1}\right)\left(1+\frac{|s-y|}{\sigma_2}\right)$.

Also, the partial modulii of continuity with respect to x and y are defined as

$$\omega^{(1)}(z;\sigma) = \sup\{|z(x_1,y)-z(x_2,y)| : y\in J \text{ and } |x_1-x_2|\leqslant\sigma\},$$
$$\omega^{(2)}(z;\sigma) = \sup\{|z(x,y_1)-z(x,y_2)| : x\in J \text{ and } |y_1-y_2|\leqslant\sigma\}.$$

The details of modulus of continuity for the bivariate case has been widely studied by Anastassiou and Gal in [3]. In what follows, let

$$\sigma_m = \sup_{(x,y)\in J^2}\phi^{i_1,j_1,i_2,j_2}_{m,n,2,0}(x,y) = \sup_{x\in J}\mathfrak{V}_{m,i_1,j_1}\left((r-x)^2;x\right),$$

$$\sigma_n = \sup_{(x,y)\in J^2} \phi^{i_1,j_1,i_2,j_2}_{m,n,0,2}(x,y) = \sup_{y\in J} \mathfrak{V}_{n,i_2,j_2}\left((s-y)^2;y\right).$$

The following theorem yields the rate of approximation in terms of the complete modulus of continuity.

Theorem 3.2. *Let* $z \in C\left(J^2\right)$*. Then,*

$$\left\|\mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}(z) - z\right\| \leqslant 4\bar{\omega}\left(z, \sqrt{\sigma_m}, \sqrt{\sigma_n}\right).$$

Proof. Using the property (ii) of complete modulus of continuity for the bivariate case as mentioned above, for linear positive operator $\mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}(;x,y)$, we have

$$\begin{aligned}\left|\mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}(z;x,y) - z(x,y)\right| &\leqslant \mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}\left(|z(r,s) - z(x,y)|;x,y\right)\\ &\leqslant \omega\left(z;\sqrt{\sigma_m},\sqrt{\sigma_n}\right)\Bigg(\mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}(1;x,y) + \frac{1}{\sqrt{\sigma_m}}\mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}\left(|r-x|;x,y\right)\\ &\quad + \frac{1}{\sqrt{\sigma_n}}\mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}\left(|s-y|;x,y\right) + \frac{1}{\sqrt{\sigma_m}\sqrt{\sigma_n}}\mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}\left(|r-x|\,|s-y|;x,y\right)\Bigg).\end{aligned}$$

Now, using the Cauchy-Schwarz inequality and Lemma 2.3, the desired result is obtained. □

Our next result provides the order of approximation for the operators (1.4) by means of the partial moduli of continuity.

Theorem 3.3. *For* $f \in C\left(J^2\right)$*, we have*

$$\left\|\mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}(z) - z\right\| \leqslant 2\left(\omega^{(1)}\left(z;\sqrt{\sigma_m}\right) + \omega^{(2)}\left(z;\sqrt{\sigma_n}\right)\right).$$

Proof. From the definition of the partial moduli of continuity and using Cauchy-Schwarz inequality, for any $\sigma_1, \sigma_2 > 0$, we have

$$\begin{aligned}\left|\mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}(z;x,y) - z(x,y)\right| &\leqslant \mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}\left(|z(r,s) - z(x,y)|;x,y\right)\\ &\leqslant \mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}\left(|z(r,s) - z(r,y)|;x,y\right) + \mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}\left(|z(r,y) - z(x,y)|;x,y\right)\\ &\leqslant \mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}\left(\omega^{(2)}\left(z;|s-y|\right);x,y\right) + \mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}\left(\omega^{(1)}\left(z;|r-x|\right);x,y\right)\\ &\leqslant \omega^{(2)}(z;\sigma_2)\left(1 + \frac{1}{\sigma_2}\mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}\left(|s-y|;x,y\right)\right)\\ &\quad + \omega^{(1)}(z;\sigma_1)\left(1 + \frac{1}{\sigma_1}\mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}\left(|r-x|;x,y\right)\right)\\ &\leqslant \omega^{(2)}(z;\sigma_2)\left(1 + \frac{1}{\sigma_2}\left(\phi^{i_1,j_1,i_2,j_2}_{m,n,0,2}(x,y)\right)^{1/2}\right)\\ &\quad + \omega^{(1)}(z;\sigma_1)\left(1 + \frac{1}{\sigma_1}\left(\phi^{i_1,j_1,i_2,j_2}_{m,n,2,0}(x,y)\right)^{1/2}\right).\end{aligned}$$

Taking $\sigma_2 = \sqrt{\sigma_n}$ and $\sigma_1 = \sqrt{\sigma_m}$, the proof is completed. □

Now, we compute the rate of convergence of the bivariate operators $\mathfrak{V}^{i_1,j_1,i_2,j_2}_{m,n}$ by means of the Lipschitz class. For $0 < \gamma_1, \gamma_2 \leqslant 1$, Lipschitz class $\mathrm{Lip}_M(\gamma_1,\gamma_2)$ for the bivariate case is defined as:

$$|z(r,s) - z(x,y)| \leqslant M|r-x|^{\gamma_1}|s-y|^{\gamma_2},$$

where M is some positive constant and $(x,y), (r,s) \in J^2$ are arbitrary.

Theorem 3.4. *If $z \in \mathrm{Lip}_M(\gamma_1, \gamma_2)$, then we have*

$$\left\|\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(z) - z\right\| \leqslant M(\sigma_m)^{\gamma_1/2}(\sigma_n)^{\gamma_2/2}.$$

Proof. Since $z \in \mathrm{Lip}_M(\gamma_1, \gamma_2)$,

$$\begin{aligned}\left|\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y) - z(x,y)\right| &\leqslant \mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\left(|z(r,s) - z(x,y)|;x,y\right)\\ &\leqslant M\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\left(|r-x|^{\gamma_1}|s-y|^{\gamma_2};x,y\right)\\ &= M\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\left(|r-x|^{\gamma_1};x,y\right)\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\left(|s-y|^{\gamma_2};x,y\right).\end{aligned}$$

Now, using Hölder's inequality with $p_1 = \frac{2}{\gamma_1}, q_1 = \frac{2}{2-\gamma_1}$ and $p_2 = \frac{2}{\gamma_2}, q_2 = \frac{2}{2-\gamma_2}$, respectively, in view of Lemma 2.3, we have

$$\begin{aligned}\left|\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y) - z(x,y)\right| &\leqslant M\left(\left(\phi_{m,n,2,0}^{i_1,j_1,i_2,j_2}(x,y)\right)\right)^{\frac{\gamma_1}{2}}\left(\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(1;x,y)\right)^{\frac{2-\gamma_1}{2}}\\ &\quad\times\left(\left(\phi_{m,n,0,2}^{i_1,j_1,i_2,j_2}(x,y)\right)\right)^{\frac{\gamma_2}{2}}\left(\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(1;x,y)\right)^{\frac{2-\gamma_2}{2}},\end{aligned}$$

which leads us to the desired assertion. □

Our forthcoming result gives a quantitative estimate in terms of the second order modulus of continuity with the aid of the Peetre's K-functional, which is defined as follows.

