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Certain nonlinear functions acting on the vector space $\mathbb{H}^n$ over the Quaternions $\mathbb{H}$

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Abstract

In this paper, we consider a certain type of nonlinear functions acting on a finite-dimensional vector space $\mathbb{H}^n$ over the ring $\mathbb{H}$ of all quaternions, for $n \in \mathbb{N}$. Our main results show that: (i) every quaternion $q \in \mathbb{H}$ is classified by its spectrum of the realization under a canonical representation on $\mathbb{C}^2$; (ii) each vector of $\mathbb{H}^n$ is classified by $\mathbb{C}^n$ in an extended set-up of (i); and (iii) the (usual linear) spectral analysis on the matricial ring $M_n(\mathbb{C})$ of all $(n \times n)$-matrices (over $\mathbb{C}$) affects some fixed point theorems for our nonlinear functions on $\mathbb{H}^n$. In conclusion, we study the connections between the "linear" spectral theory over the complex numbers $\mathbb{C}$, and fixed point theorems for "nonlinear" functions over $\mathbb{H}$.

Keywords: The quaternions, q-spectral forms, q-spectralizations, vector spaces over the quaternions.

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1. Introduction

In this paper, we consider finite-dimensional vector spaces $\mathbb{H}^n$ over the ring $\mathbb{H}$ of the *quaternions*, for $n \in \mathbb{N}$. In particular, we are interested in certain nonlinear functions acting on $\mathbb{H}^n$. Let

$$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}, \text{ and } i = \sqrt{-1}\}$$

be the set of all *complex numbers*, where $\mathbb{R}$ denotes the *real numbers*. Then the set,

$$\mathbb{H} = \left\{ x + yi + uj + vk \;\middle|\; \begin{array}{c} x,\ y,\ u,\ v \in \mathbb{R} \\ i^2 = j^2 = k^2 = -1, \\ \text{and } ijk = -1 \end{array} \right\},$$

of all *quaternions* (or *quaternion numbers*) is defined.

A representation of [19] lets us understand every quaternion $q \in \mathbb{H}$ as a matrix $[q] \in M_2(\mathbb{C})$ on the 2-dimensional complex vector space $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$. For instance,

$$[i] = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ [j] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \text{and } [k] = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

in $M_2(\mathbb{C})$. The *spectral properties of* $[q] \in M_2(\mathbb{C})$ is considered in [1]. And, by using the main results of

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[1], we formularized the solutions of monomial equations, and characterize how to solve some quadratic equations on $\mathbb{H}$, in [2]. For the self-contained-ness of the paper, we briefly introduce the main concepts and results of these preprints [1, 2] in Sections 2, 3, and 4.

1.1. Motivation

The quaternions $\mathbb{H}$ is an important object not only in mathematics (e.g., [1, 2, 9, 10, 17]), but also in scientific fields (e.g., [3, 13]). Algebra on $\mathbb{H}$ is considered in e.g., [20]; analysis on $\mathbb{H}$ is studied in e.g., [11, 18]; and physics on $\mathbb{H}$ is investigated-and-applied in e.g., [6]. Also, the matrices over the quaternions $\mathbb{H}$, and the corresponding quaternionic-eigenvalue problems have been studied in linear, or multi-linear analysis (e.g., see [4, 5, 12, 14–16, 18]).

We here emphasize that, even though our works are motivated by the recent studies, the purposes, approaches, results and applications are different from the earlier works. In this paper, we study finite-dimensional vector spaces $\mathbb{H}^n$ over the quaternions $\mathbb{H}$, for $n \in \mathbb{N}$, and certain types of nonlinear functions on $\mathbb{H}^n$. Our results may/can be applicable to geometry on $\mathbb{H}$.

1.2. Overview

In Sections 2 and 3, the spectral analysis of the realizations of quaternions is re-considered (also, see [1, 2]). And, in Section 4, we classify the quaternions $\mathbb{H}$ by their representatives, the complex numbers $\mathbb{C}$ by the spectral properties of Sections 2 and 3.

In Section 5, finite-dimensional vector spaces $\mathbb{H}^n$ are constructed-and-studied over $\mathbb{H}$, for $n \in \mathbb{N}$. The vectors of $\mathbb{H}^n$ are classified by their representatives, the complex vectors of $\mathbb{C}^n$ under our spectral classifications of Section 4. In Sections 6 and 7, we study nonlinear functions acting on $\mathbb{H}^n$, and consider a certain type of them. By collecting these nonlinear functions, we construct an algebraic structure $\sum_n(\mathbb{H})$ of such nonlinear functions, and it is shown that $\sum_n(\mathbb{H})$ forms a noncommutative ring over the real numbers $\mathbb{R}$. Basic functional properties of the ring-elements of $\sum_n(\mathbb{H})$ are considered there.

By using the results of Sections 5, 6, and 7, the relations between the usual spectral theory on the matricial ring $M_n(\mathbb{C})$ and basic fixed-point theorems on $\sum_n(\mathbb{H})$ are studied in Section 8. The results of Section 8 provide connections between "linear" analysis on $\mathbb{C}^n$ and "nonlinear" analysis on $\mathbb{H}^n$ via the spectral classification on the quaternions $\mathbb{H}$.

2. A representation $(\mathbb{C}^2, \pi)$ of $\mathbb{H}$

In this section, we review a representation of the quaternions $\mathbb{H}$. In particular, we understand each quaternion $q \in \mathbb{H}$ as a (2×2)-matrix $[q] \in M_2(\mathbb{C})$ acting on the 2-dimensional space $\mathbb{C}^2$ (e.g., see [1, 16, 20]).

2.1. Quaternions $\mathbb{H}$

Let a and b be complex numbers,

$$a = x + yi \text{ and } b = u + vi \text{ in } \mathbb{C},$$

where $x, y, u, v \in \mathbb{R}$, and $i = \sqrt{-1}$ in $\mathbb{C}$. For the complex numbers $a, b \in \mathbb{C}$, the corresponding quaternion $q \in \mathbb{H}$ is defined by

$$q = a + bj = (x + yi) + (u + vi)j = x + yi + uj + vij = x + yi + uj + vk, \tag{2.1}$$

in $\mathbb{H}$,

$$i^2 = j^2 = k^2 = ijk = -1.$$

The quaternions $\mathbb{H}$ has a well-defined *addition* $(+)$, and *multiplication* $(\cdot)$; for any

$$q_l = a_l + b_l j \in \mathbb{H}, \text{ with } a_l, b_l \in \mathbb{C},$$

in the sense of (2.1) for $l = 1, 2$, one has

$$q_1 + q_2 = (a_1 + a_2) + (b_1 + b_2)j, \quad q_1q_2 = (a_1a_2 - b_1\overline{b_2}) + (a_1b_2 + \overline{a_2}b_1)j, \tag{2.2}$$

in $\mathbb{H}$, where $\overline{z}$ are the *conjugates of* $z \in \mathbb{C}$. By (2.2),

$$q_1q_2 \neq q_2q_1 \text{ in } \mathbb{H}, \text{ in general.}$$

Under the operations of (2.2), the quaternions $\mathbb{H}$ form a ring algebraically, moreover it is a "noncommutative field" (in the sense of [20]). A *noncommutative field* $(F, +, \cdot)$ is an algebraic structure satisfying that: the algebraic pair $(F, +)$ forms an abelian group; and the pair $(F^{\times}, \cdot)$ forms a "noncommutative" group, where $F^{\times} = F \setminus \{0_F\}$, where 0_F is the $(+)$-identity of $(F, +)$; and $(+)$ and $(\cdot)$ are left-and-right distributive.

If $q \in \mathbb{H}$ is a quaternion (2.1), then one can define the *quaternion-conjugate* $\overline{q} \in \mathbb{H}$ by

$$\overline{q} = x - yi - ui - vi. \tag{2.3}$$

So, one has that

$$\overline{q}q = q\overline{q} = |a|^2 + |b|^2 = x^2 + y^2 + u^2 + v^2, \tag{2.4}$$

by (2.3). Thus, by (2.4),

$$\overline{q}q = q\overline{q} \geqslant 0 \text{ in } \mathbb{R} \subset \mathbb{H}, \ \forall q \in \mathbb{H}. \tag{2.5}$$

By (2.5), one can define the *quaternion-modulus* $\|.\|$ *on* $\mathbb{H}$ by

$$\|q\| = \sqrt{q\overline{q}}, \text{ for all } q \in \mathbb{H}. \tag{2.6}$$

This quaternion-modulus $\|.\|$ of (2.6) is a well-defined norm on $\mathbb{H}$. If $q \neq 0$ in $\mathbb{H}$, then the quaternion-reciprocal q^{-1} of q,

$$q^{-1} = \left(\frac{\overline{a}}{|a|^2 + |b|^2}\right) + \left(\frac{-b}{|a|^2 + |b|^2}\right)j$$

is well-defined in $\mathbb{H}$, by (2.4) and (2.6).

2.2. *A Representation* $(\mathbb{C}^2, \pi)$ *of* $\mathbb{H}$

In this section, we consider a representation of the quaternions $\mathbb{H}$, introduced in [20], realized on the 2-dimensional space $\mathbb{C}^2$ over the complex numbers $\mathbb{C}$. As in (2.1), let's understand each quaternion $q \in \mathbb{H}$ as

$$q = a + bj \text{ in } \mathbb{H}, \text{ with } a, \in \mathbb{C},$$

where

$$a = x + yi, \text{ and } b = u + vi \text{ in } \mathbb{C}.$$

Define an injective representation, $\pi : \mathbb{H} \to M_2(\mathbb{C})$, by

$$\pi(q) = \pi(a + bj) = \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix}, \tag{2.7}$$

where $\overline{a} = x - yi$ and $\overline{b} = u - vi$ are the complex-conjugates of a and b in $\mathbb{C}$, respectively, and $M_2(\mathbb{C})$ is the *matricial ring* of all (2×2)-matrices over $\mathbb{C}$. This morphism π of (2.7) satisfies that

$$\pi(q_1 + q_2) = \pi(q_1) + \pi(q_2), \quad \text{and} \quad \pi(q_1q_2) = \pi(q_1)\pi(q_2), \tag{2.8}$$

for all $q_1, q_2 \in \mathbb{H}$, by (2.8). Then the quaternion-conjugate $\overline{q}$ of $q \in \mathbb{H}$ satisfies that

$$\pi(\overline{q}) = \pi(\overline{a} - bj) = \begin{pmatrix} \overline{a} & b \\ -\overline{b} & a \end{pmatrix} = \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix}^{*} = \pi(q)^{*}, \tag{2.9}$$

in $M_2(\mathbb{C})$ by (2.10), where A^* are the *adjoints* (or, the conjugate-transposes) of $A \in M_2(\mathbb{C})$. Furthermore,

$$\det(\pi(q)) = \det\begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} = |a|^2 + |b|^2,$$

and hence, one can have

$$\|q\| = \sqrt{\det(\pi(q))}, \text{ for all } q \in \mathbb{H}, \tag{2.10}$$

by (2.12) and (2.13).

Proposition 2.1. *Let π be in the sense of (2.7). Then*

$$(\mathbb{C}^2, \pi) \text{ is a topological representation of } \mathbb{H}. \tag{2.11}$$

Proof. The morphism π of (2.7) is a well-defined injective ring-homomorphism from $\mathbb{H}$ into $M_2(\mathbb{C})$, by (2.8) and (2.9). Moreover, the relation (2.10) shows that the usual topology for $\mathbb{H}$, determined by the quaternion-modulus $|.|$ is preserved by the norm $\|.\|$ on $M_2(\mathbb{C})$, and hence, this representation is topological. □

Notation: Let $q \in \mathbb{H}$, and $\pi(q)$, the realization of q in $M_2(\mathbb{C})$. For convenience, we denote $\pi(q)$ by $[q]$.

Let's define a subset $\mathcal{H}_2$ of $M_2(\mathbb{C})$ by the set of all realizations of $\mathbb{H}$, i.e.,

$$\mathcal{H}_2 \overset{\text{def}}{=} \{[q] \in M_2(\mathbb{C}) : q \in \mathbb{H}\} = \pi(\mathbb{H}). \tag{2.12}$$

Theorem 2.2. *The quaternions $\mathbb{H}$ and the set $\mathcal{H}_2$ of (2.12) are isomorphic noncommutative fields, i.e.,*

$$\mathbb{H} \overset{NF}{=} \mathcal{H}_2, \tag{2.13}$$

where "$\overset{NF}{=}$" means "being noncommutative-field-isomorphic."

Proof. Take the action π of (2.7) acting on $\mathbb{C}^2$. By the injectivity of π, and by the definition (2.12), two sets $\mathbb{H}$ and $\mathcal{H}_2$ are bijective (or equipotent), i.e., $\pi : \mathbb{H} \to \mathcal{H}_2$ is a bijection. Moreover, π is a well-defined topological-ring-homomorphism from $\mathbb{H}$ onto $\mathcal{H}_2$ by (2.11), i.e., π is a continuous ring-isomorphism from $\mathbb{H}$ onto $\mathcal{H}_2$. Thus the relation (2.13) holds. □

3. Spectral analysis on $\mathbb{H}$

Let $\mathcal{H}_2$ be the noncommutative field (2.12), isomorphic to the quaternions $\mathbb{H}$. In this section, we regard each quaternion $q \in \mathbb{H}$ as a (2×2)-matrix $[q] \in \mathcal{H}_2$ in $M_2(\mathbb{C})$ by (2.13), and study spectral analysis on $\mathcal{H}_2$ (and hence, that on $\mathbb{H}$).

