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Difference Cesàro sequence space defined by a sequence of modulus function

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Abstract

The purpose of this paper is to introduce the difference sequence space $ces(B^{\mu}_{\Lambda}, F, q)$ using sequence of modulus function $F = (f_i)$. We examine some topological properties of the space and also obtain some inclusion relations.

Keywords: Cesàro sequence space, difference sequence space, paranormed space.

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1. Introduction and preliminaries

Let w, ℓ^0 denote the spaces of all scalar and real sequences, respectively. For $1 < p < \infty$, the cesàro sequence space ces_p defined by

$$ces_p = \Big\{x \in \ell^0 : \sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n}|x_k|\Big)^p < \infty\Big\},$$

is a Banach space when equipped with the norm

$$||x|| = \Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n}|x_k|\Big)^p\Big)^{\frac{1}{p}}.$$

This space was introduced by Shiue [30], which is useful in the theory of matrix operator. Some geometric properties of the cesàro sequence space ces_p were studied by many authors such as Lee [13], Leibowitz [14], Lui et. al [15], Sanhan et. al [25] and Tripathy et. al [33] and references therein. Modulus function has been discussed in [22, 23, 26–29] and references therein.

Ruckle [24] used the idea of a modulus function f to construct a class of FK spaces

$$L(f) = \Big\{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\Big\}.$$

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The space $L(f)$ is closely related to the space ℓ_1 which is an $L(f)$ space with $f(x) = x$ for all real $x \geqslant 0$.

For any set E of sequences, the space of multipliers of E, denoted by $M(E)$, is given by

$$M(E) = \{a \in w : ax \in E, \quad \text{for all} \quad x \in E\}.$$

The notion of the difference sequence space was introduced by Kızmaz [12]. It was further generalized by Et and Çolak [11] as follows

$$Z(\Delta^{\mu}) = \{x = (x_k) \in \omega : (\Delta^{\mu} x_k) \in z\},$$

for $z = \ell_\infty$, c and $c_\circ$, where μ is a non-negative integer and

$$\Delta^{\mu} x_k = \Delta^{\mu-1} x_k - \Delta^{\mu-1} x_{k+1}, \quad \Delta^{\circ} x_k = x_k, \quad \forall\, k \in \mathbb{N},$$

or equivalent to the following binomial representation

$$\Delta^{\mu} x_k = \sum_{\nu=0}^{\mu} (-1)^{\nu} \binom{\mu}{\nu} x_{k+\nu}.$$

These sequence spaces were generalized by Et and Basaşir [10] taking $z = \ell_\infty(p)$, $c(p)$ and $c_\circ(p)$.

Dutta [3] introduced the following difference sequence spaces using a new difference operator:

$$Z(\Delta_{(\eta)}) = \left\{x = (x_k) \in \omega : \Delta_{(\eta)} x \in z\right\}, \quad \text{for } z = \ell_\infty, c \text{ and } c_\circ,$$

where $\Delta_{(\eta)} x = (\Delta_{(\eta)} x_k) = (x_k - x_{k-\eta})$ for all $k, \eta \in \mathbb{N}$.

In [4], Dutta introduced the sequence spaces $\bar{c}(\|.,.\|, \Delta^{\mu}_{(\eta)}, p)$, $\bar{c_\circ}(\|.,.\|, \Delta^{\mu}_{(\eta)}, p)$, $\ell_\infty(\|.,.\|, \Delta^{\mu}_{(\eta)}, p)$, $m(\|.,.\|, \Delta^{\mu}_{(\eta)}, p)$, and $m_\circ(\|.,.\|, \Delta^{\mu}_{(\eta)}, p)$, where η, $\mu \in \mathbb{N}$ and $\Delta^{\mu}_{(\eta)} x_k = (\Delta^{\mu}_{(\eta)} x_k) = (\Delta^{\mu-1}_{(\eta)} x_k - \Delta^{\mu-1}_{(\eta)} x_{k-\eta})$, and $\Delta^{\circ}_{(\eta)} x_k = x_k$, for all $k, \eta \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta^{\mu}_{(\eta)} x_k = \sum_{\nu=0}^{\mu} (-1)^{\nu} \binom{\mu}{\nu} x_{k-\eta\nu}.$$