Let $C^k(J^2) = \left\{z \in C(J^2) : \frac{\partial^{i+j}z}{\partial x^i \partial y^j} \in C(J^2), \forall\, 0 \leqslant i+j \leqslant k\right\}$. The Peetre's K-functional for the function $z \in C(J^2)$ is defined as

$$K(z;\sigma) = \inf_{w \in C^2(J^2)}\left\{\|z-w\| + \sigma\|w\|_{C^2(J^2)}\right\},\ \sigma > 0,$$

with the norm in $C^2(J^2)$ given by

$$\|w\|_{C^2(J^2)} = \|w\| + \sum_{i=1}^{2}\left(\left\|\frac{\partial^i w}{\partial x^i}\right\| + \left\|\frac{\partial^i w}{\partial y^i}\right\|\right) + \left\|\frac{\partial^2 w}{\partial x \partial y}\right\|.$$

From [10, page 192], it is known that for any $\sigma > 0$

$$K(z;\sigma) \leqslant M\left\{\bar{\omega}_2(z;\sqrt{\sigma}) + \min(1,\sigma)\,\|z\|\right\},$$

where $M > 0$ is a constant independent of σ and f and $\bar{\omega}_2$ is a second order modulus of continuity for functions of two variables.

Theorem 3.5. *For $z \in C(J^2)$, there holds the inequality*

$$\left\|\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(z) - z\right\| \leqslant \left\{M\bar{\omega}_2\left(z;\frac{\sqrt{C_{m,n}}}{2}\right) + \min\left\{1, \frac{C_{m,n}}{4}\right\}\|z\|\right\} + \bar{\omega}\left(z;\frac{1}{2m},\frac{1}{2n}\right),$$

where $C_{m,n} = \frac{1}{2}\left\{\left(\sqrt{\sigma_m} + \sqrt{\sigma_n}\right)^2 + \frac{1}{4}\left(\frac{1}{m} + \frac{1}{n}\right)^2\right\}$.

Proof. Let us introduce an auxiliary operator as follows:

$$\tilde{\mathfrak{V}}_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y) = \mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y) - z\left(x + \frac{1}{2m}, y + \frac{1}{2n}\right) + z(x,y). \tag{3.1}$$

Then using Lemma 2.3, we can easily see that

$$\tilde{\mathfrak{V}}_{m,n}^{i_1,j_1,i_2,j_2}(1;x,y)=1,\ \tilde{\mathfrak{V}}_{m,n}^{i_1,j_1,i_2,j_2}((r-x);x,y)=0,\ \text{and}\ \tilde{\mathfrak{V}}_{m,n}^{i_1,j_1,i_2,j_2}((s-y);x,y)=0. \tag{3.2}$$

Let $w\in C^2(J^2)$ and $(x,y)\in J^2$. Using the Taylor's theorem, we may write

$$\begin{aligned} w(r,s)-w(x,y) &= w(r,y)-w(x,y)+w(r,s)-w(r,y)\\ &= \frac{\partial w(x,y)}{\partial x}(r-x)+\int_x^r (r-u)\frac{\partial^2 w(u,y)}{\partial u^2}du+\frac{\partial w(x,y)}{\partial y}(s-y)\\ &\quad+\int_y^s (s-v)\frac{\partial^2 w(x,v)}{\partial v^2}dv+\int_x^r\int_y^s\left(\frac{\partial^2 w}{\partial u\partial v}\right)dudv. \end{aligned}$$

Now applying the operators $\tilde{\mathfrak{V}}_{m,n}^{i_1,j_1,i_2,j_2}$ on both sides of the above equation and using (3.2), we get

$$\begin{aligned} \tilde{\mathfrak{V}}_{m,n}^{i_1,j_1,i_2,j_2}(w;x,y)-w(x,y) &= \mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\left(\int_x^r (r-u)\frac{\partial^2 w(u,y)}{\partial u^2}du;x,y\right)-\int_x^{x+\frac{1}{2m}}\left(x+\frac{1}{2m}-u\right)\frac{\partial^2 w(u,y)}{\partial u^2}du\\ &\quad+\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\left(\int_y^s (s-v)\frac{\partial^2 w(x,v)}{\partial v^2}dv;x,y\right)-\int_y^{y+\frac{1}{2n}}\left(y+\frac{1}{2n}-u\right)\frac{\partial^2 w(x,v)}{\partial v^2}dv\\ &\quad+\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\left(\int_x^r\int_y^s\left(\frac{\partial^2 w}{\partial u\partial v}\right)dudv;x,y\right)-\int_x^{x+\frac{1}{2m}}\int_y^{y+\frac{1}{2n}}\left(\frac{\partial^2 w}{\partial u\partial v}\right)dudv. \end{aligned}$$

Hence using Cauchy-Schwarz inequality and Lemma 2.3,

$$\begin{aligned} &|\tilde{\mathfrak{V}}_{m,n}^{i_1,j_1,i_2,j_2}(w;x,y)-w(x,y)|\\ &\leqslant \mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\left(\left|\int_x^r |r-u|\left|\frac{\partial^2 w(u,y)}{\partial u^2}\right|du\right|;x,y\right)+\left|\int_x^{x+\frac{1}{2m}}\left|x+\frac{1}{2m}-u\right|\left|\frac{\partial^2 w(u,y)}{\partial u^2}\right|du\right|\\ &\quad+\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\left(\left|\int_y^s |s-v|\left|\frac{\partial^2 w(x,v)}{\partial v^2}\right|dv\right|;x,y\right)+\left|\int_y^{y+\frac{1}{2n}}\left|y+\frac{1}{2n}-u\right|\left|\frac{\partial^2 w(x,v)}{\partial v^2}\right|dv\right|\\ &\quad+\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\left(\left|\int_x^r\int_y^s\left|\frac{\partial^2 w}{\partial u\partial v}\right|dudv;x,y\right|\right)+\left|\int_x^{x+\frac{1}{2m}}\int_y^{y+\frac{1}{2n}}\left|\frac{\partial^2 w}{\partial u\partial v}\right|dudv\right|\\ &\leqslant \frac{1}{2}\left\{\phi_{m,n,2,0}^{i_1,j_1,i_2,j_2}(x,y)+\left(x+\frac{1}{2m}-x\right)^2+\phi_{m,n,0,2}^{i_1,j_1,i_2,j_2}(x,y)+\left(y+\frac{1}{2n}-y\right)^2\right\}\|w\|_{C^2(J^2)}\\ &\quad+\left\{\sqrt{\phi_{m,n,2,0}^{i_1,j_1,i_2,j_2}(x,y)}\sqrt{\phi_{m,n,0,2}^{i_1,j_1,i_2,j_2}(x,y)}+\frac{1}{4mn}\right\}\|w\|_{C^2(J^2)}\\ &\leqslant \frac{1}{2}\left\{(\sqrt{\sigma_m}+\sqrt{\sigma_n})^2+\frac{1}{4}\left(\frac{1}{m}+\frac{1}{n}\right)^2\right\}\|w\|_{C^2(J^2)}\\ &= C_{m,n}\|w\|_{C^2(J^2)}. \end{aligned} \tag{3.3}$$