3.1. Quaternion-spectral forms of $\mathbb{H}$

In this section, we consider the spectra spec$([q])$ of the realizations $[q] \in \mathcal{H}_2$ of quaternions $q \in \mathbb{H}$ canonically, by regarding $[q]$ as the usual (2×2)-matrices of $M_2(\mathbb{C})$. Let $q = a + bj \in \mathbb{H}$ be a quaternion with

$$a = x + yi, \; b = u + vi \in \mathbb{C},$$

and

$$[q] = \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} = \begin{pmatrix} x+yi & -u-vi \\ u-vi & x-yi \end{pmatrix} \in \mathcal{H}_2, \tag{3.1}$$

the realization of q. The realization $[q] \in \mathcal{H}_2 \subset M_2(\mathbb{C})$ of (3.1) has its characteristic polynomial,

$$\operatorname{def}([q] - zI_2) \in \mathbb{C}[z],$$

in the polynomial ring $\mathbb{C}[z]$ in a variable $z \in \mathbb{C}$, and the corresponding equation,

$$\det([q] - zI_2) = 0 \iff z^2 - 2xz + \left(x^2 + y^2 + u^2 + v^2\right) = 0,$$

has its solutions,

$$z = x \pm i\sqrt{y^2 + u^2 + v^2} \text{ in } \mathbb{C}. \tag{3.2}$$

(See [1, 2] for details).

Theorem 3.1. *Let* $q = a + bj \in \mathbb{H}$ *be a quaternion, realized to be* $[q] \in \mathcal{H}_2$. *Then the spectrum* spec $([q])$ *of* $[q]$ *is the subset,*

$$\operatorname{spec}([q]) = \left\{\lambda, \overline{\lambda}\right\} \text{ of } \mathbb{C},$$

where

$$\lambda = x + i\sqrt{y^2 + u^2 + v^2} \text{ in } \mathbb{C}. \tag{3.3}$$

Proof. The spectrum (3.3) is obtained by (3.2). □

Motivated by (3.3), we define the following concept.

Definition 3.2. Let $q = x + yi + uj + vk \in \mathbb{H}$ be a quaternion, realized to be $[q] \in \mathcal{H}_2$. If $u = 0 = v$ in $\mathbb{R}$, equivalently, if $q = x + yi + 0j + 0k$ in $\mathbb{H}$, equivalently, if $q \in \mathbb{C} \subset \mathbb{H}$, then the matrix,

$$\mathbf{q} \overset{\text{denote}}{=} \begin{pmatrix} x + yi & 0 \\ 0 & x - yi \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & \overline{q} \end{pmatrix} = [q] \in \mathcal{H}_2$$

is called the quaternion-spectral form (in short, the q-spectral form) of q. Meanwhile, if either $u \neq 0$, or $v \neq 0$ in $\mathbb{R}$, equivalently, if $q \in (\mathbb{H} \setminus \mathbb{C}) \subset \mathbb{H}$, then the matrix,

$$\mathbf{q} \overset{\text{denote}}{=} \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} \in \mathcal{H}_2,$$

with

$$\lambda = x + i\sqrt{y^2 + u^2 + v^2} \in \mathbb{C},$$

is called the quaternion-spectral form (in short, the q-spectral form) of q.

By definition, the q-spectral form $\mathbf{q} \in \mathcal{H}_2$ of a quaternion $q \in \mathbb{H}$ is the diagonal matrix of $M_2(\mathbb{C})$ whose diagonal entries are the eigenvalues of the realization $[q] \in \mathcal{H}_2$, which is "contained in $\mathcal{H}_2$," by (3.3). Note that if $q = x + yi + 0j + 0k \in \mathbb{C}$ in $\mathbb{H}$, then the realization $[q] \in \mathcal{H}_2$ has its spectrum,

$$\left\{\lambda, \overline{\lambda}\right\},$$

with

$$\lambda = x + i\sqrt{y^2 + 0^2 + 0^2} = x + yi,$$

and the corresponding q-spectral form becomes

$$\mathbf{q} = \begin{pmatrix} x + yi & 0 \\ 0 & x - yi \end{pmatrix} = [q]$$

is well-determined in $\mathcal{H}_2$. However, if $q = x + yi + uj + vk \in \mathbb{H}$, with either $u \neq 0$, or $v \neq 0$ in $\mathbb{R}$, then

$$\mathbf{q} = \begin{pmatrix} x + i\sqrt{y^2 + u^2 + v^2} & 0 \\ 0 & x - i\sqrt{y^2 + u^2 + v^2} \end{pmatrix},$$

by Definition 3.2.

For example, if $q_1 = 2 - i + j - 2k$ and $q_2 = 2 - i + 0j + 0k$ in $\mathbb{H}$, then their q-spectral forms are

$$\mathbf{q}_1 = \begin{pmatrix} 2 + \sqrt{6}i & 0 \\ 0 & 2 - \sqrt{6}i \end{pmatrix}, \quad \mathbf{q}_2 = \begin{pmatrix} 2 - i & 0 \\ 0 & 2 + i \end{pmatrix} = [q_2],$$

respectively, in $\mathcal{H}_2$, where $[q_2] \in \mathcal{H}_2$ is the realization of q_2, by Definition 3.2.

3.2. Similarity on $\mathcal{H}_2$

Throughout this section, we let

$$a = x + yi, b = u + vi \in \mathbb{C}, \text{ with } x, y, u, v \in \mathbb{R}, \text{ and } q = a + bj = x + yi + uj + vk \in \mathbb{H}. \tag{3.4}$$

We showed in Section 3.1 that each quaternion $q \in \mathbb{H}$ of (3.4) is realized to be $[q]$ in $\mathcal{H}_2$, having its q-spectral form,

$$\mathbf{q} = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}, \text{ with } \lambda = x + i\sqrt{y^2 + u^2 + v^2}, \tag{3.5}$$

if either $u \neq 0$, or $v \neq 0$ in $\mathbb{R}$, and

$$\mathbf{q} = [q] = \begin{pmatrix} x + yi & 0 \\ 0 & x - yi \end{pmatrix}, \tag{3.6}$$

in $\mathcal{H}_2$, if $u = 0 = v$ in $\mathbb{R}$.

Suppose $b \in \mathbb{C}^\times$ in (3.4). For $t \in \mathbb{C}^\times$, define a (2×2)-matrix $Q_t(q)$ by

$$Q_t(q) = \begin{pmatrix} t & -\overline{t\left(\frac{a-\lambda}{b}\right)} \\ t\left(\frac{a-\lambda}{b}\right) & \overline{t} \end{pmatrix}, \tag{3.7}$$

in $M_2(\mathbb{C})$, where $q \in \mathbb{H}$ is in the sense of (3.4).

By the assumption that $t, b \in \mathbb{C}^\times$, the nonzero matrix $Q_t(q)$ of (3.7) is well-defined in $M_2(\mathbb{C})$. Note that this matrix $Q_t(q)$ is invertible, since

$$\det\left(Q_t(q)\right) = |t|^2 \left(1 + \left|\frac{a - \lambda}{b}\right|^2\right) \neq 0 \text{ in } \mathbb{C}. \tag{3.8}$$

By the straightforward computations, one can get that

$$[q]Q_t(q) = Q_t(q)\mathbf{q}, \text{ whenever } t, b \in \mathbb{C}^\times, \tag{3.9}$$

in $M_2(\mathbb{C})$ (e.g., see [1, 2] for details). Note here that the (2×2)-matrix $Q_t(q)$ of (3.7) is contained in the noncommutative field $\mathcal{H}_2$ by (2.12)) (which implies the invertibility (3.8) in $M_2(\mathbb{C})$ automatically), whenever $t, b \in \mathbb{C}^\times$.

Theorem 3.3. *Let $q = a + bj \in \mathbb{H}$ be a quaternion (3.4), realized to be $[q]$ in $\mathcal{H}_2$, and let $\mathbf{q} \in \mathcal{H}_2$ be the q-spectral form of q. If $b \neq 0$ in $\mathbb{C}$, then*

$$\mathbf{q} = Q_t(q)^{-1}[q]Q_t(q) \Longleftrightarrow [q] = Q_t(q)\mathbf{q}Q_t(q)^{-1}$$

"in $\mathcal{H}_2$," where

$$Q_t(q) = \begin{pmatrix} t & -\overline{\left(\frac{a-\lambda}{b}\right) t} \\ \left(\frac{a-\lambda}{b}\right) t & \overline{t} \end{pmatrix} \in \mathcal{H}_2, \tag{3.10}$$

for all $t \in \mathbb{C}^\times$. Meanwhile, if $b = 0$ in $\mathbb{C}$, then

$$\mathbf{q} = [w]^{-1}\mathbf{q}[w] = [w]^{-1}[q][w], \text{ in } \mathcal{H}_2, \tag{3.11}$$

where

$$w = w + 0j + 0k \in \mathbb{C}^\times \text{ in } \mathbb{H}. \tag{3.12}$$

Proof. First, suppose that $b = 0$ in $\mathbb{C}$ as in (3.12), and hence, $q = a + 0j$ in $\mathbb{H}$. Then, by (3.5), the quaternion q has its q-spectral form,

$$\mathbf{q} = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} = [q] \text{ in } \mathcal{H}_2,$$

by (3.6). Suppose $w \in \mathbb{C}^{\times}$, and $w = w + 0j + 0k \in \mathbb{H}$, realized to be $[w] \in \mathcal{H}_2$. Then

$$\mathbf{q} = [q] = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} = \begin{pmatrix} \frac{wa}{w} & 0 \\ 0 & \overline{\left(\frac{wa}{w}\right)} \end{pmatrix} = \begin{pmatrix} w & 0 \\ 0 & \overline{w} \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix}\begin{pmatrix} w^{-1} & 0 \\ 0 & \overline{w^{-1}} \end{pmatrix}$$
$$= [w][q][w^{-1}] = [w]\mathbf{q}[w]^{-1},$$

in $\mathcal{H}_2$. Therefore, the relation (3.11) holds true under (3.12) $'$.

Assume now that $b \neq 0$ in $\mathbb{C}$. Then, for any $t \in \mathbb{C}^{\times}$, the corresponding matrices $Q_t(q)$ of (3.7) satisfy

$$Q_t(q)\mathbf{q} = [q]Q_t(q),$$

by (3.5) and (3.9). Thus, by the invertibility (3.8) of $Q_t(q)$,

$$Q_t(q)^{-1}\left(Q_t(q)\mathbf{q}\right) = Q_t(q)^{-1}[q]Q_t(q) \text{ in } \mathcal{H}_2,$$

if and only if

$$\mathbf{q} = Q_t(q)^{-1}[q]Q_t(q) \text{ in } \mathcal{H}_2,$$

implying the relation (3.10). □

The importance of (3.10) and (3.11) is that these formulas hold not only in $M_2(\mathbb{C})$, but also in $\mathcal{H}_2$, i.e., Theorem 5 shows that, for a quaternion $q \in \mathbb{H}$ with its q-spectral form $\mathbf{q} \in \mathcal{H}_2$, there exists at least one nonzero matrix $A \in \mathcal{H}_2$, such that

$$\mathbf{q} = A^{-1}[q]A, \text{ or } [q] = A\mathbf{q}A^{-1},$$

in $\mathcal{H}_2$.

Corollary 3.4. *Let* $q = a + bj \in \mathbb{H}$ *be a quaternion* (3.4) *with* $b \neq 0$ *in* $\mathbb{C}$, *and let*

$$\lambda = x + i\sqrt{y^2 + u^2 + v^2} \in \mathbb{C} \text{ in } \mathbb{H}.$$

Then there exist

$$y_t = t + \left(-t\overline{\left(\frac{a-\lambda}{b}\right)}\right) j \in \mathbb{H},$$

for any $t \in \mathbb{C}^{\times}$, *such that*

$$q = y_t \lambda y_t^{-1} \text{ in } \mathbb{H}. \tag{3.13}$$

Meanwhile, if $b = 0$ *in* $\mathbb{C}$, *then there exists non-zero* $h \in \mathbb{C} \subset \mathbb{H}$, *such that*

$$q = hqh^{-1} \text{ in } \mathbb{H}. \tag{3.14}$$

Proof. The relations (3.13) and (3.14) hold by (3.10) and (3.11), respectively, by (2.13). See [2] for details. □

Corollary 3.4 shows that, for any $q \in \mathbb{H}$ with its realization $[q] \in \mathcal{H}_2$, there exists at least one nonzero $q_0 \in \mathbb{H}$, such that

$$q = q_0 \lambda q_0^{-1} \text{ in } \mathbb{H}, \tag{3.15}$$

where $\operatorname{spec}([q]) = \{\lambda, \overline{\lambda}\}$ in $\mathbb{C}$, by (3.13) and (3.14).

Definition 3.5. Let $q \in \mathbb{H}$ be a quaternion with its realization $[q] \in \mathcal{H}_2$, and let

$$\mathbf{q} = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} = [\lambda] \in \mathcal{H}_2$$

be the q-spectral form. Then the $(1, 1)$-entry $\lambda \in \mathbb{C}$ of $\mathbf{q}$ is called the quaternion-spectral value (in short, q-spectral value) of q.