The difference sequence space have been studied by authors [5–9, 18–21, 23, 31, 32, 35] and references therein. Başar and Altay [1] introduced the generalized difference matrix $B = (b_{mk})$ for all $k, m \in \mathbb{N}$, which is a generalization of $\Delta_{(1)}$-difference operator, by

$$b_{mk} = \begin{cases} r, & k = m, \\ s, & k = m-1, \\ 0, & (k > m) \text{ or } (0 \leqslant k < m-1). \end{cases}$$

Başarir and Kayikçi [2] defined the matrix $B^{\mu}(b^{\mu}_{mk})$ which reduced the difference matrix $\Delta^{\mu}_{(1)}$ incase $r = 1$, $s = -1$. The generalized B^{μ}-difference operator is equivalent to the following binomial representation:

$$B^{\mu} x = B^{\mu}(x_k) = \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} r^{\mu-\nu} s^{\nu} x_{k-\nu}.$$

Let $F = (f_i)$ be a sequence of modulus functions, $q = (q_n)$ be a bounded sequence of strictly positive real numbers, then we define the cesàro sequence space as follows

$$ces(B^{\mu}_{\Lambda}, F, q) = \left\{x \in w : \sum_{n=1}^{\infty} \left[f_i\left(\frac{1}{n} \sum_{i=1}^{n} |B^{\mu}_{\Lambda} x_i|\right)\right]^{q_n} < \infty\right\}.$$

Taking modulus function F^{ν} instead of F in the space $ces(B^{\mu}_{\Lambda}, F, q)$, we can define the composite space $ces(B^{\mu}_{\Lambda}, F^{\nu}, q)$ as follow

$$ces(B^{\mu}_{\Lambda}, F^{\nu}, q) = \Big\{x \in w : \sum_{n=1}^{\infty} \Big[f_i^{\nu}\Big(\frac{1}{n}\sum_{i=1}^{n}|B^{\mu}_{\Lambda}x_i|\Big)\Big]^{q_n} < \infty\Big\}.$$

The following inequality will be used throughout the paper. If $0 \leqslant p_i \leqslant \sup p_i = H$, $K = \max(1, 2^{H-1})$, then

$$|a_i + b_i|^{p_i} \leqslant K\{|a_i|^{p_i} + |b_i|^{p_i}\}, \tag{1.1}$$

for all i and $a_i, b_i \in \mathbb{C}$. Also $|a|^{p_i} \leqslant \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

We examine some topological properties of the space $ces(B^{\mu}_{\Lambda}, F, q)$ and also obtain some inclusion relations.

2. Topological properties

Theorem 2.1. *Let $F = (f_i)$ be a sequence of modulus function and $q = (q_n)$ be a bounded sequence of positive real numbers. Then $ces(B^{\mu}_{\Lambda}, F, q)$ is a linear space over the field of complex number $\mathbb{C}$.*

Proof. Let $x, y \in ces(B^{\mu}_{\Lambda}, F, q)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive number M_{α} and N_{β} such that $|\alpha| \leqslant M_{\alpha}$ and $|\beta| \leqslant N_{\beta}$. From condition (ii) and (iii) of definition of modulus function and by using inequality (1.1), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\Big[f_i\Big(\frac{1}{n}\sum_{i=1}^{\infty}|B^{\mu}_{\Lambda}(\alpha x_i + \beta y_i)|\Big)\Big]^{q_n} &\leqslant \max(1, 2^{H-1})\Big(\max(1, M^H_{\alpha})\sum_{n=1}^{\infty}\Big[f_i\Big(\frac{1}{n}\sum_{i=1}^{n}|B^{\mu}_{\Lambda}x_i|\Big)\Big]^{q_n} \\ &\quad + \max(1, N^H_{\beta})\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\Big[f_i\Big(\frac{1}{n}\sum_{i=1}^{n}|B^{\mu}_{\Lambda}y_i|\Big)\Big]^{q_n}\Big).\end{aligned}$$