From equation (3.1) we have

$$|\tilde{\mathfrak{V}}_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y)|\leqslant|\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y)|+\left|z\left(x+\frac{1}{2m},y+\frac{1}{2n}\right)\right|+|z(x,y)|\leqslant 3\|z\|. \tag{3.4}$$

Hence from equations (3.1), (3.3), and (3.4), $\forall w\in C^2(J^2)$, we get

$$\begin{aligned} \left|\tilde{\mathfrak{V}}_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y)-z(x,y)\right| &\leqslant |\tilde{\mathfrak{V}}_{m,n}^{i_1,j_1,i_2,j_2}(z-w;x,y)|+|\tilde{\mathfrak{V}}_{m,n}^{i_1,j_1,i_2,j_2}(w;x,y)-w(x,y)|\\ &\quad+|w(x,y)-z(x,y)|+\left|z\left(x+\frac{1}{2m},y+\frac{1}{2n}\right)-z(x,y)\right| \end{aligned}$$

$$\leqslant 4\|z-w\| + |\tilde{\mathfrak{V}}_{m,n}^{i_1,j_1,i_2,j_2}(w;x,y) - w(x,y)| + \left|z\left(x+\frac{1}{2m}, y+\frac{1}{2n}\right) - z(x,y)\right|$$
$$\leqslant 4\|z-w\| + C_{m,n}\|w\|_{C^2(J^2)} + \bar{\omega}\left(z;\frac{1}{2m},\frac{1}{2n}\right).$$

Hence, using the relation (3.4),

$$\left\|\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(z) - z\right\| \leqslant 4\inf_{w\in C^2(J^2)}\left\{\|z-w\| + \frac{C_{m,n}}{4}\|w\|_{C^2(J^2)}\right\} + \bar{\omega}\left(z;\frac{1}{2m},\frac{1}{2n}\right)$$
$$\leqslant M\left\{\bar{\omega}_2\left(z;\frac{\sqrt{C_{m,n}}}{2}\right) + \min\left(1,\frac{C_{m,n}}{4}\right)\|z\|\right\} + \bar{\omega}\left(z;\frac{1}{2m},\frac{1}{2n}\right).$$

□

In the next theorem, we determine an error estimate for continuously differentiable functions by the operators (1.4).

Theorem 3.6. *For $z \in C^1(J^2)$, we have*

$$\left\|\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(z) - z\right\| \leqslant \left\|z'_x\right\|\sqrt{\sigma_m} + \left\|z'_y\right\|\sqrt{\sigma_n}.$$

Proof. Let $(x,y) \in J^2$ be a fixed point. Then, we may write

$$z(r,s) - z(x,y) = \int_x^r z'_u(u,s)\,du + \int_y^s z'_v(x,v)\,dv.$$

On both sides of the above equation, applying $\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(;x,y)$, we get

$$\left|\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y) - z(x,y)\right| \leqslant \mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\left(\left|\int_x^r z'_u(u,s)\,du\right|;x,y\right) + \mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\left(\left|\int_y^s z'_v(x,v)\,dv\right|;x,y\right).$$

Since $\left|\int_x^r z'_u(u,s)\,du\right| \leqslant \|z'_x\|\,|r-x|$ and $\left|\int_y^s z'_v(x,v)\,dv\right| \leqslant \left\|z'_y\right\||s-y|$, we have,

$$\left|\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y) - z(x,y)\right| \leqslant \left\|z'_x\right\|\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(|r-x|;x,y) + \left\|z'_y\right\|\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(|s-y|;x,y).$$

Using Cauchy-Schwarz inequality

$$\left|\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y) - z(x,y)\right| \leqslant \left\|z'_x\right\|\left(\phi_{m,n,2,0}^{i_1,j_1,i_2,j_2}(x,y)\right)^{1/2}\left(\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(1;x,y)\right)^{1/2}$$
$$+ \left\|z'_y\right\|\left(\phi_{m,n,0,2}^{i_1,j_1,i_2,j_2}(x,y)\right)^{1/2}\left(\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(1;x,y)\right)^{1/2}$$
$$\leqslant \left\|z'_x\right\|\sqrt{\sigma_m} + \left\|z'_y\right\|\sqrt{\sigma_n},\ \forall\,(x,y) \in J^2,$$

from which the required result is obvious. □

Now, we prove Voronovskaja type theorem for the operators $\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}$.

Theorem 3.7. *For $z \in C^2(J^2)$, we have*

$$\lim_{n\to\infty} n\left(\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(z;x,y) - z(x,y)\right) = \frac{1}{2}z_x(x,y) + \frac{1}{2}z_y(x,y) + \frac{1}{2}\left\{x(1-x)\,z_{xx}(x,y) + y(1-y)\,z_{yy}(x,y)\right\}$$

uniformly on J^2.

Proof. Let $(x,y) \in J^2$ be arbitrary. Using Taylor's expansion formula, we have

$$z(r,s) = z(x,y) + z_x(x,y)(r-x) + z_y(x,y)(s-y) + \frac{1}{2}\left\{z_{xx}(x,y)(r-x)^2 + 2z_{xy}(x,y)(r-x)(s-y) + z_{yy}(x,y)(s-y)^2\right\} + \varphi(r,s;x,y)\sqrt{(r-x)^4+(s-y)^4},$$

where $(r,s) \in J^2$ and $\varphi(r,s;x,y) \in C\left(J^2\right)$ and $\varphi(r,s;x,y) \to 0$, as $(r,s) \to (x,y)$. Applying $\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(.;x,y)$ on both sides of the above equation,

$$\begin{aligned}\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(z;x,y) &= z(x,y) + z_x(x,y)\,\phi_{n,n,1,0}^{i_1,j_1,i_2,j_2}(x,y) + z_y(x,y)\,\phi_{n,n,0,1}^{i_1,j_1,i_2,j_2}(x,y)\\ &\quad + \frac{1}{2}\Big\{z_{xx}(x,y)\,\phi_{n,n,2,0}^{i_1,j_1,i_2,j_2}(x,y) + 2z_{xy}(x,y)\,\phi_{n,n,1,1}^{i_1,j_1,i_2,j_2}(x,y)\\ &\quad + z_{yy}(x,y)\,\phi_{n,n,0,2}^{i_1,j_1,i_2,j_2}(x,y)\Big\} + \mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}\left(\varphi(r,s;x,y)\sqrt{(r-x)^4+(s-y)^4};x,y\right).\end{aligned}$$