For example, if $q_1 = 2 - i - j + k \in \mathbb{H}$, then the q-spectral value is

$$2 + i\sqrt{(-1)^2 + (-1)^2 + 1^2} = 2 + \sqrt{3}i;$$

while if $q_2 = 2 - i + 0j + 0k \in \mathbb{H}$, then the q-spectral value is

$$2 - i = q_2 \text{ in } \mathbb{C},$$

by Definitions 3.2 and 3.5.

3.3. *Equivalence on* $\mathbb{H}$

In this section, we let $q \in \mathbb{H}$ be in the sense of (3.4). Define a relation $\mathcal{R}$ on $\mathbb{H}$ by

$$q_1 \mathcal{R} q_2 \overset{\text{def}}{\Longleftrightarrow} \lambda_1 = \lambda_2 \text{ in } \mathbb{C}, \tag{3.16}$$

where λ_l are the q-spectral values of q_l, for $l = 1, 2$. It is not hard to check that this relation $\mathcal{R}$ of (3.16) is an equivalence relation on $\mathbb{H}$ (e.g., [2]).

Definition 3.6. The equivalence relation $\mathcal{R}$ of (3.16) is called the quaternion-spectral equivalence relation (in short, the q-spectral relation) on $\mathbb{H}$. If the relation (3.16) holds, then the two q-spectral equivalent quaternions q_1 and q_2 are said to be q-spectral related in $\mathbb{H}$.

Let $q_l = a_l + b_l j$ be q-spectral related quaternions in $\mathbb{H}$, and let $\lambda \in \mathbb{C}$ be the identical q-spectral value of q_l, for $l = 1, 2$. Then there exists $y_l \in \mathbb{H}$ such that

$$q_l = y_l \lambda y_l^{-1} \text{ in } \mathbb{H}, \ \forall l = 1, 2, \tag{3.17}$$

by (3.15). In particular, if $b_l \neq 0$ in $\mathbb{C}$, then

$$y_l = t + \left(\overline{-t\left(\frac{a_l - \lambda}{b_l}\right)} \right) j \in \mathbb{H}, \ \forall l = 1, 2,$$

by (3.13); meanwhile, if $b_l = 0$ in $\mathbb{C}$, then $y_l \in \mathbb{C}^{\times}$ in $\mathbb{H}$, by (3.14). So, one can have that

$$\begin{aligned} q_2 = y_2 \lambda y_2^{-1} &= y_2 \left(y_1^{-1} y_1\right) \lambda \left(y_1^{-1} y_1\right) y_2^{-1} \\ &= \left(y_2 y_1^{-1}\right) \left(y_1 \lambda y_1^{-1}\right) \left(y_1 y_2^{-1}\right) \text{ by (3.17)} = \left(y_2 y_1^{-1}\right) q_1 \left(y_2 y_1^{-1}\right)^{-1} \text{ by (2.13)}, \end{aligned} \tag{3.18}$$

in $\mathbb{H}$.

Recall that two matrices A_1 and A_2 are *similar* in a matricial ring $M_n(\mathbb{C})$, for $n \in \mathbb{N}$, if there exists an invertible matrix $U \in M_n(\mathbb{C})$, such that

$$A_2 = U A_1 U^{-1}, \text{ in } M_n(\mathbb{C}). \tag{3.19}$$

It is also well-know that if two matrices A_1 and A_2 are similar in the sense of (3.19), then

$$\operatorname{spec}(A_1) = \operatorname{spec}(A_2) \text{ in } \mathbb{C}, \tag{3.20}$$

and vice versa (e.g., [7, 8]).

Definition 3.7. Let $q_l \in \mathbb{H}$ be quaternions realized to be $[q_l] \in \mathcal{H}_2$, for $l = 1, 2$. The realizations $[q_1]$ and $[q_2]$ are said to be similar "in $\mathcal{H}_2$," if there exists a nonzero matrix U "in $\mathcal{H}_2$," such that

$$[q_2] = U[q_1]U^{-1}\text{“ in } \mathcal{H}_2.\text{''} \tag{3.21}$$

Also, two quaternions q_1 and q_2 are said to be similar in $\mathbb{H}$, if their realizations $[q_1]$ and $[q_2]$ are similar in the sense of (3.21).

Note that the similarity on $\mathcal{H}_2$ (and hence, that on $\mathbb{H}$) is an equivalence relation, because the similarity (3.19), or (3.20), is a well-defined equivalence relation on $M_2(\mathbb{C})$.

Theorem 3.8. *Two quaternions q_1 and q_2 are q-spectral related, if and only if they are similar in $\mathbb{H}$, i.e., as equivalence relations,*

$$\textit{the q-spectral relation on } \mathbb{H} = \textit{the similarity on } \mathbb{H}. \tag{3.22}$$

Proof.

($\Rightarrow$) If q_1 and q_2 are q-spectral related in $\mathbb{H}$, then they are similar in $\mathbb{H}$ by (3.18), (3.19), and (3.21).

($\Leftarrow$) Suppose q_1 and q_2 are similar in $\mathbb{H}$, equivalently, their realizations $[q_1]$ and $[q_2]$ are similar in $\mathcal{H}_2$ by (3.22). If λ_l are the q-spectral values of q_l, then $[q_l]$ and $[\lambda_l]$ are similar in the sense of (3.21) in $\mathcal{H}_2$, too, for all $l = 1, 2$. Since the similarity on $\mathcal{H}_2$ is an equivalence relation, the q-spectral forms $[\lambda_1]$ and $[\lambda_2]$ are similar in $\mathcal{H}_2$ by (3.20). Because

$$[\lambda_l] = \begin{pmatrix} \lambda_l & 0 \\ 0 & \overline{\lambda_l} \end{pmatrix} \in \mathcal{H}_2, \text{ for } l = 1, 2,$$

we have

$$[\lambda_1] = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \overline{\lambda_1} \end{pmatrix} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \overline{\lambda_2} \end{pmatrix} = [\lambda_2],$$

by (3.21), and hence,

$$\lambda_1 = \lambda = \lambda_2 \text{ in } \mathbb{C}.$$

Therefore, if q_1 and q_2 are similar in $\mathbb{H}$, then they are q-spectral related in $\mathbb{H}$. □

3.4. *Quaternion-spectral mapping theorem*

Throughout this section, we let

$$q = x + yi + uj + vk \in \mathbb{H}$$

be a quaternion with its q-spectral value,

$$\lambda = x + i\sqrt{y^2 + u^2 + v^2},$$

if either $u \neq 0$ or $v \neq 0$ in $\mathbb{R}$, or

$$\lambda = x + yi,$$

if $u = 0 = v$ in $\mathbb{R}$. Now, let $\mathbb{C}[z]$ be the polynomial ring over a field $\mathbb{C}$ in a variable z,

$$\mathbb{C}[z] = \{f(z) : f \text{ is a polynomial in } z \text{ over } \mathbb{C}\},$$

i.e., $f(z) \in \mathbb{C}[z]$, if and only if

$$\sum_{n=0}^{k} a_n z^n, \text{ for } a_n \in \mathbb{C}, \ \forall n = 1, \dots, k, \tag{3.23}$$

for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. It is well-known that if A is a matrix in $M_n(\mathbb{C})$ for $n \in \mathbb{N}$, and if $f \in \mathbb{C}[z]$ is a polynomial (3.23), then

$$\mathrm{spec}\left(f(A)\right) = f\left(\mathrm{spec}(A)\right) \text{ in } \mathbb{C}, \tag{3.24}$$

by the *spectral mapping theorem*, where the right-hand side of (3.24) means that

$$f\left(\mathrm{spec}(A)\right) = \{f(t) : t \in \mathrm{spec}(A)\},$$

set-theoretically, and in the left-hand side of (3.24), a new matrix $f(A) \in M_n(\mathbb{C})$ is

$$a_k A^k + a_{k-1} A^{k-1} + \cdots + a_2 A^2 + a_1 A + a_0 I_n,$$

where I_n is the identity $(n \times n)$-matrix of $M_n(\mathbb{C})$, whenever $f(z)$ is in the sense of (3.23). By (3.24), one can get that

$$\mathrm{spec}\left(f\left([q]\right)\right) = f\left(\mathrm{spec}\left([q]\right)\right), \ \forall f \in \mathbb{C}[z],$$

"in $M_2(\mathbb{C})$," for all $q \in \mathbb{H}$, realized to be $[q] \in \mathcal{H}_2$ in $M_2(\mathbb{C})$.

Now, define the subset $\mathbb{C}_r[z]$ of $\mathbb{C}[z]$ by

$$\mathbb{C}_r[z] = \bigcup_{N=0}^{\infty} \{\sum_{n=0}^{N} a_n z^n \in \mathbb{C}[z] : a_0, a_1, \ldots, a_N \in \mathbb{R}\}. \tag{3.25}$$

Theorem 3.9. *Let* $q \in \mathbb{H}$ *be a quaternion* (3.4) *with its q-spectral value* $\lambda \in \mathbb{C}$. *If*

$$f(z) = \sum_{n=0}^{N} a_n z^n \in \mathbb{C}_r[z],$$

then

$$f(\lambda) \in \mathbb{C} \textit{ is the q-spectral value of } f(q) \in \mathbb{H}, \tag{3.26}$$

where $\mathbb{C}_r[z]$ *is the subset* (3.25) *of* $\mathbb{C}[z]$, *and* $f(q) = \sum_{n=0}^{N} a_n q^n$ *in* $\mathbb{H}$.

Proof. Let $q \in \mathbb{H}$ be a quaternion (3.4) with its q-spectral value $\lambda \in \mathbb{C}$, and let $h(z) \in \mathbb{C}[z]$. If $[q] \in \mathcal{H}_2$ is the realization of q, then

$$\mathrm{spec}\left(h\left([q]\right)\right) = \left\{h\left(\lambda\right), h\left(\overline{\lambda}\right)\right\}, \text{ in } \mathbb{C},$$

by (3.24). Note however that, for $h(z) \in \mathbb{C}[z]$,

$$h\left(\overline{\lambda}\right) \neq \overline{h(\lambda)} \text{ in } \mathbb{C}, \text{ in general.}$$

For instance, if $h(z) = iz$ in $\mathbb{C}[z]$, then $\overline{h(1+i)} = -1 - i \neq 1 + i = h\left(\overline{1+i}\right)$. However, if $f(z) = \sum_{n=0}^{N} a_n z^n \in \mathbb{C}_r[z]$ with $a_0, a_1, \ldots, a_N \in \mathbb{R}$, then

$$f\left(\overline{\lambda}\right) = \sum_{n=0}^{N} a_n \left(\overline{\lambda}\right)^n = \sum_{n=0}^{N} a_n \left(\overline{\lambda^n}\right) = \sum_{n=0}^{N} \overline{\left(a_n \lambda^n\right)} = \overline{\sum_{n=0}^{N} a_n \lambda^n} = \overline{f(\lambda)},$$

in $\mathbb{C}$. It shows that, if $f(z) \in \mathbb{C}_r[z]$, then

$$\mathrm{spec}\left(f\left([q]\right)\right) = \left\{f(\lambda), f\left(\overline{\lambda}\right)\right\} = \left\{f\left(\lambda\right), \overline{f\left(\lambda\right)}\right\},$$

in $\mathbb{C}$, satisfying that

$$\text{the q-spectral form of } f\left([q]\right) = f\left(\mathbf{q}\right)$$

in $\mathcal{H}_2$, if and only if the q-spectral value of $f\left(q\right)$ is identified with $f\left(\lambda\right)$ in $\mathbb{C} \subset \mathbb{H}$, where $\mathbf{q}$ is the q-spectral form of $[q]$ in $\mathcal{H}_2$. Therefore, the statement (3.26) holds. □

Remark that Theorem 3.9 holds for the polynomials of $\mathbb{C}_r[z]$, not for those of $\mathbb{C}[z]$ (in general). Now, let $\mathbb{R}[x]$ be the polynomial ring over $\mathbb{R}$ in a variable x, i.e.,

$$\mathbb{R}[x] = \bigcup_{N=0}^{\infty} \{\sum_{n=0}^{N} a_n x^n : a_0, a_1, \ldots, a_N \in \mathbb{R}\}. \tag{3.27}$$

Then, the above theorem can be re-stated as follows.

Corollary 3.10. *Let $f(x) \in \mathbb{R}[x]$, where $\mathbb{R}[x]$ is the polynomial ring (3.27). If $q \in \mathbb{H}$ is a quaternion with its q-spectral value $\lambda \in \mathbb{C}$, realized to be $[q] \in \mathcal{H}_2$, then*

$$\operatorname{spec}(f([q])) = \{f(\lambda), \overline{f(\lambda)}\} \text{ in } \mathbb{C}. \tag{3.28}$$

Proof. The set-equality (3.28) holds by (3.26) and (3.27). □

The relation (3.28) is called the *quaternion-spectral mapping theorem*.