This implies that $\alpha x + \beta y \in ces(B^{\mu}_{\Lambda}, F, q)$. This proves that $ces(B^{\mu}_{\Lambda}, F, q)$ is a linear space. This completes the proof of the theorem. □

Theorem 2.2. *Let $F = (f_i)$ be a sequence of modulus function and $q = (q_n)$ be a bounded sequence of positive real numbers, $ces(B^{\mu}_{\Lambda}, F, q)$ is a topological linear space, paranormed by*

$$g(x) = \Big(\sum_{n=1}^{\infty}\Big[f_i\Big(\frac{1}{n}\sum_{i=1}^{n}|B^{\mu}_{\Lambda}x_i|\Big)\Big]^{q_n}\Big)^{\frac{1}{K}},$$

where $H = \sup q_n < \infty$ and $K = \max(1, H)$.

Proof. Clearly $g(x) = g(-x)$. It is trivial $B^{\mu}_{\Lambda}x_i = 0$ for $x = 0$. Since $f_i(0) = 0$, we get $g(x) = 0$ for $x = 0$. Since $\frac{p_i}{i} \leqslant 1$, Using the Minkowski's inequality, we have

$$\begin{aligned}\Big(\sum_{n=1}^{\infty}\Big[f_i\Big(\frac{1}{n}\sum_{i=1}^{\infty}|B^{\mu}_{\Lambda}(x_i + y_i)|\Big)\Big]^{q_n}\Big)^{\frac{1}{K}} &\leqslant \Big(\sum_{n=1}^{\infty}\Big[f_i\Big(\frac{1}{n}\sum_{i=1}^{\infty}(|B^{\mu}_{\Lambda}x_i| + |B^{\mu}_{\Lambda}y_i|)\Big)\Big]^{q_n}\Big)^{\frac{1}{K}} \\ &\leqslant \Big(\sum_{n=1}^{\infty}\Big[f_i\Big(\frac{1}{n}\sum_{i=1}^{\infty}|B^{\mu}_{\Lambda}x_i|\Big)\Big]^{q_n}\Big)^{\frac{1}{K}} \\ &\quad + \Big(\sum_{n=1}^{\infty}\Big[f_i\Big(\frac{1}{n}\sum_{i=1}^{\infty}|B^{\mu}_{\Lambda}y_i|\Big)\Big]^{q_n}\Big)^{\frac{1}{K}}.\end{aligned}$$

Hence $g(x)$ is subadditive. For the continuity of multiplication, let us take any complex number α. By definition, we have

$$\begin{aligned} g(\alpha x) &= \Big(\sum_{n=1}^{\infty} \Big[f_i \Big(\frac{1}{n} \sum_{i=1}^{\infty} |B_{\Lambda}^{\mu}(\alpha x_i)| \Big) \Big]^{q_n} \Big)^{\frac{1}{K}} \\ &\leqslant C_{\alpha}^{\frac{H}{K}} g(x), \end{aligned}$$

where C_{α} is a positive integer such that $|\alpha| \leqslant C_{\alpha}$. Now, let $\alpha \to 0$ for any fixed x with $g(x) \neq 0$. By definition for $|\alpha| < 1$, we have

$$\sum_{n=1}^{\infty} \Big[f_i \Big(\frac{1}{n} \sum_{k=1}^{\infty} |\alpha B_{\Lambda}^{\mu} x_i| \Big) \Big]^{q_n} < \epsilon, \quad \text{for } n > n_0(\epsilon). \tag{2.1}$$

Also, for $1 \leqslant n \leqslant n_0$, taking α small enough, since $F = (f_i)$ is continuous, we have

$$\sum_{n=1}^{\infty} \Big[f_i \Big(\frac{1}{n} \sum_{i=1}^{\infty} |\alpha B_{\Lambda}^{\mu} x_i| \Big) \Big]^{q_n} < \epsilon. \tag{2.2}$$