Now using Lemma 2.5, we get

$$\begin{aligned}\lim_{n\to\infty} n\left(\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(z;x,y) - z(x,y)\right) &= \frac{1}{2}z_x(x,y) + \frac{1}{2}z_y(x,y)\\ &\quad + \frac{1}{2}\Big\{x(1-x)z_{xx}(x,y) + y(1-y)z_{yy}(x,y)\Big\} + \lim_{n\to\infty} nQ,\end{aligned}$$

uniformly in J^2, where $Q \equiv \mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}\left(\varphi(r,s;x,y)\sqrt{(r-x)^4+(s-y)^4};x,y\right)$. Applying Cauchy-Schwarz inequality, we have

$$\begin{aligned}|Q| &\leqslant \left\{\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}\left(\varphi^2(r,s;x,y);x,y\right)\right\}^{1/2}\left\{\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}\left((r-x)^4+(s-y)^4;x,y\right)\right\}^{1/2}\\ &\leqslant \left\{\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}\left(\varphi^2(r,s;x,y);x,y\right)\right\}^{1/2}\left\{\phi_{n,n,4,0}^{i_1,j_1,i_2,j_2}(x,y) + \phi_{n,n,0,4}^{i_1,j_1,i_2,j_2}(x,y)\right\}^{1/2}.\end{aligned}$$

Since $\varphi(r,s;x,y) \in C\left(J^2\right)$ and $\varphi^2(r,s;x,y) \to 0$ as $(r,s) \to (x,y)$, using Theorem 2.6, we have

$$\lim_{n\to\infty} \mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}\left(\varphi^2(r,s;x,y);x,y\right) = 0,$$

uniformly with respect to $(x,y) \in J^2$. Further, using Lemma 2.5, $n\left\{\phi_{n,n,4,0}^{i_1,j_1,i_2,j_2}(x,y) + \phi_{n,n,0,4}^{i_1,j_1,i_2,j_2}(x,y)\right\}^{1/2}$ is bounded for all $(x,y) \in J^2$. Hence, $\lim_{n\to\infty} nQ = 0$, uniformly in $(x,y) \in J^2$. Thus,

$$\lim_{n\to\infty} n\left(\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(z;x,y) - z(x,y)\right) = \frac{1}{2}z_x(x,y) + \frac{1}{2}z_y(x,y) + \frac{1}{2}\Big\{x(1-x)z_{xx}(x,y) + y(1-y)z_{yy}(x,y)\Big\},$$

uniformly in $(x,y) \in J^2$. □

Next we establish a quantitative Voronovskaja type theorem by means of modulus of continuity.

Theorem 3.8. *Let $w \in C^2\left(J^2\right)$ then we have the following inequality:*

$$\begin{aligned}&\Bigg|n\left(\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(w;x,y) - w(x,y)\right) - \frac{1}{2}w'_x(x,y) - \frac{1}{2}w'_y(x,y) - \frac{1}{2}\Big\{(x-x^2) + \frac{1}{n}\{i_1^2j_1x - i_1j_1x - i_1^2j_1x^2 + i_1j_1x^2\\ &\quad + \frac{1}{3}\}\Big\}w''_{xx}(x,y) - \frac{1}{2}\Big\{(y-y^2) + \frac{1}{n}\{i_2^2j_2y - i_2j_2y - i_2^2j_2y^2 + i_2j_2y^2 + \frac{1}{3}\}\Big\}w''_{yy}(x,y) - \frac{1}{4n}w''_{xy}(x,y)\Bigg|\\ &\leqslant C\left\{\bar{\omega}\left(w''_{xx};\frac{1}{\sqrt{n}}\right) + \bar{\omega}\left(w''_{yy};\frac{1}{\sqrt{n}}\right) + \bar{\omega}\left(w''_{xy};\frac{1}{\sqrt{n}}\right)\right\},\end{aligned}$$

where C is some positive constant.

Proof. Let $w \in C^2\left(J^2\right)$, then by Taylor's expansion, we have

$$\begin{aligned} w(r,s) &= w(x,y)+(r-x)w'_x(x,y)+(s-y)w'_y(x,y) \\ &\quad +\frac{1}{2}\left\{(r-x)\frac{\partial}{\partial x}+(s-y)\frac{\partial}{\partial y}\right\}^2 w(x+\theta_1(r-x),y+\theta_2(s-y)) \\ &= w(x,y)+(r-x)w'_x(x,y)+(s-y)w'_y(x,y)+\frac{1}{2}\left\{(r-x)\frac{\partial}{\partial x}+(s-y)\frac{\partial}{\partial y}\right\}^2 w(x,y) \\ &\quad +\frac{1}{2}\left\{(r-x)\frac{\partial}{\partial x}+(s-y)\frac{\partial}{\partial y}\right\}^2\Big(w(x+\theta_1(r-x),y+\theta_2(s-y))-w(x,y)\Big),\ 0<\theta_1,\theta_2<1. \end{aligned} \tag{3.5}$$

Now applying the operator $\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}$ on both sides of (3.5), we are led to

$$\begin{aligned} &\Big|\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(w;x,y)-w(x,y)-w'_x(x,y)\phi_{n,n,1,0}^{i_1,j_1,i_2,j_2}(x,y)-w'_y(x,y)\phi_{n,n,0,1}^{i_1,j_1,i_2,j_2}(x,y) \\ &\quad -\frac{1}{2}w''_{xx}(x,y)\phi_{n,n,2,0}^{i_1,j_1,i_2,j_2}(x,y)-\frac{1}{2}w''_{yy}(x,y)\phi_{n,n,0,2}^{i_1,j_1,i_2,j_2}(x,y)-w''_{xy}(x,y)\phi_{n,n,1,1}^{i_1,j_1,i_2,j_2}(x,y)\Big| \\ &\leqslant \frac{1}{2}\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}\left(\left|\left\{(r-x)\frac{\partial}{\partial x}+(s-y)\frac{\partial}{\partial y}\right\}^2\left\{w(x+\theta_1(r-x),y+\theta_2(s-y))-w(x,y)\right\}\right|;x,y\right) \\ &\leqslant \frac{1}{2}\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}\Big((r-x)^2|w''_{xx}(x+\theta_1(r-x),y+\theta_2(s-y))-w''_{xx}(x,y)|;x,y\Big) \\ &\quad +\frac{1}{2}\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}\Big((s-y)^2|w''_{yy}(x+\theta_1(r-x),y+\theta_2(s-y))-w''_{yy}(x,y)|;x,y\Big) \\ &\quad +\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}\Big(|r-x||s-y||w''_{xy}(x+\theta_1(r-x),y+\theta_2(s-y))-w''_{xy}(x,y)|;x,y\Big) \\ &\leqslant \frac{1}{2}\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}\left((r-x)^2+\frac{(r-x)^4+(r-x)^2(s-y)^2}{\sigma^2};x,y\right)\bar{\omega}(w''_{xx};\sigma) \\ &\quad +\frac{1}{2}\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}\left((s-y)^2+\frac{(r-x)^2(s-y)^2+(s-y)^4}{\sigma^2};x,y\right)\bar{\omega}(w''_{yy};\sigma) \\ &\quad +\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}\left(|r-x||s-y|+\frac{|r-x|^3|s-y|+|r-x||s-y|^3}{\sigma^2};x,y\right)\bar{\omega}(w''_{xy};\sigma),\ \sigma>0. \end{aligned}$$