Theorem 3.11. *Let q_1 and q_2 be q-spectral related in $\mathbb{H}$, with their q-spectral value $\lambda \in \mathbb{C}$. If $f(x) \in \mathbb{R}[x]$, then $f(q_1)$ and $f(q_2)$ are q-spectral related in $\mathbb{H}$, too, with their identical q-spectral value $f(\lambda) \in \mathbb{C}$. Equivalently, if q_1 and q_2 are similar in $\mathbb{H}$, then $f(q_1)$ and $f(q_2)$ are similar in $\mathbb{H}$, for all $f(x) \in \mathbb{R}[x]$.*

Proof. Let q_1 and q_2 be q-spectral related quaternions in $\mathbb{H}$. Assume that $\lambda \in \mathbb{C}$ is the q-spectral value of both q_1 and q_2. Then, for any $f(x) \in \mathbb{R}[x]$, the quantity $f(\lambda) \in \mathbb{C}$ is the q-spectral value of both $f(q_1)$ and $f(q_2)$ by (3.26) and (3.28). Therefore, two quaternions $f(q_1)$ and $f(q_2)$ are q-spectral related in $\mathbb{H}$. By (3.22), the q-spectral relation and the similarity are equivalent on $\mathbb{H}$. So, if q_1 and q_2 are similar, then $f(q_1)$ and $f(q_2)$ are similar in $\mathbb{H}$, for all $f(x) \in \mathbb{R}[x]$. □

3.5. Quaternion-Spectralization σ

Define now a function $\sigma : \mathbb{H} \to \mathbb{H}$ by

$$\sigma(q) \stackrel{\text{def}}{=} \text{the q-spectral value of } q, \forall q \in \mathbb{H}. \tag{3.29}$$

For instance,

$$\sigma(1 + 0i + 2j - 3k) = 1 + i\sqrt{0^2 + 2^2 + (-3)^2} = 1 + \sqrt{13}i,$$

and

$$\sigma(-2 - i + 0j + 0k) = -2 - i,$$

etc.

Definition 3.12. We call the function σ of (3.29), the quaternion-spectralization (in short, the q-spectralization).

Let's consider the range of the q-spectralization σ.

Proposition 3.13. *If σ is the q-spectralization (3.29), then*

$$\sigma(\mathbb{H}) = \mathbb{C}. \tag{3.30}$$

Proof. Let $q = x + yi + uj + vk \in \mathbb{H}$ be an arbitrary quaternion. If σ is the q-spectralization (3.29), then

$$\sigma(q) = x + i\sqrt{y^2 + u^2 + v^2} \in \mathbb{C},$$

(if either $u \neq 0$, or $v \neq 0$), or

$$\sigma(q) = x + yi \in \mathbb{C},$$

(if $u = 0 = v$) in $\mathbb{H}$. So, one has

$$\sigma(\mathbb{H}) \subseteq \mathbb{C}.$$

Now, let $t + si \in \mathbb{C}$, with $t, s \in \mathbb{R}$. If $s \geqslant 0$ in $\mathbb{R}$, then there exists

$$h = t + yi + uj + vk \in \mathbb{H}, \text{ with } t, y, u, v \in \mathbb{R},$$

such that

$$\sigma(h) = t + i\sqrt{y^2 + u^2 + v^2} \in \mathbb{C},$$

satisfying

$$\sqrt{y^2 + u^2 + v^2} = s \text{ in } \mathbb{R},$$

by (3.29). Meanwhile, if $s < 0$ in $\mathbb{R}$, then there exists

$$h = t + si + 0j + 0k \in \mathbb{H},$$

such that

$$\sigma(h) = t + si \text{ in } \mathbb{C},$$

and hence,

$$\mathbb{C} \subseteq \sigma(\mathbb{H}).$$

Therefore, the set-equality (3.30) holds. □

4. Classification of $\mathbb{H}$

Let σ be the q-spectralization (3.29). For a fixed quaternion $q \in \mathbb{H}$, define the subset,

$$q^o = \{h \in \mathbb{H} : \sigma(h) = \sigma(q)\}, \tag{4.1}$$

in $\mathbb{H}$. Then

$$q^o = (\sigma(q))^o \text{ in } \mathbb{H},$$

set-theoretically. Thus,

$$\sigma(q) \in \mathbb{C} \subset \mathbb{H}$$

becomes a representative of all quaternions of q^o in $\mathbb{H}$, by (3.22) and (3.30), i.e., the subset q^o of (4.1) forms an equivalence class of q in $\mathbb{H}$ for the q-spectral relation, or the similarity. Define now the quotient set $\mathbb{H}^o$ by

$$\mathbb{H}^o \overset{\text{def}}{=} \{q^o : q \in \mathbb{H}\}, \tag{4.2}$$

where q^o are the equivalence classes (4.1).

Theorem 4.1. *The following set-equality holds;*

$$\mathbb{H}^o = \mathbb{C}. \tag{4.3}$$

Proof. Note first that $q^o = (\sigma(q))^o$ in $\mathbb{H}^o$ by (4.1), for all $q \in \mathbb{H}$. Therefore, by (3.30) and (4.2),

$$\mathbb{H}^o = \{\lambda^o : \lambda \in \mathbb{C}, \exists q \in \mathbb{H}, \text{ s.t., } \sigma(q) = \lambda\},$$

and hence,

$$\mathbb{H}^o = \{\lambda^o : \lambda \in \mathbb{C}\}. \tag{4.4}$$

Define a function $\varphi : \mathbb{H}^o \to \mathbb{C}$ by

$$\varphi(\lambda^o) = \lambda, \text{ for all } \lambda^o \in \mathbb{H}^o,$$

where λ^o are in the sense of (4.4) for $\lambda \in \mathbb{C}$. Then, by (3.30), (4.1), (4.3), and (4.4), the function φ is surjective from $\mathbb{H}^o$ onto $\mathbb{C}$.

Also, for $\lambda_1^o, \lambda_2^o \in \mathbb{H}^o$ (in the sense of (4.4)), if

$$\varphi(\lambda_1^o) = \lambda_1 = \lambda_2 = \varphi(\lambda_2^o) \text{ in } \mathbb{C},$$

then $\lambda_1^o = \lambda_2^o$ in $\mathbb{H}^o$, by (3.28), (4.2) and (4.4), i.e., φ is injective, too. Therefore, the function φ is a bijection, implying the set-equality (4.3). □

Theorem 4.1 shows that the quaternions $\mathbb{H}$ is classified by the q-spectral relation (or, the similarity, or the action of the q-spectralization σ). And the corresponding classification is characterized by the set $\mathbb{C}$ in the sense that: every quantity $\lambda \in \mathbb{C}$ represents all quaternions $q \in \mathbb{H}$ satisfying $\sigma(q) = \lambda$.

5. Quaternionic vector spaces

In this section, we consider a vector space $\mathbb{H}^n$ over the quaternions $\mathbb{H}$, for $n \in \mathbb{N}$. Since $\mathbb{H}$ is a a noncommutative field (and hence, a ring), vector spaces over $\mathbb{H}$ are well-determined algebraically.

5.1. *Vector spaces $\mathbb{H}^n$ over $\mathbb{H}$*

For $n \in \mathbb{N}$, define a Cartesian product set $\mathbb{H}^n$ of the n-copies of the quaternions $\mathbb{H}$ by

$$\mathbb{H}^n \overset{\text{def}}{=} \{(q_1, \dots, q_n) : q_1, \dots, q_n \in \mathbb{H}\}, \tag{5.1}$$

consisting of n-tuples of quaternions. Define now a binary operation $(+)$ on $\mathbb{H}^n$ by

$$(q_1, \dots, q_n) + (h_1, \dots, h_n) = (q_1 + h_1, \dots, q_n + h_n), \tag{5.2}$$

for all $(q_1, \dots, q_n), (h_1, \dots, h_n) \in \mathbb{H}^n$, where $(+)$ in the right-hand side of (5.2) is the quaternion-addition of (2.2). Define now the left scalar product, and the right scalar product on $\mathbb{H}^n$ by

$$q(q_1, \dots, q_n) = (qq_1, \dots, qq_n), \quad \text{and} \quad (q_1, \dots, q_n)q = (q_1q, \dots, q_nq), \tag{5.3}$$

respectively, for all $q \in \mathbb{H}$, $(q_1, \dots, q_n) \in \mathbb{H}^n$, where $(\cdot)$ in the right-hand sides of (5.3) is the quaternion-multiplication of (2.2). From below, if there is no confusion, it is said that "$(\cdot)$ is the scalar product (5.3)", which means the scalar products from both left and right as in (5.3).

Definition 5.1. The mathematical triple $(\mathbb{H}^n, +, \cdot)$ is called the n-dimensional quaternion-vector space over $\mathbb{H}$ (in short, the n-dimensional $\mathbb{H}$-vector space), where $\mathbb{H}^n$ is the set (5.1), $(+)$ is the addition (5.2), and $(\cdot)$ is the scalar product (5.3). For convenience, we denote the triple $(\mathbb{H}^n, +, \cdot)$ simply by $\mathbb{H}^n$ from below.

It is clear that the algebraic structure $\mathbb{H}^n$ of Definition 17 is indeed a vector space over a ring $\mathbb{H}$.

Definition 5.2. Let V be a set containing its subset $\mathbb{H}$. Assume that V is equipped with a well-defined addition $(+)$, and a scalar product $(\cdot)$ over $\mathbb{H}$, in the sense that:

$$v_1 + v_2 \in V, \ \forall v_1, v_2 \in V,$$

respectively,

$$qv, \ vq \in V, \ \forall q \in \mathbb{H}, \text{ and } v \in V.$$

Then the triple $(V, +, \cdot)$ is called a vector space over the quaternions $\mathbb{H}$ (in short, a $\mathbb{H}$-vector space). All elements of $(V, +, \cdot)$ are said to be $\mathbb{H}$-vectors.

By Definitions 5.1 and 5.2, all n-dimensional $\mathbb{H}$-vector spaces $\mathbb{H}^n$ are $\mathbb{H}$-vector spaces, for all $n \in \mathbb{N}$.

Definition 5.3. Let V_1 and V_2 be $\mathbb{H}$-vector spaces. A function $T : V_1 \to V_2$ is said to be a linear transformation over the quaternions $\mathbb{H}$ (or, in short, $\mathbb{H}$-linear transformation), if

$$T(v_1 + v_2) = T(v_1) + T(v_2) \quad \text{and} \quad T(qv) = qT(v) \text{ and } T(vq) = T(v)\, q, \tag{5.4}$$

for all $q \in \mathbb{H}$, and $v, v_1, v_2 \in V_1$. A bijective $\mathbb{H}$-linear transformation T of (5.4) is called a $\mathbb{H}$-vector-space-isomorphism (or, in short, a $\mathbb{H}$-isomorphism). In particular, if T is a $\mathbb{H}$-isomorphism, then the $\mathbb{H}$-vector spaces V_1 and V_2 are said to be $\mathbb{H}$-isomorphic.

Let $\mathcal{H}_2$ be the isomorphic noncommutative field (2.12) of the quaternions $\mathbb{H}$ for the representation $(\mathbb{C}^2, \pi)$ by (2.13). Define now a set $\mathcal{H}_2^n$ by the Cartesian product set of the n-copies of $\mathcal{H}_2$,

$$\mathcal{H}_2^n = \{([q_1], \ldots, [q_n]) : [q_1], \ldots, [q_n] \in \mathcal{H}_2\}. \tag{5.5}$$

Define a binary operation $(+)$, and a (left-and-right) scalar-product(s) $(\cdot)$ on the set $\mathcal{H}_2^n$ of (5.5) by

$$\begin{aligned} ([q_1], \ldots, [q_n]) + ([h_1], \ldots, [h_n]) &\overset{\text{def}}{=} ([q_1] + [h_1], \ldots, [q_n] + [n_n]) = ([q_1 + h_1], \ldots, [q_n + h_n]), \\ q\,([q_1], \ldots, [q_n]) &\overset{\text{def}}{=} ([q][q_1], \ldots, [q][q_n]) = ([qq_1], \ldots, [qq_n]), \\ ([q_1], \ldots, [q_n])\, q &= ([q_1 q], \ldots, [q_n q]), \end{aligned} \tag{5.6}$$

for all $q, q_1, \ldots, q_n, h_1, \ldots, h_n \in \mathbb{H}$.

Lemma 5.4. *The triple* $\mathcal{H}_2^n \overset{\text{denote}}{=} (\mathcal{H}_2^n, +, \cdot)$ *of the set* $\mathcal{H}_2^n$ *of* (5.5) *and the operations* $(+)$ *and* $(\cdot)$ *of* (5.6) *is a* $\mathbb{H}$*-vector space.*

Proof. Since the operations of (5.6) are well-defined on the set $\mathcal{H}_2^n$ by (2.13) and (5.5), the triple $\mathcal{H}_2^n$ forms a $\mathbb{H}$-vector space in the sense of Definition 5.2. □

Furthermore, one can verify that two $\mathbb{H}$-vector spaces $\mathbb{H}^n$ and $\mathcal{H}_2^n$ are related as follows by (2.13).