Now, (2.1) and (2.2) together imply that $g(\alpha x) \to 0$ as $\alpha \to 0$. This completes the proof of the theorem. □

Theorem 2.3. *Let $F = (f_i)$ be a sequence of modulus function and $q = (q_n)$ be a bounded sequence of positive real numbers, $ces(B_{\Lambda}^{\mu}, F, q)$ is a complete paranormed space with paranorm defined by*

$$g(x) = \Big(\sum_{n=1}^{\infty} \Big[f_i \Big(\frac{1}{n} \sum_{i=1}^{n} |B_{\Lambda}^{\mu} x_i| \Big) \Big]^{q_n} \Big)^{\frac{1}{K}},$$

where $H = \sup q_n < \infty$ and $K = \max(1, H)$.

Proof. In view of Theorem 2.2 it suffices to prove the completeness of $ces(B_{\Lambda}^{\mu}, F, q)$. Let $(x^{(s)})$ be a Cauchy sequence in $ces(B_{\Lambda}^{\mu}, F, q)$. Then $g(x^{(s)} - x^{(t)}) \to 0$ as $t \to \infty$, that is

$$\sum_{n=1}^{\infty} \Big[f_i \Big(\frac{1}{n} \sum_{i=1}^{\infty} |B_{\Lambda}^{\mu}(x_i^{(s)} - x_i^{(t)})| \Big) \Big]^{q_n} \to 0, \quad \text{as } s, t \to \infty, \tag{2.3}$$

which implies that for each i, $|x_i^{(s)} - x_i^{(t)}| \to 0$ as $s, t \to \infty$ and so $(x_i^{(s)})$ is a Cauchy sequence in $\mathbb{C}$ for each fixed i. Since $\mathbb{C}$ is complete, as $s \to \infty$, $x_i^{(s)} \to x_i$, for each i. Now from (2.3), we have that for $\epsilon > 0$, there exists a natural number N such that

$$\sum_{n=1}^{\infty} \Big[f_i \Big(\frac{1}{n} \sum_{i=1}^{\infty} |B_{\Lambda}^{\mu}(x_i^{(s)} - x_i^{(t)})| \Big) \Big]^{q_n} < \epsilon^K, \quad \text{for } s, t > N. \tag{2.4}$$

Since for any fixed natural number M, we have from (2.4)

$$\sum_{n=1}^{M} \Big[f_i \Big(\frac{1}{n} \sum_{i=1}^{\infty} |B_{\Lambda}^{\mu}(x_i^{(s)} - x_i^{(t)})| \Big) \Big]^{q_n} < \epsilon^K, \quad \text{for } s, t > N,$$

by taking $t \to \infty$ in the above expression we obtain

$$\sum_{n=1}^{M} \Big[f_i \Big(\frac{1}{n} \sum_{i=1}^{\infty} |B_{\Lambda}^{\mu}(x_i^{(s)} - x_i)| \Big) \Big]^{q_n} < \epsilon^K, \quad \text{for } s > N.$$

Since M is arbitrary, by taking $M \to \infty$, we obtain

$$\sum_{n=1}^{\infty}\left[f_i\Big(\frac{1}{n}\sum_{i=1}^{\infty}|B_{\Lambda}^{\mu}(x_i^{(s)}-x_i)|\Big)\right]^{q_n} < \epsilon^K, \quad \text{for } s > N,$$

i.e., $g(x^s - x) < \epsilon$ for $s > N$. To show that $x \in ces(B_{\Lambda}^{\mu}, F, q)$, let $t > M$ and fix n_0. Since $\frac{p_i}{K} \leqslant 1$ and $K \geqslant 1$, using Minkowski's inequality and the definition of modulus function, we have