Applying Cauchy-Schwarz inequality and taking $\sigma=\frac{1}{\sqrt{n}}$, in view of Lemma 2.5, we get

$$\begin{aligned} &\Big|n\Big(\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(w;x,y)-w(x,y)\Big)-\frac{1}{2}w'_x(x,y)-\frac{1}{2}w'_y(x,y)-\frac{1}{2}\Big\{(x-x^2)+\frac{1}{n}\{i_1^2j_1x-i_1j_1x \\ &\quad -i_1^2j_1x^2+i_1j_1x^2+\frac{1}{3}\}\Big\}w''_{xx}(x,y)-\frac{1}{2}\Big\{(y-y^2)+\frac{1}{n}\{i_2^2j_2y-i_2j_2y-i_2^2j_2y^2+i_2j_2y^2+\frac{1}{3}\}\Big\}w''_{yy}(x,y)\Big| \\ &\leqslant C\left\{\bar{\omega}\left(w''_{xx};\frac{1}{\sqrt{n}}\right)+\bar{\omega}\left(w''_{yy};\frac{1}{\sqrt{n}}\right)\right\}+\bar{\omega}\left(w''_{xy};\frac{1}{\sqrt{n}}\right). \end{aligned}$$

□

Now, we study the Grüss-Voronovskaja theorem.

Theorem 3.9. *Let $z,w \in C^2\left(J^2\right)$, then the following equality holds true:*

$$\begin{aligned} \lim_{n\to\infty} n\{\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(zw;x,y)-\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(z;x,y)\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(w;x,y)\} =& x(1-x)z'_x(x,y)w'_x(x,y) \\ &+y(1-y)z'_y(x,y)w'_y(x,y), \end{aligned}$$

uniformly in $(x,y) \in J^2$.

Proof. From the Taylor's expansion of z, w, and zw, we get

$$
\begin{aligned}
&n\{\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(zw;x,y)-\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(z;x,y)\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(w;x,y)\}\\
&=n\Big[\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(zw;x,y)-z(x,y)w(x,y)-(z'_x(x,y)w(x,y)+z(x,y)w'_x(x,y))\phi_{n,n,1,0}^{i_1,j_1,i_2,j_2}(x,y)\\
&\quad-(z'_y(x,y)w(x,y)+z(x,y)w'_y(x,y))\phi_{n,n,0,1}^{i_1,j_1,i_2,j_2}(x,y)-\frac{1}{2}(z''_{xx}(x,y)w(x,y)+2z'_x(x,y)w'_x(x,y)\\
&\quad+z(x,y)w''_{xx}(x,y))\phi_{n,n,2,0}^{i_1,j_1,i_2,j_2}(x,y)-\frac{1}{2}(z''_{yy}(x,y)w(x,y)+2z'_y(x,y)w'_y(x,y)+z(x,y)w''_{yy}(x,y))\\
&\phi_{n,n,0,2}^{i_1,j_1,i_2,j_2}(x,y)-(z(x,y)w''_{xy}(x,y)+z'_y(x,y)w'_x(x,y)+z'_x(x,y)w'_y(x,y)+z''_{xy}(x,y)w(x,y))\\
&\phi_{n,n,1,1}^{i_1,j_1,i_2,j_2}(x,y)-w(x,y)\Big\{\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(z;x,y)-z(x,y)-z'_x(x,y)\phi_{n,n,1,0}^{i_1,j_1,i_2,j_2}(x,y)-z'_y(x,y)\\
&\phi_{n,n,0,1}^{i_1,j_1,i_2,j_2}(x,y)-\frac{1}{2}z''_{xx}(x,y)\phi_{n,n,2,0}^{i_1,j_1,i_2,j_2}(x,y)-\frac{1}{2}z''_{yy}\phi_{n,n,0,2}^{i_1,j_1,i_2,j_2}(x,y)-z''_{xy}(x,y)\phi_{n,n,1,1}^{i_1,j_1,i_2,j_2}(x,y)\Big\}\\
&\quad-\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(z;x,y)\Big\{\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(w;x,y)-w(x,y)-w'_x(x,y)\phi_{n,n,1,0}^{i_1,j_1,i_2,j_2}(x,y)\\
&\quad-w'_y(x,y)\phi_{n,n,0,1}^{i_1,j_1,i_2,j_2}(x,y)-\frac{1}{2}w''_{xx}(x,y)\phi_{n,n,2,0}^{i_1,j_1,i_2,j_2}(x,y)-\frac{1}{2}w''_{yy}(x,y)\phi_{n,n,0,2}^{i_1,j_1,i_2,j_2}(x,y)\\
&\quad-w''_{xy}(x,y)\phi_{n,n,1,1}^{i_1,j_1,i_2,j_2}(x,y)\Big\}+w'_x(x,y)\phi_{n,n,1,0}^{i_1,j_1,i_2,j_2}(x,y)\Big\{z(x,y)-\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(z;x,y)\Big\}\\
&\quad+w'_y(x,y)\phi_{n,n,0,1}^{i_1,j_1,i_2,j_2}(x,y)\Big\{z(x,y)-\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(z;x,y)\Big\}\\
&\quad+\frac{1}{2}w''_{xx}(x,y)\phi_{n,n,2,0}^{i_1,j_1,i_2,j_2}(x,y)\Big\{z(x,y)-\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(z;x,y)\Big\}+\frac{1}{2}w''_{yy}(x,y)\phi_{n,n,0,2}^{i_1,j_1,i_2,j_2}(x,y)\Big\{z(x,y)\\
&\quad-\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(z;x,y)\Big\}+w''_{xy}(x,y)\phi_{n,n,1,1}^{i_1,j_1,i_2,j_2}(x,y)\Big\{z(x,y)-\mathfrak{V}_{n,n}^{i_1,j_1,i_2,j_2}(z;x,y)\Big\}+z'_x(x,y)w'_x(x,y)\\
&\phi_{n,n,2,0}^{i_1,j_1,i_2,j_2}(x,y)+z'_x(x,y)w'_y(x,y)\phi_{n,n,1,1}^{i_1,j_1,i_2,j_2}(x,y)+z'_y(x,y)w'_x(x,y)\phi_{n,n,1,1}^{i_1,j_1,i_2,j_2}(x,y)\\
&\quad+z'_y(x,y)w'_y(x,y)\phi_{n,n,0,2}^{i_1,j_1,i_2,j_2}(x,y)\Big].
\end{aligned}
$$

Applying Theorem 3.1, Lemma 2.5, and Theorem 3.8, we reach the assertion. □

4. GBS operator associated to $\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}$

Bögel ([7, 8]) introduced the concepts of B-continuous and B-differentiable functions and gave some significant theorems. Badea et al. [5] proved a Korovkin type theorem for approximation of B-continuous functions. The term "GBS operator"(Generalized Boolean Sum Operator) was introduced by Badea and Cottin [6] wherein quantitative and non-quantitative variants of the Korovkin type theorem were developed. Agrawal et al. [2] constructed the GBS operator of bivariate Lupas-Durrmeyer type operators and derived some approximation theorems. We refer the reader to the book [12] and the references therein, for a detailed study of the research work in this direction.