Theorem 5.5. *For any* $n \in \mathbb{N}$*, we have*

$$\mathbb{H}^n \overset{q\text{-iso}}{=} \mathcal{H}_2^n, \tag{5.7}$$

where "$\overset{q\text{-iso}}{=}$" *means "being* $\mathbb{H}$*-isomorphic."*

Proof. Define a function,

$$\Pi : \mathbb{H}^n \to \mathcal{H}_2^n$$

by

$$\Pi((q_1, \ldots, q_n)) \overset{\text{def}}{=} (\pi(q_1), \ldots, \pi(q_n)) = ([q_1], \ldots, [q_n]), \tag{5.8}$$

for all $q_1, \ldots, q_n \in \mathbb{H}$, where $(\mathbb{C}^2, \pi)$ is the representation (2.11) of $\mathbb{H}$. Since the action $\pi : \mathbb{H} \to \mathcal{H}_2$ is a bijection, the function Π is bijective from $\mathbb{H}^n$ onto $\mathcal{H}_2^n$ by (5.8), and,

$$\begin{aligned} \Pi((q_1, \ldots, q_n) + (h_1, \ldots, h_n)) &= \Pi((q_1 + h_1, \ldots, q_n + h_n)) \\ &= ([q_1 + h_1], \ldots, [q_n + h_n]) \text{ by } (5.2) \\ &= ([q_1] + [h_1], \ldots, [q_n] + [h_n]) \text{ by } (5.8) \\ &= ([q_1], \ldots, [q_n]) + ([h_1], \ldots, [h_n]) \text{ by } (2.8) \\ &= \Pi((q_1, \ldots, q_n)) + \Pi((h_1, \ldots, h_n)) \text{ by } (5.6) \end{aligned}$$

for all $(q_1, \ldots, q_n), (h_1, \ldots, h_n) \in \mathbb{H}^n$. Also, one can get that

$$\begin{aligned}\Pi\left(q(q_1, \ldots, q_n)\right) &= \Pi\left((qq_1, \ldots, qq_n)\right) \\ &= \left([qq_1], \ldots, [qq_n]\right) \text{ by (5.3)} \\ &= \left([q][q_1], \ldots, [q][q_n]\right) \text{ by (5.8)} \\ &= q\left([q_1], \ldots, [q_n]\right) \text{ by (2.8)} \\ &= q\Pi\left((q_1, \ldots, q_n)\right) \text{ by (5.6)},\end{aligned}$$

and similarly,

$$\Pi\left((q_1, \ldots, q_n)\, q\right) = \Pi\left((q_1, \ldots, q_n)\right) q,$$

for all $q \in \mathbb{H}$, and $(q_1, \ldots, q_n) \in \mathbb{H}^n$.

Therefore, the isomorphic relation (5.7) holds. □

By (5.7), one can use two $\mathbb{H}$-isomorphic $\mathbb{H}^n$ and $\mathcal{H}_2^n$ alternatively as the n-dimensional $\mathbb{H}$-vector space from below.

Definition 5.6. Let V be a $\mathbb{H}$-vector space, and let W be a R-vector space (i.e., a vector space over a ring R), where R is a subring of $\mathbb{H}$. If a function $T : V \to W$ satisfies

$$T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w}), \quad \text{and} \quad T(r\mathbf{v}) = rT(\mathbf{v}), \quad \text{and} \quad T(\mathbf{v}r) = T(\mathbf{v})\, r,$$

in W, for all $\mathbf{v}, \mathbf{w} \in V$ and $r \in R$, then this function T is called a linear transformation over R (or, in short, a R-linear transformation). If a R-linear transformation T is bijective, then it is said to be an R-isomorphism; and, in such a case, V and W are said to be R-isomorphic.

We are interested in the cases where a subring R of $\mathbb{H}$ in Definition 22 is $\mathbb{C}$, or $\mathbb{R}$.

Theorem 5.7. *The n-dimensional $\mathbb{H}$-vector space $\mathbb{H}^n$ is $\mathbb{R}$-isomorphic to $\mathbb{C}^{2n}$, and it is also $\mathbb{R}$-isomorphic to $\mathbb{R}^4$, i.e.,*

$$\mathbb{H}^n \overset{r\text{-}iso}{=} \mathbb{C}^{2n}, \text{ and } \mathbb{H}^n \overset{r\text{-}iso}{=} \mathbb{R}^{4n}, \tag{5.9}$$

where "$\overset{r\text{-}iso}{=}$ "means "being $\mathbb{R}$-isomorphic."

Proof. Define a function $T : \mathbb{H}^n \to \mathbb{C}^{2n}$ by

$$T(q_1, q_2 \ldots, q_n) = (a_1, b_1, a_2, b_2, \ldots, a_n, b_n),$$

for all $(q_1, \ldots, q_n) \in \mathbb{H}^n$, where $q_l = a_l + b_l j$, with $a_l, b_l \in \mathbb{C}$, for all $l = 1, \ldots, n$. Then it is not difficult to show this function T is a $\mathbb{R}$-isomorphism, i.e., the first $\mathbb{R}$-isomorphic relation of (5.9) holds. Now, define a function $S : \mathbb{H}^n \to \mathbb{R}^{4n}$ by

$$S(q_1, \ldots, q_n) = (x_1, y_1, u_1, v_1, \ldots, x_n, y_n, u_n, v_n),$$

for all $(q_1, \ldots, q_n) \in \mathbb{H}^n$, with

$$q_l = x_l + y_l i + u_l j + v_l k \in \mathbb{H}, \text{ with } x_l, y_l, u_l, v_l \in \mathbb{R},$$

for all $l = 1, \ldots, n$. Then, similarly, it is shown that it is a $\mathbb{R}$-isomorphism, i.e., the second $\mathbb{R}$-isomorphic relation of (5.9) holds, too. □

5.2. The function $\sum^n : \mathbb{H}^n \to \mathcal{H}_2^n$

In Section 5.1, we considered $\mathbb{H}$-vector spaces $\mathbb{H}^k$ and $\mathbb{H}$-isomorphic vector spaces $\mathcal{H}_2^k$, for $k \in \mathbb{N}$. Here, motivated by the classification of Section 4, we study a certain function,

$$\Sigma^n : \mathbb{H}^n \to \mathcal{H}_2^n,$$

implying our spectral analysis of Sections 2 and 3 on the quaternions $\mathbb{H}$.

Let $\sigma : \mathbb{H} \to \mathbb{H}$ be the q-spectralization (3.29), i.e., for all $q = x + yi + uj + vk \in \mathbb{H}$,

$$\sigma (x + yi + uj + vk) = x + i\sqrt{y^2 + u^2 + v^2},$$

if either $u \neq 0$, or $v \neq 0$; and

$$\sigma (x + yi + 0j + 0k) = x + yi,$$

if $u = 0 = v$ in $\mathbb{R}$. Then it induces the q-spectral forms of quaternions, i.e., the following diagram assigns a function;

$$q \overset{\sigma}{\longmapsto} \sigma(q) \overset{\pi}{\longmapsto} \mathbf{q} = \begin{pmatrix} \sigma(q) & 0 \\ 0 & \overline{\sigma(q)} \end{pmatrix},$$

in $\mathcal{H}_2$. Define a function $\sum : \mathbb{H} \to \mathcal{H}_2$ by

$$\sum \overset{\text{def}}{=} \pi \circ \sigma,$$

i.e.,

$$\sum (q) = \begin{pmatrix} \sigma(q) & 0 \\ 0 & \overline{\sigma(q)} \end{pmatrix} = [\sigma(q)], \tag{5.10}$$

for all $q \in \mathbb{H}$. Since $\sigma : \mathbb{H} \to \mathbb{H}$ is a well-defined function whose range is $\mathbb{C}$ by (3.30), and the action $\pi : \mathbb{H} \to \mathcal{H}_2$ is a well-defined bijection, the function $\sum$ of (5.10) is well-defined.

Consider now that: if $q_1 = 2 + i - j + k$ and $q_2 = 2 - i + j + k$ are distinct quaternions, then

$$\sum (q_1) = \begin{pmatrix} 2 + \sqrt{3}i & 0 \\ 0 & 2 - \sqrt{3}i \end{pmatrix} = \sum (q_2),$$

in $\mathcal{H}_2$. It shows that the function $\sum$ is not injective. Moreover, since $\sigma (\mathbb{H}) = \mathbb{C}$ in $\mathbb{H}$, this function $\sum$ is not surjective either. Also, observe that: if

$$q_1 = 1 + 0i + j + 0k,$$

and

$$q_2 = 2 - i + j + 0k,$$

in $\mathbb{H}$, then

$$\sum (q_1) = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}, \quad \text{and} \quad \sum (q_2) = \begin{pmatrix} 2 + \sqrt{2}i & 0 \\ 0 & 2 - \sqrt{2}i \end{pmatrix}, \tag{5.11}$$

respectivelt, in $\mathcal{H}_2$. If q_1 and q_2 are as above, then

$$q_1 + q_2 = 3 - i + 2j + 0k,$$

satisfying

$$\sum (q_1 + q_2) = \begin{pmatrix} 3 + \sqrt{5}i & 0 \\ 0 & 3 - \sqrt{5}i \end{pmatrix}. \tag{5.12}$$

The formulas (5.11) and (5.12) show that

$$\sum (q_1 + q_2) \neq \sum (q_1) + \sum (q_2),$$

in $\mathcal{H}_2$, implying that the function $\sum$ is not linear either.

Lemma 5.8. *The nonlinear function $\sum$ of (5.10) is a function from $\mathbb{H}$ into $\mathcal{H}_2$ "over $\mathbb{R}$," in the sense that: it is a well-defined function from $\mathbb{H}$ into $\mathcal{H}_2$ satisfying*

$$\sum(tq) = t\sum(q) = \sum(q)\,t = \sum(qt), \tag{5.13}$$

for all $q \in \mathbb{H}$, and "$t \in \mathbb{R}$," where $zA \in \mathcal{H}_2$ means

$$(zI_2)(A) \text{ in } \mathcal{H}_2 \subset M_2(\mathbb{C}),$$

for all $z \in \mathbb{C}$ and $A \in M_2(\mathbb{C})$.

Proof. We discussed that $\sum = \pi \circ \sigma : \mathbb{H} \to \mathcal{H}_2$ is a well-defined nonlinear function. But this function is over $\mathbb{R}$ in the sense of (5.13). Indeed, if $t \in \mathbb{R}$ and $q \in \mathbb{H}$, then

$$\sigma(tq) = t\sigma(q) = \sigma(q)\,t = \sigma(qt) \text{ in } \mathbb{H},$$

by the q-spectral mapping theorem (3.28). □

The above lemma shows that the function $\sum$ is a nonlinear function over $\mathbb{R}$ from $\mathbb{H}$ into $\mathcal{H}_2$.

Definition 5.9. Let V_1 and V_2 be $\mathbb{H}$-vector spaces. A function $f : V_1 \to V_2$ is said to be over $\mathbb{R}$, if

$$f(tw) = tf(w), \text{ and } f(wt) = f(w)\,t,$$

in V_2, for all $w \in V_1$ and $t \in \mathbb{R}$. Similarly, f is said to be over $\mathbb{C}$, or, over $\mathbb{H}$, if the above equalities hold for all $t \in \mathbb{C}$, respectively, for all $t \in \mathbb{H}$.

Now, let $\mathbb{H}^n$ be the n-dimensional $\mathbb{H}$-vector space, and let $\mathcal{H}_2^n$ be an isomorphic $\mathbb{H}$-vector space of $\mathbb{H}^n$. Define a function, $\sum^n : \mathbb{H}^n \to \mathcal{H}_2^n$, by

$$\sum^n((q_1, \ldots, q_n)) = \left(\sum(q_1), \ldots, \sum(q_n)\right) \text{ in } \mathcal{H}_2^n, \tag{5.14}$$

for all $(q_1, \ldots, q_n) \in \mathbb{H}^n$. By Lemma 5.8, one can verify that the function $\sum^n$ of (5.14) is over $\mathbb{R}$, too, in the sense of Definition 5.9, i.e.,

$$\Sigma^n(tw) = t\Sigma^n(w) = \Sigma^n(w)\,t = \Sigma^n(wt),$$

in $\mathcal{H}_2^n$, for all $w \in \mathbb{H}^n$, and $t \in \mathbb{R}$.

Proposition 5.10. *The function $\sum^n : \mathbb{H}^n \to \mathcal{H}_2^n$ of (5.14) is a nonlinear function over $\mathbb{R}$.*

Proof. Since a function $\sum : \mathbb{H} \to \mathcal{H}_2$ of (5.10) is a well-defined nonlinear function, the morphism $\sum^n$ is nonlinear from $\mathbb{H}^n$ to $\mathcal{H}_2^n$ by (5.14). Now, let $t \in \mathbb{R}$, and $w = (q_1, \ldots, q_n) \in \mathbb{H}^n$. Then

$$\begin{aligned}\sum^n(tw) &= \sum^n((tq_1, \ldots, tq_n)) \\ &= \left(\sum(tq_1), \ldots, \sum(tq_n)\right) \text{ by (5.3)} \\ &= \left(t\sum(q_1), \ldots, t\sum(q_n)\right) \text{ by (5.14)} \\ &= t\left(\sum(q_1), \ldots, \sum(q_n)\right) \text{ by (5.13)} = t\sum^n((q_1, \ldots, q_n)) = t\sum^n(w),\end{aligned} \tag{5.15}$$

and similar to (5.15),

$$\Sigma_n(wt) = \Sigma_n(w)\,t.$$

So, the function $\sum^n$ is a nonlinear function over $\mathbb{R}$. □

5.3. The n-quaternion-spectralization on $\mathbb{H}^n$

Throughout this section, we fix $n \in \mathbb{N}$. Motivated by the main results of Section 5.2, define here a function, $\sigma^n : \mathbb{H}^n \to \mathbb{H}^n$, by

$$\sigma^n\left((q_1, \ldots, q_n)\right) = (\sigma(q_1),\ \ldots,\ \sigma(q_n)) \text{ in } \mathbb{C}^n \subset \mathbb{H}^n, \tag{5.16}$$

for all $(q_1, \ldots, q_n) \in \mathbb{H}^n$. The well-definedness of this function σ^n of (5.16) is guaranteed by that of $\sum^n$ by (5.14). By (4.3), we have

$$\sigma^n\left(\mathbb{H}^n\right) = \mathbb{C}^n \text{ in } \mathbb{H}^n.$$

Note that, since the q-spectralization σ is nonlinear, the function σ^n on $\mathbb{H}^n$ is not linear either by (5.16). But it is over $\mathbb{R}$ in the sense of Definition 5.9 because $\sum^n$ is.