$$\begin{aligned}\Big(\sum_{n=1}^{n_0}\Big[f_i\Big(\frac{1}{n}\sum_{i=1}^{\infty}|B_{\Lambda}^{\mu}(x_i)|\Big)\Big]^{q_n}\Big)^{\frac{1}{K}} &= \Big(\sum_{n=1}^{n_0}\Big[f_i\Big(\frac{1}{n}\sum_{i=1}^{\infty}|B_{\Lambda}^{\mu}(x_i - x_i^{(t)} + x_i^{(t)})|\Big)\Big]^{q_n}\Big)^{\frac{1}{K}}\\ &\leqslant \Big(\sum_{n=1}^{n_0}\Big[f_i\Big(\frac{1}{n}\sum_{i=1}^{\infty}|B_{\Lambda}^{\mu}(x_i - x_i^{(t)})|\Big) + f_i\Big(\frac{1}{n}\sum_{i=1}^{n}|B_{\Lambda}^{\mu}x_i^{(t)}|\Big)\Big]^{q_n}\Big)^{\frac{1}{K}}\\ &\leqslant \Big(\sum_{n=1}^{n_0}\Big[f_i\Big(\frac{1}{n}\sum_{i=1}^{\infty}|B_{\Lambda}^{\mu}(x_i - x_i^{(t)})|\Big)\Big]^{q_n}\Big)^{\frac{1}{K}}\\ &\quad + \Big(\sum_{n=1}^{n_0}\Big[f_i\Big(\frac{1}{n}\sum_{i=1}^{\infty}|B_{\Lambda}^{\mu}x_i^{(t)}|\Big)\Big]^{q_n}\Big)^{\frac{1}{K}}\\ &< \epsilon + g(x^{(t)}).\end{aligned}$$

It follows that $\sum_{n=1}^{\infty}\left[f_i\left(\frac{1}{n}\sum_{i=1}^{n}|B_{\Lambda}^{\mu}x_i|\right)\right]^{q_n}$ converges, so that $x = (x_i) \in ces(B_{\Lambda}^{\mu}, F, q)$ and the space is complete. This completes the proof of the theorem. □

3. Inclusion relations

Theorem 3.1. *If $q = (q_n)$ and $p = (p_n)$ are bounded sequences of positive real numbers with $0 < q_n \leqslant p_n < \infty$, for each n and $F = (f_i)$ be a sequence of modulus function, then $ces(B_{\Lambda}^{\mu}, F, q) \subseteq ces(B_{\Lambda}^{\mu}, F, p)$.*

Proof. Let $x \in ces(B_{\Lambda}^{\mu}, F, q)$. Then

$$\sum_{n=1}^{\infty}\left[f_i\Big(\frac{1}{n}\sum_{i=1}^{n}|B_{\Lambda}^{\mu}x_i|\Big)\right]^{q_n} < \infty.$$

This implies that $f_i\left(\frac{1}{n}\sum_{i=1}^{n}|B_{\Lambda}^{\mu}x_i|\right) \leqslant 1$ for sufficiently large values of n, say $n \geqslant n_0$ for some fixed $n_0 \in \mathbb{N}$. Since $F = (f_i)$ is increasing and $q_n \leqslant p_n$, we have

$$\sum_{n\geqslant n_0}^{\infty}\left[f_i\Big(\frac{1}{n}\sum_{i=1}^{n}|B_{\Lambda}^{\mu}x_i|\Big)\right]^{p_n} \leqslant \sum_{n\geqslant n_0}^{\infty}\left[f_i\Big(\frac{1}{n}\sum_{i=1}^{n}|B_{\Lambda}^{\mu}x_i|\Big)\right]^{q_n} < \infty,$$

which implies that $x \in ces(B_{\Lambda}^{\mu}, F, p)$ and this completes the proof of the theorem. □

Theorem 3.2. *If $u = (u_n)$ and $v = (v_n)$ are bounded sequences of positive real numbers with $0 < u_n, v_n < \infty$, and $q_n = \min(u_n, v_n)$, then*

$$ces(B_{\Lambda}^{\mu}, F, q) = ces(B_{\Lambda}^{\mu}, F, u) \cap ces(B_{\Lambda}^{\mu}, F, v).$$