In the following, we recall some basic notations and definitions which will be needed later.

Let Z_1 and Z_2 be compact real intervals. At a point $(x_0, y_0) \in Z_1 \times Z_2$, a function $z : Z_1 \times Z_2 \to \mathbb{R}$ is called B-continuous iff $\lim\limits_{(x,y)\to(x_0,y_0)} \Delta z\left[(x,y),(x_0,y_0)\right] = 0$, where $\Delta z\left[(x,y),(x_0,y_0)\right] = z(x,y) - z(x_0,y) - z(x,y_0) + z(x_0,y_0)$ denotes the mixed difference of z. A function $z : Z_1 \times Z_2 \to \mathbb{R}$ is called B-continuous on $Z_1 \times Z_2$ iff it is B-continuous at every point of $Z_1 \times Z_2$.

A function $z: Z_1 \times Z_2 \to \mathbb{R}$ is called B-differentiable at $(x_0, y_0) \in Z_1 \times Z_2$ iff

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta z\left[(x,y),(x_0,y_0)\right]}{(x-x_0)(y-y_0)},$$

exists and is finite. This limit is named as the B-differential of z at the point (x_0, y_0) and is denoted by $D_B z(x_0, y_0)$. The function $z: Z_1 \times Z_2 \to \mathbb{R}$ is called B-differentiable if it is B-differentiable at every point of $Z_1 \times Z_2$.

The function $z : Z_1 \times Z_2 \to \mathbb{R}$ is B-bounded on $Z_1 \times Z_2$ iff there exists some $k > 0$ such that $|\Delta z[(x,y),(r,s)]| \leqslant k$ for any $(x,y),(r,s) \in Z_1 \times Z_2$. Let us define the sets:

$$\begin{aligned} B(Z_1 \times Z_2) =&\{z : Z_1 \times Z_2 \to \mathbb{R} : z \text{ is bounded on } Z_1 \times Z_2\};\\ C(Z_1 \times Z_2) =&\{z : Z_1 \times Z_2 \to \mathbb{R} : z \text{ is continuous on } Z_1 \times Z_2\};\\ B_b(Z_1 \times Z_2) =&\{z : Z_1 \times Z_2 \to \mathbb{R} : z \text{ is B-bounded on } Z_1 \times Z_2\};\\ C_b(Z_1 \times Z_2) =&\{z : Z_1 \times Z_2 \to \mathbb{R} : z \text{ is B-continuous on } Z_1 \times Z_2\};\\ \text{and } D_b(Z_1 \times Z_2) =&\{z : Z_1 \times Z_2 \to \mathbb{R} : z \text{ is B-differentiable on } Z_1 \times Z_2\}. \end{aligned}$$

The norm in $B(Z_1 \times Z_2)$ is given by $\|z\|_\infty = \sup_{(x,y)\in Z_1\times Z_2} |z(x,y)|$, $\forall z \in B(Z_1 \times Z_2)$. From [9, page 52], it is known that $C(Z_1 \times Z_2) \subset C_b(Z_1 \times Z_2)$. Let $z \in B_b(z_1 \times z_2)$. For all $(x,y),(r,s) \in Z_1 \times Z_2$ and for any $(\sigma_1, \sigma_2) \in [0,\infty) \times [0,\infty)$, the mixed modulus of smoothness is defined as the function $\omega_{\text{mixed}} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ such that

$$\omega_{\text{mixed}}(z; \sigma_1, \sigma_2) := \sup\{|\Delta z[(r,s),(x,y)]| : |x-r| < \sigma_1, |y-s| < \sigma_2\}.$$

The usual modulus of continuity and ω_{mixed} have similar basic properties, which were obtained by Badea et al. in [4].

A function $z \in C_b(z_1 \times z_2)$ is said to belong to the Lipschitz class $Lip_{M,b}(\varepsilon,\eta)$, $0 < \varepsilon, \eta \leqslant 1$ if

$$|\Delta z\left[(x,y),(x_0,y_0)\right]| \leqslant M(x-x_0)^{\varepsilon}(y-y_0)^{\eta},\ \forall (x,y),(x_0,y_0) \in z_1 \times z_2$$

for some $M > 0$. For any $z \in C_b(J^2)$, we define the GBS operator associated to $\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}$ as

$$P_{m,n}^{i_1,j_1,i_2,j_2}(z(r,s);x,y) := \mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(z(r,y)+z(x,s)-z(r,s);x,y)$$

for all $(x,y) \in J^2$. More precisely,

$$\begin{aligned} P_{m,n}^{i_1,j_1,i_2,j_2}\Big(z(r,s);x,y\Big) = &\sum_{\alpha_1=0}^{m-i_1j_1} b_{m-i_1j_1,\alpha_1}(x) \sum_{\beta_1=0}^{i_1} b_{i_1,\beta_1}(x) \sum_{\alpha_2=0}^{n-i_2j_2} b_{n-i_2j_2\alpha_2}(y) \sum_{\beta_2=0}^{i_2} b_{i_2,\beta_2}(y)\\ &\times \int_0^1\int_0^1 \left[z\left(\frac{\alpha_1+\beta_1 i_1+r}{m},y\right)+z\left(x,\frac{\alpha_2+\beta_2 i_2+s}{n}\right)\right.\\ &\left.-z\left(\frac{\alpha_1+\beta_1 i_1+r}{m},\frac{\alpha_2+\beta_2 i_2+s}{n}\right)\right]drds. \end{aligned} \tag{4.1}$$

Clearly, $P_{m,n}^{i_1,j_1,i_2,j_2}$ is a linear operator from the space $C_b(J^2)$ into $C(J^2)$.

For Lipschitz class of Bögel continuous functions, the following theorem provides the degree of approximation by the operator $P_{m,n}^{i_1,j_1,i_2,j_2}$.

Theorem 4.1. *Let* $z \in Lip_{M,b}(\varepsilon,\eta)$, *then we have*

$$\left\|P_{m,n}^{i_1,j_1,i_2,j_2}(z)-z\right\| \leqslant M\sigma_m^{\varepsilon/2}\sigma_n^{\eta/2},$$

where $M > 0$ *and* $0 < \varepsilon, \eta \leqslant 1$.