Proposition 5.11. *The function $\sigma^n : \mathbb{H}^n \to \mathbb{H}^n$ of (5.16) is nonlinear over $\mathbb{R}$.*

Proof. The function σ^n is nonlinear over $\mathbb{R}$, since the function $\sum^n$ of (5.14) is nonlinear over $\mathbb{R}$ by Proposition 5.10. □

For example, if

$$q_1 = 2 + 0i + j - 1k, \quad q_2 = 1 - i + 0j + 0k, \quad q_3 = 4 + i - j - 2k,$$

in $\mathbb{H}$, inducing

$$(q_1, q_2, q_3) \in \mathbb{H}^3,$$

then

$$\sigma^3\left((q_1,\ q_2,\ q_3)\right) = \left(2 + \sqrt{2}i,\ 1 - i,\ 4 + \sqrt{6}i\right),$$

in $\mathbb{C}^3 \subset \mathbb{H}^3$.

Definition 5.12. For any $n \in \mathbb{N}$, the function $\sigma^n : \mathbb{H}^n \to \mathbb{H}^n$ of (5.16) is called the n-quaternion-spectralization (in short, the q-spectralization) on $\mathbb{H}^n$. Our q-spectralization σ of (3.29) is redefined by this general definition, i.e., $\sigma = \sigma^1$ on $\mathbb{H} = \mathbb{H}^1$.

6. Noncommutative unital rings $\sum_n(\mathbb{H})$ over $\mathbb{R}$

In Section 5, we introduced two types of functions acting on the $\mathbb{H}$-vector space $\mathbb{H}^n$, for $n \in \mathbb{N}$. The first one is the function,

$$\Sigma^n : \mathbb{H}^n \to \mathcal{H}_2^n \overset{\text{q-iso}}{=} \mathbb{H}^n,$$

of (5.14), and the second one is the q-spectralization,

$$\sigma^n : \mathbb{H}^n \to \mathbb{H}^n.$$

These functions are nonlinear, but they are over $\mathbb{R}$. Also, by (4.3), one has

$$\sigma^n\left(\mathbb{H}^n\right) = \mathbb{C}^n,$$

set-theoretically, i.e., all $\mathbb{H}$-vectors $(q_1, \ldots, q_n)$ of $\mathbb{H}^n$ are classified by the $\mathbb{C}$-vectors,

$$\left(\sigma\left(q_1\right), \ldots, \sigma\left(q_n\right)\right) \in \mathbb{C}^n.$$

Theorem 6.1. *The q-spectralization σ^n is idempotent in the sense that:*

$$\sigma^n \circ \sigma^n = \sigma^n \text{ on } \mathbb{H}^n. \tag{6.1}$$

Proof. For an $\mathbb{H}$-vector $w = (q_1, \ldots, q_n)$,

$$(\sigma^n \circ \sigma^n)(w) = \sigma^n (\sigma^n ((q_1, \ldots, q_n))) = \sigma^n ((\sigma(q_1), \ldots, \sigma(q_n))) = (\sigma(\sigma(q_1), \ldots, \sigma(\sigma(q_n)))). \tag{6.2}$$

Since the q-spectral values $\sigma(q_l)$ of q_l are $\mathbb{C}$-quantities in $\mathbb{H}$, for all $l = 1, \ldots, n$,

$$\sigma \circ \sigma(q_l) = \sigma(\sigma(q_l)) = \sigma(q_l) \text{ in } \mathbb{C} \subset \mathbb{H},$$

for all $l = 1, \ldots, n$. Thus the equality (6.2) is identified with

$$\sigma^n(\sigma^n(w)) = (\sigma(q_1), \ldots, \sigma(q_n)) = \sigma^n(w),$$

in $\mathbb{C}^n \subset \mathbb{H}^n$, i.e., the idempotence (6.1) holds on $\mathbb{H}^n$ by σ^n. □

The above theorem shows our q-spectralizations σ^k are nonlinear, idempotent on $\mathbb{H}^k$ over $\mathbb{R}$, for all $k \in \mathbb{N}$, by (6.1).

Now, let $M_n(\mathbb{C})$ be the matricial ring acting on $\mathbb{C}^n$. Define now a set $\sum_n(\mathbb{H})$ by

$$\sum_n(\mathbb{H}) \overset{\text{def}}{=} \{\alpha \circ \sigma^n : \alpha \in M_n(\mathbb{C})\}. \tag{6.3}$$

By (6.3), all elements $\alpha \circ \sigma^n$ are well-defined nonlinear functions on $\mathbb{H}^n$ over $\mathbb{R}$. Remark that, if

$$w_1 = (q_1, \ldots, q_n), w_2 = (h_1, \ldots, h_n) \in \mathbb{H}^n, \text{ and } t \in \mathbb{R},$$

then

$$\begin{aligned} \alpha \circ \sigma^n(w_1 + w_2) &= \alpha(\sigma^n((q_1 + h_1, \ldots, q_n + h_n))) \\ &= \alpha((\sigma(q_1 + h_1), \ldots, \sigma(q_n + h_n))) \\ &\neq \alpha(\sigma^n((q_1, \ldots, q_n))) + \alpha(\sigma^n((h_1, \ldots, h_n))) = \alpha \circ \sigma^n(w_1) + \alpha \circ \sigma^n(w_2), \end{aligned} \tag{6.4}$$

in general, because of the nonlinearity of σ^n. However,

$$\begin{aligned} \alpha \circ \sigma^n(tw_1) = \alpha(\sigma^n((tq_1, \ldots, tq_n))) &= \alpha((t\sigma(q_1), \ldots, t\sigma(q_n))) \\ &= \alpha(t\sigma^n((q_1, \ldots, q_n))) \\ &= \alpha \circ (t\sigma^n)(w_1) = (t\alpha) \circ \sigma^n(w_1) = t(\alpha \circ \sigma^n)((q_1, \ldots, q_n)), \end{aligned} \tag{6.5}$$

for all $t \in \mathbb{R}$, since $\alpha \in M_n(\mathbb{C})$ is (linear, and hence, it is) over $\mathbb{R}$, too.

Proposition 6.2. *Every element $\alpha \circ \sigma^n \in \Sigma_n(\mathbb{H})$ is a nonlinear function over $\mathbb{R}$.*

Proof. Each element $\alpha \circ \sigma^n \in \sum_n(\mathbb{H})$ is not linear by (6.4), but it is over $\mathbb{R}$ by (6.5). □

Notation: From below, we denote $\alpha \circ \sigma^n \in \sum_n(\mathbb{H})$ by $\alpha^{(n)}$.

For instance, if

$$q_1 = 2 + 0i - j + k, \quad q_2 = 1 - i, \quad q_3 = -1,$$

inducing

$$w = (q_1, q_2, q_3) \in \mathbb{H}^3,$$

and if

$$\alpha = \begin{pmatrix} 1 & 0 & i \\ 1 & -i & 0 \\ 0 & 1 & 0 \end{pmatrix} \in M_3(\mathbb{C}),$$

then, for the corresponding $\alpha^{(3)} \in \sum_3(\mathbb{H})$, one has

$$\alpha^{(3)}(w) = \alpha\left(2+\sqrt{2}i,\ 1-i,\ -1\right) = \left(2+(\sqrt{2}-1)i,\ 1+(\sqrt{2}-1)i,\ 1+i\right),$$

in $\mathbb{H}^3$. On the set $\sum_n(\mathbb{H})$ of (6.3), define the operations $(+)$ and $(\cdot)$ by

$$\begin{aligned} \alpha_1^{(n)} + \alpha_2^{(n)} &\stackrel{\text{def}}{=} (\alpha_1+\alpha_2)\circ\sigma^{(n)} = (\alpha_1+\alpha_2)^{(n)}, \\ \alpha_1^{(n)}\alpha_2^{(n)} &\stackrel{\text{def}}{=} (\alpha_1\alpha_2)\circ\sigma^{(n)} = (\alpha_1\alpha_2)^{(n)}, \end{aligned} \tag{6.6}$$

for all $\alpha_1^{(n)}, \alpha_2^{(n)} \in \sum_n(\mathbb{H})$, respectively, where the addition $(+)$ and the multiplication $(\cdot)$ on the far right-hand sides of (6.6) are the usual matricial addition, and matricial multiplication on $M_n(\mathbb{C})$, respectively. These operations of (6.6) are well-defined on $\sum_n(\mathbb{H})$ by (4.3) and (6.1). So, the triple,

$$\sum_n(\mathbb{H}) \stackrel{\text{denote}}{=} \left(\sum_n(\mathbb{H}),\ +,\ \cdot\right), \tag{6.7}$$

forms an algebraic structure, where $(+)$ and $(\cdot)$ of (6.7) are in the sense of (6.6). Observe that if

$$w = (q_1, \ldots, q_n) \in \mathbb{H}^n,$$

then

$$\left(\alpha_1^{(n)}\alpha_2^{(n)}\right)(w) = \alpha^{(1)}\left(\alpha_2\left(\sigma(q_1), \ldots, \sigma(q_n)\right)\right) = \alpha_1\left(\sigma^n\left(z_1, \ldots, z_n\right)\right),$$

where

$$(z_1, \ldots, z_n) = \alpha_2\left(\sigma(q_1), \ldots, \sigma(q_n)\right) \in \mathbb{C}^n \ (\text{in } \mathbb{H}^n) = \alpha_1\left(z_1, \ldots, z_n\right) = (\alpha_1\alpha_2)\left(\sigma(q_1), \ldots, \sigma(q_n)\right),$$

by (6.1)

$$= (\alpha_1\alpha_2\circ\sigma^n)(w) = (\alpha_1\alpha_2)^{(n)}(w),$$

in $\mathbb{H}^n$. Since $w \in \mathbb{H}^n$ is arbitrary,

$$\alpha_1^{(n)}\alpha_2^{(n)} = (\alpha_1\alpha_2)^{(n)} \ \text{ in } \Sigma_n(\mathbb{H}).$$

It shows that the multiplication $(\cdot)$ of (6.6) is indeed well-defined on $\sum_n(\mathbb{H})$.

Moreover, one can define a $\mathbb{R}$-scalar product on $\sum_n(\mathbb{H})$ by

$$t\cdot\alpha^{(n)} = t\cdot\left(\alpha\circ\sigma^{(n)}\right) \stackrel{\text{def}}{=} t\alpha\circ\sigma^n = \alpha\circ t\sigma^n = \alpha^{(n)}\cdot t, \tag{6.8}$$

for all $t \in \mathbb{R}$, and $\alpha^{(n)} \in \sum_n(\mathbb{H})$. The last two equalities of (6.8) say that the definition (6.8) covers both left and right scalar products. Also, they demonstrates that this $\mathbb{R}$-scalar product is well-defined by (6.5).

Theorem 6.3. *The triple $\sum_n(\mathbb{H})$ of (6.7) is a noncommutative unital ring over $\mathbb{R}$, in the sense that: (i) it is a noncommutative ring with its unity, and (ii) there is a well-defined $\mathbb{R}$-scalar product on $\sum_n(\mathbb{H})$.*

Proof. Let $\sum_n(\mathbb{H})$ be the algebraic triple (6.7). For the addition $(+)$ of (6.6),

$$\begin{aligned} \left(\alpha_1^{(n)} + \alpha_2^{(n)}\right) + \alpha_3^{(n)} = (\alpha_1+\alpha_2)^{(n)} + \alpha_3^{(n)} &= ((\alpha_1+\alpha_2)+\alpha_3)^{(n)} \\ &= (\alpha_1+(\alpha_2+\alpha_3))^{(n)} \\ &= \alpha_1^{(n)} + (\alpha_2+\alpha_3)^{(n)} = \alpha_1^{(n)} + \left(\alpha_2^{(n)} + \alpha_3^{(n)}\right), \end{aligned}$$

for $\alpha_l^{(n)} \in \sum_n(\mathbb{H})$, for all $l = 1, 2, 3$. Also, there exists the zero matrix O_n of $M_n(\mathbb{C})$ inducing

$$0_n \overset{\text{denote}}{=} O_n \circ \sigma^n \in \Sigma_n(\mathbb{H}),$$

such that

$$\alpha^{(n)} + 0^{(n)} = (\alpha + O_n)^{(n)} = \alpha^{(n)} = (O_n + \alpha)^{(n)} = 0^{(n)} + \alpha^{(n)},$$

for all $\alpha^{(n)} \in \sum_n(\mathbb{H})$. For the $(+)$-identity $0^{(n)}$ and arbitrary $\alpha^{(n)} \in \sum_n(\mathbb{H})$, there exists a unique $(-\alpha)^{(n)}$ $\in \sum_n(\mathbb{H})$, such that

$$\alpha^{(n)} + (-\alpha)^{(n)} = (\alpha - \alpha)^{(n)} = 0^{(n)} = (-\alpha + \alpha)^{(n)} = (-\alpha)^{(n)} + \alpha^{(n)},$$

in $\sum_n(\mathbb{H})$. Also,

$$\alpha_1^{(n)} + \alpha_2^{(n)} = (\alpha_1 + \alpha_2)^{(n)} = (\alpha_2 + \alpha_1)^{(n)} = \alpha_2^{(n)} + \alpha_1^{(n)},$$

for all $\alpha_1^{(n)}, \alpha_2^{(n)} \in \sum_n(\mathbb{H})$. So, the algebraic pair $(\sum_n(\mathbb{H}), +)$ is an abelian. For the multiplication $(\cdot)$, one has that

$$\left(\alpha_1^{(n)}\alpha_2^{(n)}\right)\alpha_3^{(n)} = ((\alpha_1\alpha_2)\alpha_3)^{(n)} = (\alpha_1(\alpha_2\alpha_3))^{(n)} = \alpha_1^{(n)}\left(\alpha_2^{(n)}\alpha_3^{(n)}\right),$$

for $\alpha_l^{(n)} \in \sum_n(\mathbb{H})^{\times} = \sum_n(\mathbb{H}) \setminus \left\{0^{(n)}\right\}$, for all $l = 1, 2, 3$.