Proof. From the definition of the operator $P_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y)$ and the linearity of the operator $\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}$, for any $(x,y)\in J^2$, we have

$$P_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y)=\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\Big(z(r,y)+z(x,s)-z(r,s);x,y\Big)$$

or

$$P_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y)=z(x,y)-\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\Big(\Delta z[(r,s);(x,y)];x,y\Big),$$

which implies that

$$\left|P_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y)-z(x,y)\right|\leqslant \mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\Big(|\Delta z[(r,s);(x,y)]|;x,y\Big). \tag{4.2}$$

Since $z\in \mathrm{Lip}_M(\varepsilon,\eta)$, we get

$$\begin{aligned}\left|P_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y)-z(x,y)\right| &\leqslant M\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\Big(|r-x|^{\varepsilon}|s-y|^{\eta};x,y\Big)\\ &\leqslant M\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\Big(|r-x|^{\varepsilon};x,y\Big)\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\Big(|s-y|^{\eta};x,y\Big).\end{aligned}$$

Now using Hölder's inequality with $p_1=\frac{2}{\varepsilon}$, $q_1=\frac{2}{2-\varepsilon}$, and $p_2=\frac{2}{\eta}$, $q_2=\frac{2}{2-\eta}$, we get

$$\begin{aligned}\left|P_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y)-z(x,y)\right| &\leqslant M\Big(\phi_{m,n,2,0}^{i_1,j_1,i_2,j_2}(x,y)\Big)^{\varepsilon/2}\Big(\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(z_{00};x,y)\Big)^{(2-\varepsilon)/2}\\ &\quad\times\Big(\phi_{m,n,0,2}^{i_1,j_1,i_2,j_2}(x,y)\Big)^{\eta/2}\Big(\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(z_{00};x,y)\Big)^{(2-\eta)/2}\leqslant M\sigma_m^{\varepsilon/2}\sigma_n^{\eta/2},\end{aligned}$$

from which required result is immediate. □

Next, we shall estimate the rate of convergence of the operator (4.1) for $z\in C_b(J^2)$, using mixed modulus of smoothness.

Theorem 4.2. *For $z\in C_b(J^2)$, the operator $P_{m,n}^{i_1,j_1,i_2,j_2}$ verifies the following inequality:*

$$\left\|P_{m,n}^{i_1,j_1,i_2,j_2}(z)-z\right\|\leqslant 4\omega_{\text{mixed}}\left(z;m^{-1/2},n^{-1/2}\right).$$

Proof. Using the definition of $\omega_{\text{mixed}}(z;\sigma_1,\sigma_2)$ and the elementary inequality

$$\omega_{\text{mixed}}(z;\lambda_1\sigma_1,\lambda_2\sigma_2)\leqslant(1+\lambda_1)(1+\lambda_2)\,\omega_{\text{mixed}}(z;\sigma_1,\sigma_2);\sigma_1,\sigma_2,\lambda_1,\lambda_2>0,$$

we have

$$|\sigma z[(r,s);(x,y)]|\leqslant\omega_{\text{mixed}}(z;|r-x|,|s-y|)\leqslant\left(1+\frac{|r-x|}{\sigma_1}\right)\left(1+\frac{|s-y|}{\sigma_2}\right)\omega_{\text{mixed}}(z;\sigma_1,\sigma_2), \tag{4.3}$$

$\forall\,(x,y),(r,s)\in J^2$ and for any $\sigma_1,\sigma_2>0$. In view of (4.2) and (4.3),

$$\begin{aligned}\left|P_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y)-z(x,y)\right| &\leqslant \mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\Big(|\Delta z[(r,s);(x,y)]|;x,y\Big)\\ &\leqslant \mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\left(\left(1+\frac{|r-x|}{\sigma_1}\right)\left(1+\frac{|s-y|}{\sigma_2}\right)\omega_{\text{mixed}}(z;\sigma_1,\sigma_2);x,y\right).\end{aligned}$$

Now, applying Cauchy-Schwarz inequality,

$$\left|P_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y)-z(x,y)\right| \leqslant \Big\{\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(e_{00};x,y)+\sigma_1^{-1}\sqrt{\phi_{m,n,2,0}^{i_1,j_1,i_2,j_2}(x,y)}+\sigma_2^{-1}\sqrt{\phi_{m,n,0,2}^{i_1,j_1,i_2,j_2}(x,y)}$$
$$+\sigma_1^{-1}\sigma_2^{-1}\sqrt{\phi_{m,n,2,0}^{i_1,j_1,i_2,j_2}(x,y)\phi_{m,n,0,2}^{i_1,j_1,i_2,j_2}(x,y)}\Big\}\omega_{\text{mixed}}(z;\sigma_1,\sigma_2).$$

Hence, using Lemma 2.5 and taking $\sigma_1=\frac{1}{\sqrt{m}}, \sigma_2=\frac{1}{\sqrt{n}}$, we get the desired result. □

Now, we determine the order of approximation for the B-differentiable functions by the operator $P_{m,n}^{i_1,j_1,i_2,j_2}$.

Theorem 4.3. *For the function $z\in D_b(J^2)$ with $D_Bz\in C_b(J^2)\cap B(J^2)$, there holds the following inequality:*

$$\left\|P_{m,n}^{i_1,j_1,i_2,j_2}(z)-z\right\| \leqslant M\left[3\|D_Bz\|_\infty+4\omega_{\text{mixed}}\left(D_B;m^{-1},n^{-1}\right)\right]m^{-1}n^{-1},$$

where M is some positive constant.

Proof. For $z\in D_b(J^2)$, by mean value theorem,

$$\Delta z\left[(r,s);(x,y)\right]=(r-x)(s-y)D_Bz(\theta_1,\theta_2) \tag{4.4}$$

with $x<\theta_1<r, y<\theta_2<s$. Now,

$$D_Bz(\theta_1,\theta_2)=\Delta D_Bz[(\theta_1,\theta_2);(x,y)]+D_Bz(\theta_1,y)+D_Bf(x,\theta_2)-D_Bz(x,y).$$

Since $D_Bz\in C_b(J^2)\cap B(J^2)$, from (4.4), we have

$$\begin{aligned}
|\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(\Delta z[(r,s);(x,y)];x,y)| &= \left|\phi_{m,n,1,1}^{i_1,j_1,i_2,j_2}(x,y)D_Bz(\theta_1,\theta_2);x,y\right|\\
&\leqslant \mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\Big(|r-x||s-y||\Delta D_Bz[(\theta_1,\theta_2);(x,y)]|;x,y\Big)\\
&\quad+\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\Big(|r-x||s-y|\Big(|D_Bz(\theta_1,y)|+|D_Bz(x,\theta_2)|\\
&\quad+|D_Bz(x,y)|\Big);x,y\Big)\\
&\leqslant \mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\left(|r-x||s-y|\,\omega_{\text{mixed}}(D_Bz;|\theta_1-x|,|\theta_2-y|);x,y\right)\\
&\quad+3\|D_Bz\|_\infty\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(|r-x||s-y|;x,y).
\end{aligned} \tag{4.5}$$

Using the properties of ω_{mixed}, we get

$$\begin{aligned}
\omega_{\text{mixed}}\Big(D_Bz;|\theta_1-x|,|\theta_2-y|\Big) &\leqslant \omega_{\text{mixed}}\Big(D_Bz;|r-x|,|s-y|\Big)\\
&\leqslant\left(1+\frac{|r-x|}{\sigma_1}\right)\left(1+\frac{|s-y|}{\sigma_2}\right)\omega_{\text{mixed}}\Big(D_Bz;\sigma_1,\sigma_2\Big).
\end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6), we obtain

$$\begin{aligned}
\left|P_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y)-z(x,y)\right| &= \left|\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(\Delta z[(r,s);(x,y)];x,y)\right|\\
&\leqslant \mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}\left(|r-x||s-y|\left(1+\sigma_1^{-1}|r-x|\right)\left(1+\sigma_2^{-1}|s-y|\right);x,y\right)\\
&\quad\times\omega_{\text{mixed}}(D_Bz;\sigma_1,\sigma_2)+3\|D_Bz\|_\infty\mathfrak{V}_{m,n}^{i_1,j_1,i_2,j_2}(|r-x||s-y|;x,y).
\end{aligned}$$

Now, applying Cauchy-Schwarz inequality and Lemma 2.3,

$$\begin{aligned}\left|P_{m,n}^{i_1,j_1,i_2,j_2}(z;x,y)-z(x,y)\right| \leqslant & \left[\sqrt{\phi_{m,n,2,2}^{i_1,j_1,i_2,j_2}(x,y)}+\frac{1}{\sigma_2}\sqrt{\phi_{m,n,2,4}^{i_1,j_1,i_2,j_2}(x,y)}\right.\\ &\left.+\frac{1}{\sigma_1}\sqrt{\phi_{m,n,4,2}^{i_1,j_1,i_2,j_2}(x,y)}+\frac{1}{\sigma_1}\frac{1}{\sigma_2}\phi_{m,n,2,2}^{i_1,j_1,i_2,j_2}(x,y)\right]\omega_{\text{mixed}}\left(D_B f;\sigma_1,\sigma_2\right)\\ &+3\|D_B z\|_\infty\sqrt{\phi_{m,n,2,2}^{i_1,j_1,i_2,j_2}(x,y)}.\end{aligned}$$

From Lemma 2.4 and for $u,v\in\{1,2\}$,

$$\phi_{m,n,2u,2v}^{i_1,j_1,i_2,j_2}(x,y)=\phi_{m,n,2u,0}^{i_1,j_1,i_2,j_2}(x,y)\phi_{m,n,0,2v}^{i_1,j_1,i_2,j_2}(x,y)\leqslant\left(\frac{M}{m^u n^v}\right),\ \forall\, m,n\in\mathbb{N}\text{ and }(x,y),(r,s)\in J^2,$$

where M is some positive constant. Hence taking $\sigma_1=m^{-1/2}$ and $\sigma_2=n^{-1/2}$, the desired result is obtained. □

Acknowledgement

The authors are extremely thankful to Science and Engineering Research Board (SERB), Govt. of India, for providing the financial support under Teachers Associateship for Research Excellence (TARE) Award (TAR/2018/000356).

References

[1] U. Abel, M. Ivan, R. Păltănea, *The Durrmeyer variant of an operator defined by DD Stancu*, Appl. Math. Comput., **259** (2015), 116–123. 1

[2] P. N. Agrawal, N. Ispir, A. Kajla, *GBS operators of Lupaş-Durrmeyer type based on Polya distribution*, Results Math., **69** (2016), 397–418. 4

[3] G. A. Anastassiou, S. G. Gal, *Approximation Theory:Moduli of Continuity and Global Smoothness Preservation*, Birkhauser, Boston, (2000). 3

[4] C. Badea, I. Badea, C. Cottin, H. H. Gonska, *Notes on the degree of approximation of B-continuous and B-differentiable functions*, Approx. Theory Appl., **4** (1988), 95–108. 4

[5] C. Badea, I. Badea, H. H. Gonska, *A test function theorem and approximation by pseudo polynomials*, Bull. Aust. Math. Soc., **34** (1986), 53–64. 4

[6] C. Badea, C. Cottin, *Korovkin-type theorems for generalized Boolean sum operators*, Colloq. Math. Soc. Janos Bolyai, **58** (1991), 51–68. 4

[7] K. Bögel, *Mehrdimensionale Differentiation von Funktionen mehrer reeller Veränderlichen*, J. Reine Angew. Math., **170** (1934), 197–217. 4

[8] K. Bögel, *Über mehrdimensionale Differentiation, Integration und beschränkte Variation*, J. Reine Angew Math., **173** (1935), 5–23. 4

[9] K. Bögel, *Über die mehrdimensionale Differentiation*, Jber. Deutsch. Math.-Verein., **65** (1962/63), 45–71. 4

[10] P. L. Butzer, H. Berens, *Semi-groups of operators and approximation*, Springer-Verlag, New York, (1967). 3

[11] N. Deo, N. Bhardwaj, *Some approximation theorems for multivariate Bernstein operators*, Southeast Asian Bull. Math., **34** (2010), 1023–1034. 1

[12] V. Gupta, T. M. Rassias, P. N. Agrawal, A. M. Acu, *Recent Advances in Constructive Approximation Theory*, Springer, Cham, (2018). 4

[13] A. Kajla, *The Kantorovich variant of an operator defined by D. D. Stancu*, Appl. Math. Comput., **316** (2018), 400–408. 1, 2.1, 2.2

[14] A. Kajla, N. Ispir, P. N. Agrawal, M. Goyal, q-*Bernstein-Schurer-Durrmeyer type operators for functions of one and two variables*, Appl. Math. Comput., **275** (2016), 372–385. 1

[15] D. D. Stancu, *Some Bernšteĭn polynomials in two variables and their applications*, Soviet Math., **1** (1961), 1025–1028. 1

[16] D. D. Stancu, *The remainder in the approximation by a generalized Bernstein operator : a representation by a convex combination of second-order divided differences*, Calcolo, **35** (1998), 53–62. 1

[17] V. I. Volkov, *On the convergence of sequences of linear positive operators in the space of continuous functions of two variables*, Dokl. Akad. Nauk SSSR, **115** (1957), 17–19. 2.6

[18] D.-X. Zhou, *Inverse theorems for multidimensional Bernstein-Durrmeyer operators in* L_p^*, J. Approx. Theory, **70** (1992), 68–93. 1