Since the matricial ring $M_n(\mathbb{C})$ is noncommutative, the above associative multiplication is noncommutative. So, the algebraic pair $\left(\sum_n(\mathbb{H})^{\times}, \cdot\right)$ forms a noncommutative semigroup. Moreover, it has its multiplication-identity,

$$1^{(n)} \overset{\text{denote}}{=} I_n \circ \sigma^n \in \Sigma_n(\mathbb{H}),$$

where I_n is the identity matrix of $M_n(\mathbb{C})$, satisfying

$$\alpha^{(n)} \cdot 1^{(n)} = (\alpha \circ I_n)^{(n)} = \alpha^{(n)} = (I_n \circ \alpha)^{(n)} = 1^{(n)} \cdot \alpha^{(n)},$$

for all $\alpha^{(n)} \in \sum_n(\mathbb{H})^{\times}$.

It is not hard to check the left-and-right distributiveness of the operations $(+)$ and $(\cdot)$ of (6.6) on $\sum_n(\mathbb{H})$, by those of matricial addition and multiplication on $M_2(\mathbb{C})$. So, the algebraic triple $\sum_n(\mathbb{H})$ of (6.7) is a unital ring with its unity $1^{(n)}$.

Finally, the $\mathbb{R}$-scalar product (6.8) is well-defined on $\sum_n(\mathbb{H})$, because all matrices of $M_2(\mathbb{C})$ and our q-spectralization σ^n are over $\mathbb{R}$. □

Let $\alpha^{(n)} = \alpha \circ \sigma^n \in \sum_n(\mathbb{H})$, with $\alpha \in M_n(\mathbb{C})$. If α is invertible in $M_n(\mathbb{C})$ (as a matrix), then $\alpha^{(n)}$ is invertible in $\sum_n(\mathbb{H})$ (as a ring-element) in the sense that: there exists a unique element $\beta^{(n)} = \beta \circ \sigma^n \in \sum_n(\mathbb{H})$, such that

$$\alpha^{(n)}\beta^{(n)} = 1^{(n)} = \beta^{(n)}\alpha^{(n)},$$

in $\sum_n(\mathbb{H})$, where $1^{(n)} = I_n \circ \sigma^n$ is the unity of $\sum_n(\mathbb{H})$, i.e., the invertibility of α in $M_n(\mathbb{C})$ implies the invertibility of $\alpha^{(n)}$ in $\sum_n(\mathbb{H})$;

$$\left(\alpha^{(n)}\right)^{-1} = \left(\alpha^{-1}\right)^{(n)} \text{ in } \Sigma_n(\mathbb{H}), \tag{6.9}$$

if α^{-1} exists in $M_n(\mathbb{C})$.

Remark 6.4. Remark that "$\alpha^{(n)}$ is invertible in $\sum_n(\mathbb{H})$" does not mean $\alpha^{(n)}$ is invertible "on $\mathbb{H}^n$," as a function. Indeed, by definition, $\alpha^{(n)}$ is neither injective nor surjective on $\mathbb{H}^n$, implying that it cannot be invertible on $\mathbb{H}^n$ "as a function," i.e., the invertibility (6.9) on $\sum_n(\mathbb{H})$ is the ring-invertibility on the noncommutative unital ring $\sum_n(\mathbb{H})$.

By the invertibility (6.9) on $\sum_n (\mathbb{H})$, one can define the subring $S_n(\mathbb{H})$ by

$$S_n(\mathbb{H}) \overset{\text{def}}{=} \{\alpha^{(n)} \in \sum_n (\mathbb{H}) : \alpha^{(n)} \text{ is invertible}\}. \tag{6.10}$$

It is not difficult to check that the sub-structure $(S_n(\mathbb{H}), \cdot)$ of this subring $S_n(\mathbb{H})$ of (6.10) forms a non-abelian group, where $(\cdot)$ is the multiplication (6.6).

Definition 6.5. We call the noncommutative unital ring $\sum_n(\mathbb{H})$ of (6.7), the quaternion-spectral matricial ring acting on $\mathbb{H}^n$ (in short, the q-spectral ring). And all elements of $\sum_n (\mathbb{H})$ are called the quaternion-matricial functions (in short, q-spectral functions) on $\mathbb{H}^n$. The subring $S_n(\mathbb{H})$ of (6.10) is called the invertible quaternion-spectral matricial ring (in short, the invertible q-spectral ring). In particular, the algebraic pair $(S_n(\mathbb{H}), \cdot)$ is said to be the invertible q(uaternion)-spectral (matricial) group.

7. Spectral theoretic properties on $\sum_n (\mathbb{H})$

In this section, we fix $n \in \mathbb{N}$, and the corresponding q-spectral ring $\sum_n (\mathbb{H})$. Let

$$\alpha^{(n)} = \alpha \circ \sigma^{(n)} \in \Sigma_n (\mathbb{H}), \text{ with } \alpha \in M_n (\mathbb{C}),$$

be a q-spectral function on $\mathbb{H}^n$. Note-and-recall that the q-spectral ring $\sum_n (\mathbb{H})$ is a noncommutative unital ring over $\mathbb{R}$ with the $\mathbb{R}$-scalar product (6.8);

$$\left(t, \alpha^{(n)}\right) \in \mathbb{R} \times \Sigma_n (\mathbb{H}) \to t\alpha^{(n)} \in \Sigma_n (\mathbb{H}),$$

where

$$t\alpha^{(n)} = (t\alpha)^{(n)} = t\alpha \circ \sigma^n \text{ in } \Sigma_n (\mathbb{H}). \tag{7.1}$$

By (7.1), for any $t \in \mathbb{R}$ and $\alpha^{(n)} \in \sum_n (\mathbb{H})$,

$$\alpha^{(n)} - t \cdot 1^{(n)} = \alpha \circ \sigma^n - t (I_n \circ \sigma^n),$$

where $1^{(n)} = I_n \circ \sigma^n \in \sum_n (\mathbb{H})$ is the unity of $\sum_n (\mathbb{H})$,

$$= \alpha \circ \sigma^n + ((-t)I_n \circ \sigma^n)$$

by (7.1),

$$= (\alpha - tI_n) \circ \sigma^n = (\alpha - tI_n)^{(n)} \tag{7.2}$$

in $\sum_n (\mathbb{H})$. By (7.2), one can get that

$$\alpha^{(n)} - t \cdot 1^{(n)} = 0^{(n)} \text{ in } \Sigma_n (\mathbb{H}),$$

where $0^{(n)} = O_n \circ \sigma^n \in \Sigma_n (\mathbb{H})$ is the zero element, if and only if

$$\alpha - tI_n = O_n \text{ in } M_n (\mathbb{C}), \tag{7.3}$$

because the q-spectralization σ^n is a nonzero function on $\mathbb{H}^n$.

Proposition 7.1. *Let $\alpha^{(n)} \in \Sigma_n (\mathbb{H})$, and let $t \in \mathbb{R}$. Then*

$$\alpha^{(n)} - t \cdot 1^{(n)} = 0^{(n)} \text{ in } \Sigma_n (\mathbb{H}),$$

if and only if

$$\alpha - tI_n = O_n \Longleftrightarrow \alpha = tI_n \text{ in } M_n (\mathbb{C}). \tag{7.4}$$

Proof. The relation (7.4) holds by (7.2) and (7.3). □

Now, consider the equality,

$$\alpha - tI_n = O_n \text{ in } M_n(\mathbb{C}), \tag{7.5}$$

in the relation (7.4), both "globally" and "locally" as a function on $\mathbb{C}^n$. Note that

$$t = t + 0i \in \mathbb{R} \subset \mathbb{C} \text{ in (7.5)}.$$

Of course, the equality (7.5) holds globally, if and only if the relation (7.4) holds by Proposition 7.1. Meanwhile, the equality (7.5) holds "locally" on $\mathbb{C}^n$ in the sense that: there exists a subspace $\mathcal{E}$ of $\mathbb{C}^n = \sigma(\mathbb{H}^n)$ such that

$$\alpha - tI_n = O_n \text{ "on } \mathcal{E}\text{," in } \mathbb{C}^n,$$

if and only if

$$\alpha(\mathbf{w}) = t\mathbf{w}, \text{ for all } \mathbf{w} \in \mathcal{E},$$

if and only if $t \in \mathbb{R}$ is contained in the spectrum of α,

$$\mathrm{spec}(\alpha) = \{\lambda \in \mathbb{C} : \exists \mathbf{w} \in \mathbb{C}^n, \text{ s.t., ff}(\mathbf{w}) = \check{\ }\mathbf{w}\},$$

or

$$\mathrm{spec}(\alpha) = \{\lambda \in \mathbb{C} : \alpha - \lambda I_n \text{ is not invertible on } \mathbb{C}^n\},$$

in $\mathbb{C}$, if and only if t is an eigenvalue of α.

Indeed, $t \in \mathrm{spec}(\alpha)$, if and only if the nonempty eigenspace $\mathcal{E}_t$ of t is well-determined in $\mathbb{C}^n$ as a subspace, if and only if

$$\alpha(\mathbf{w}) = t\mathbf{w} \Longleftrightarrow (\alpha - tI_n)(\mathbf{w}) = 0_n, \ \forall \mathbf{w} \in \mathcal{E}_t,$$

if and only if

$$\alpha - tI_n = O_n \text{ "on } \mathcal{E}_t\text{,"} \tag{7.6}$$

"locally" in $\mathbb{C}^n$. So, different from the global relation (7.4), we have the following local result.

Theorem 7.2. *Let $\alpha^{(n)} \in \sum_n(\mathbb{H})$ be a q-spectral function and $t \in \mathbb{R}$. Then there exists non-zero $\mathbb{H}$-vector $\mathbf{w} \in \mathbb{H}^n$ such that*

$$\left(\alpha^{(n)} - t \cdot 1^{(n)}\right)(\mathbf{w}) = 0_n \text{ in } \mathbb{H}^n,$$

if and only if

$$t \in \mathrm{spec}(\alpha), \text{ and } \sigma^n(\mathbf{w}) \in \mathcal{E}_t, \tag{7.7}$$

where $\mathcal{E}_t$ is the eigenspace of an eigenvalue t.

Proof.
($\Leftarrow$) By (7.4), if $\alpha = tI_n$ in $M_n(\mathbb{C})$, then the identity,

$$\alpha^{(n)} - t \cdot 1^{(n)} = (\alpha - tI_n)^{(n)} = 0^{(n)},$$

holds in the q-spectral ring $\sum_n(\mathbb{H})$ (globally), i.e., for all $\mathbf{w} \in \mathbb{H}^n$, the equality in (7.7) holds. Remark that if $\alpha = tI_n$ in $M_n(\mathbb{C})$, then

$$\mathrm{spec}(\alpha) = \{t\}, \text{ and } \mathcal{E}_t = \mathbb{C}^n.$$

It shows that if the relation (7.4) holds, then

$$t \in \mathrm{spec}(\alpha), \text{ and } \mathcal{E}_t = \mathbb{C}^n = \sigma^n(\mathbb{H}^n),$$

and hence, the equality of (7.7) holds as a special case. Generally, if $t \in \text{spec}(\alpha)$ and $\sigma^n(\mathbf{w}) \in \mathcal{E}_t$, then the equality of (7.7) holds true by (7.6). Therefore, if there exists $\mathbf{w} \in \mathbb{H}^n$, such that

$$t \in \text{spec}(\alpha), \text{ and } \sigma^n(\mathbf{w}) \in \mathcal{E}_t,$$

then the equality of (7.7) holds.

($\Rightarrow$) Now, assume that either

$$t \notin \text{spec}(\alpha), \text{ or } \sigma^n(\mathbf{w}) \notin \mathcal{E}_t.$$

First, assume that $t \notin \text{spec}(\alpha)$. Then there does not exist $w \in \mathbb{C}^n$, such that

$$\alpha(w) = tw \Longleftrightarrow (\alpha - tI_n)(w) = 0_n,$$

implying that

$$(\alpha - tI_n)^{(n)}(\mathbf{w}) \neq 0_n, \text{ for all } \mathbf{w} \in \mathbb{H}^n,$$

by (7.2), since $\sigma^n(\mathbf{w}) \in \mathbb{C}^n$ in $\mathbb{H}^n$, for all $w \in \mathbb{H}^n$, i.e., if $t \notin \text{spec}(\alpha)$, then the equality of (7.7) does not hold. Suppose now that $w \overset{\text{denote}}{=} \sigma^n(\mathbf{w}) \notin \mathcal{E}_t$, for $\mathbf{w} \in \mathbb{H}^n$. Then

$$\alpha(w) \neq tw \Longleftrightarrow (\alpha - tI_n)(w) \neq 0_n,$$

in $\mathbb{C}^n$, if and only if

$$(\alpha - tI_n)^{(n)}(\mathbf{w}) = \left(\alpha^{(n)} - t \cdot 1^{(n)}\right)(\mathbf{w}) \neq 0_n,$$

in $\mathbb{C}^n$, because $\mathbf{w} \in \mathbb{H}^n$ is assumed to be non-zero, i.e., if $\sigma^n(w) \notin \mathcal{E}_t$, then the equality of (7.7) does not hold. Thus, if the equality of (7.7) holds for non-zero $\mathbf{w} \in \mathbb{H}^n$, then

$$t \in \text{spec}(\alpha), \text{ and } \sigma^n(\mathbf{w}) \in \mathcal{E}_t.$$

□

The above theorem characterizes the existence of nonzero $\mathbb{H}$-vectors $\mathbf{w} \in \mathbb{H}^n$ satisfying

$$\left(\alpha^{(n)} - t \cdot 1^{(n)}\right)(\mathbf{w}) = 0_n \text{ on } \mathbb{H}^n,$$

by the spectral property of α on $\mathbb{C}^n$, by (7.7).

8. Fixed-point theorems in $\sum_n(\mathbb{H})$

In this section, motivated by the main results of Section 7, we study fundamental fixed-point theorems on the n-dimensional vector space $\mathbb{H}^n$ over the quaternions $\mathbb{H}$ induced by q-spectral functions of the q-spectral ring $\sum_n(\mathbb{H})$, and characterize them in terms of the usual spectral theory on $M_n(\mathbb{C})$. Throughout this section, we fix $n \in \mathbb{N}$, and the corresponding q-spectral ring $\sum_n(\mathbb{H})$. Recall that, in a usual (Hilbert-space-operator) spectral theory, a matrix $\alpha \in M_n(\mathbb{C})$ is self-adjoint, if

$$\alpha^* = \alpha \text{ in } M_n(\mathbb{C}),$$

where α^* is the adjoint (or, the conjugate transpose) of α; and it is said to be a projection, if it is both self-adjoint and idempotent, i.e.,

$$\alpha^* = \alpha = \alpha^2 \text{ in } M_n(\mathbb{C});$$

and a matrix α is normal, if

$$\alpha^*\alpha = \alpha\alpha^* \text{ in } M_n(\mathbb{C});$$

and α is an isometry, if

$$\alpha^*\alpha = I_n \text{ in } M_n(\mathbb{C}),$$

etc. And the corresponding spectral properties are well-characterized (e.g., [7, 8]).

Here, we focus on the cases where the matrix parts $\alpha \in M_n(\mathbb{C})$ of q-spectral functions $\alpha^{(n)} = \alpha \circ \sigma^n \in \sum_n(\mathbb{H})$ are self-adjoint, or projections on $\mathbb{C}^n$.

Definition 8.1. A q-spectral function $\alpha^{(n)} = \alpha \circ \sigma^n \in \sum_n (\mathbb{H})$ is said to be self-adjoint (or, a projection), if the matrix part $\alpha \in M_n(\mathbb{C})$ is self-adjoint (resp., a projection) on $\mathbb{C}^n$.

Similar to Definition 8.1, one can define normal elements, or isometries on the q-spectral ring $\sum_n (\mathbb{H})$ by the (usual) spectral properties of the matrix parts of q-spectral functions of $\sum_n (\mathbb{H})$.

For instance, the unity $1^{(n)} = I_n \circ \sigma^n$, and the zero element $0^{(n)} = O_n \circ \sigma^n$ of $\sum_n (\mathbb{H})$ are not only self-adjoint, but also projections in $\sum_n (\mathbb{H})$, because the identity matrix I_n and the zero matrix O_n of the matricial ring $M_n(\mathbb{C})$ are both self-adjoint and projections on $\mathbb{C}^n$.

Proposition 8.2. *Let $\alpha_l^{(n)} = \alpha_l \circ \sigma^n \in \sum_n (\mathbb{H})$, for $l = 1, 2$.*

(1) *If $\alpha_l \in M_n(\mathbb{C})$ are self-adjoint and $t_l \in \mathbb{R}$, for $l = 1, 2$, then $t_1\alpha_1^{(n)} + t_2\alpha_2^{(n)}$ is self-adjoint in $\sum_n (\mathbb{H})$.*

(2) *If $\alpha_l \in M_n(\mathbb{C})$ are self-adjoint for $l = 1, 2$, satisfying*

$$\alpha_1\alpha_2 = \alpha_2\alpha_1 \text{ in } M_n(\mathbb{C}),$$

and if $t \in \mathbb{R}$, then $t\alpha_1^{(n)}\alpha_2^{(n)}$ is self-adjoint in $\sum_n (\mathbb{H})$.

(3) *If $\alpha_l \in M_n(\mathbb{C})$ are projections for all $l = 1, 2$, and if*

$$\alpha_1\alpha_2 = O_n = \alpha_2\alpha_1 \text{ in } M_n(\mathbb{C}),$$

then $\alpha_1^{(n)} + \alpha_2^{(n)}$ is a projection in $\sum_n (\mathbb{H})$.

(4) *If $\alpha_l \in M_n(\mathbb{C})$ are projections for all $l = 1, 2$, and if*

$$\alpha_1\alpha_2 = \alpha_2\alpha_1 \text{ in } M_n(\mathbb{C}),$$

then $\alpha_1^{(n)}\alpha_2^{(n)}$ and $\alpha_2^{(n)}\alpha_1^{(n)}$ are projections in $\sum_n (\mathbb{H})$.

Proof. Under the conditions of (1), the matrix $\alpha = t_1\alpha_1 + t_2\alpha_2$ is self-adjoint, i.e.,

$$\alpha^* = \overline{t_1}\alpha_1^* + \overline{t_2}\alpha_2^* = t_1\alpha_1 + t_2\alpha_2 = \alpha,$$

in $M_n(\mathbb{C})$, implying the self-adjointness of α. Therefore, the q-spectral function,

$$\alpha^{(n)} = (t_1\alpha_1 + t_2\alpha_2) \circ \sigma^n = t_1\alpha_1^{(n)} + t_2\alpha_2^{(n)},$$

is self-adjoint in $\sum_n (\mathbb{H})$ by (6.6) and (6.8). So, the statement (1) holds.

If the conditions of (2) hold in $M_n(\mathbb{C})$, then the matrix $\beta = t\alpha_1\alpha_2$ is self-adjoint, i.e.,

$$\beta^* = \overline{t}\alpha_2^*\alpha_1^* = t\alpha_2\alpha_1 = t\alpha_1\alpha_2 = \beta,$$

in $M_n(\mathbb{C})$, implying the self-adjointness of β. So, the corresponding q-spectral function,

$$\beta^{(n)} = (t\alpha_1\alpha_2) \circ \sigma^n = t\alpha_1^{(n)}\alpha_2^{(n)},$$

is self-adjoint in $\sum_n (\mathbb{H})$ by (6.6) and (6.8). Thus, the statement (2) holds.

Under the condition of (3), the matrix $\gamma = \alpha_1 + \alpha_2 \in M_n(\mathbb{C})$ is a projection, since

$$\gamma^* = \alpha_1^* + \alpha_2^* = \alpha_1 + \alpha_2 = \gamma,$$

and

$$\gamma^2 = \alpha_1^2 + \alpha_1\alpha_2 + \alpha_2\alpha_1 + \alpha_2^* = \alpha_1 + \alpha_2 = \gamma,$$

on $\mathbb{C}^n$. So, the corresponding element,

$$\gamma^{(n)} = (\alpha_1 + \alpha_2) \circ \sigma^n = \alpha_1^{(n)} + \alpha_2^{(n)},$$

is a projection in $\sum_n(\mathbb{H})$ by (6.6) and (6.8). It shows that the statement (3) holds.

Finally, if the conditions of (4) hold, then the matrix $\theta = \alpha_1\alpha_2 \in M_n(\mathbb{C})$ is a projection because

$$\theta^* = \alpha_2^*\alpha_1^* = \alpha_2\alpha_1 = \alpha_1\alpha_2 = \theta,$$

and

$$\theta^2 = \alpha_1\alpha_2\alpha_1\alpha_2 = \alpha_1^2\alpha_2^2 = \alpha_1\alpha_2 = \theta,$$

on $\mathbb{C}^n$. So, the ring element,

$$\theta^{(n)} = (\alpha_1\alpha_2)\circ\sigma^n = \alpha_1^{(n)}\alpha_2^{(n)},$$

is a projection in $\sum_n(\mathbb{H})$ by (6.6) and (6.8). Therefore, the statement (4) holds. □

Recall now that every matrix of $M_n(\mathbb{C})$ has its nonempty spectrum as a subset of $\mathbb{C}$ (e.g., [7, 8]). Note that, in particular, a self-adjoint matrix $\alpha \in M_n(\mathbb{C})$ has its nonempty spectrum,

$$\operatorname{spec}(\alpha) \subset \mathbb{R}, \text{ in } \mathbb{C}.$$

Note also that a projection $\beta \in M_n(\mathbb{C})$ has its spectrum satisfying

$$\operatorname{spec}(\beta) \subseteq \{0,1\}, \text{ in } \mathbb{C},$$

(e.g., see [7, 8]). So, by (7.7), we obtain the following fixed-point theorem.

Theorem 8.3. *For any arbitrary "self-adjoint" q-spectral function $\alpha^{(n)} \in \sum_n(\mathbb{H})$, there always exist $t \in \mathbb{R}$, and $\mathbf{w} \in \mathbb{H}^n$ such that*

$$\alpha^{(n)}(\mathbf{w}) = t\sigma^n(\mathbf{w}) \text{ in } \mathbb{C}^n. \tag{8.1}$$

Proof. Suppose a q-spectral function $\alpha^{(n)} = \alpha\circ\sigma^n$ is self-adjoint in $\sum_n(\mathbb{H})$, i.e., the matrix part $\alpha \in M_n(\mathbb{C})$ is self-adjoint. Then it has its spectrum $\operatorname{spec}(\alpha)$, contained in $\mathbb{R}$. Suppose $t \in \operatorname{spec}(\alpha)$. Then, as a $\mathbb{C}$-quantity, this $\mathbb{R}$-quantity t has its eigenspace $\mathcal{E}_t$ in $\mathbb{C}^n$. For an eigenvector $\mathbf{v} \in \mathcal{E}_t$, one can have $\mathbb{H}$-vector $\mathbf{w} \in \mathbb{H}^n$, satisfying

$$\sigma^n(\mathbf{w}) = \mathbf{v},$$

by (4.3). Therefore, by (7.7),

$$\left(\alpha^{(n)} - t\cdot 1^{(n)}\right)(\mathbf{w}) = 0_n,$$

if and only if

$$\alpha^{(n)}(\mathbf{w}) = t\cdot 1^{(n)}(\mathbf{w}) = t\sigma^n(\mathbf{w}),$$

in $\mathbb{C}^n$. Therefore, the equality (8.1) holds. □

Theorem 8.3 shows that the self-adjointness of a q-spectral function $\alpha^{(n)} \in \sum_n(\mathbb{H})$ guarantees the fixed-point formula (8.1).

Theorem 8.4. *Let $\alpha^{(n)} = \alpha\circ\sigma^n \in \sum_n(\mathbb{H})$ be a "nonzero" projection. Then there exists $\mathbf{w} \in \mathbb{H}^n$ such that*

$$\alpha^{(n)}(\mathbf{w}) = \sigma^n(\mathbf{w}) \text{ in } \mathbb{C}^n. \tag{8.2}$$

Proof. Let $\alpha^{(n)} \in \sum_n(\mathbb{H})$ be a nonzero projection, i.e., the matrix part $\alpha \in M_n(\mathbb{C})$ is a nonzero projection on $\mathbb{C}^n$, satisfying

$$\operatorname{spec}(\alpha) \subseteq \{0,1\} \text{ in } \mathbb{C}^n.$$

In particular, the nonzero-ness of α implies that

$$1 \in \operatorname{spec}(\alpha).$$

So, by (8.1), there exists $\mathbf{w} \in \mathbb{H}^n$, such that

$$\alpha^{(n)}(\mathbf{w}) = 1\cdot\sigma^n(\mathbf{w}) \text{ in } \mathbb{C}^n,$$

showing the relation (8.2). □

Theorem 8.4 provides a special example of the fixed-point theorem (8.1) under self-adjointness on $\sum_n (\mathbb{H})$. However, it also fully characterizes the nonzero-projection-property on $\sum_n (\mathbb{H})$ in terms of the fixed-point formula (8.2).

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