Proof. It follows from Theorem 3.1 that

$$ces(B_{\Lambda}^{\mu}, F, q) \subset ces(B_{\Lambda}^{\mu}, F, u) \cap ces(B_{\Lambda}^{\mu}, F, v).$$

For any complex number λ, $|\lambda|^{q_n} \leqslant \max(|\lambda|^{u_n}, |\lambda|^{v_n})$, thus

$$ces(B_{\Lambda}^{\mu}, F, u) \cap ces(B_{\Lambda}^{\mu}, F, v) \subseteq ces(B_{\Lambda}^{\mu}, F, q),$$

and the proof of the theorem is complete. □

Theorem 3.3. *If* $H = \sup p_k < \infty$ *and* $F = (f_i)$ *be a sequence of modulus function, then* $\ell_\infty \subset M(ces(B^\mu_\Lambda, F, q))$.

Proof. $a \in \ell_\infty$ implies $|a_i| < 1 + [i]$ for some $i > 0$ and all i. Hence, $x \in ces(B^\mu_\Lambda, F, q)$ implies

$$\sum_{n=1}^{\infty}\left[f_i\Big(\frac{1}{n}\sum_{i=1}^{n}|a_i x_i|\Big)\right]^{q_n} < (1+[i])^H \sum_{n=1}^{\infty}\left[f_i\Big(\frac{1}{n}\sum_{i=1}^{n}|B^\mu_\Lambda x_i|\Big)\right]^{q_n},$$

which gives $\ell_\infty \subset M(ces(B^\mu_\Lambda, F, q))$. This completes the proof of the theorem. □

Theorem 3.4. *For any sequence of modulus function* $F = (f_i)$ *and* $\nu \in \mathbb{N}$,

(i) $ces(B^\mu_\Lambda, F^\nu, q) \subseteq ces(B^\mu_\Lambda, q)$, *if* $\lim_{t\to\infty}\frac{f(t)}{t} = \beta > 0$.

(ii) $ces(B^\mu_\Lambda, q) \subseteq ces(B^\mu_\Lambda, f^\nu, q)$, *if there exists a positive constants* α *such that* $f(t) \leqslant \alpha t$, *for all* $t \geqslant 0$.

Proof. (i) By Maddox [12, Proposition 1], we have

$$\beta = \lim_{t\to\infty}\frac{f(t)}{t} = \inf\Big\{\frac{f(t)}{t} : t > 0\Big\},$$

so that $0 \leqslant \beta \leqslant f(1)$. Let $\beta > 0$, by definition of β, we have $\beta t \leqslant f(t), \forall t \geqslant 0$. Since $F = (f_i)$ is increasing we have $\beta^2 t \leqslant f^2(t)$. So by induction we have $\beta^\nu t \leqslant f^\nu(t)$. Let $x \in ces(F^\nu, q, B^\mu_\Lambda)$, Using inequality $|\lambda|^{q_i} \leqslant \max(1, |\lambda|^H)$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{i=1}^{n}|B^\mu_\Lambda x_i|\Big)^{q_n} &\leqslant \sum_{n=1}^{\infty}\left[\beta^{-\nu} f_i^\nu\Big(\frac{1}{n}\sum_{i=1}^{n}|B^\mu_\Lambda x_i|\Big)\right]^{q_n}\\ &\leqslant \max(1, \beta^{-\nu H})\sum_{n=1}^{\infty}\left[f_k^\nu\Big(\frac{1}{n}\sum_{i=1}^{n}|B^\mu_\Lambda x_i|\Big)\right]^{q_n},\end{aligned}$$

and hence $x \in ces(B^\mu_\Lambda, q)$.

(ii) Since $f_i(t) \leqslant \alpha t$, for all $t \geqslant 0$ and $F = (f_i)$ is an increasing function, we have $f_i^\nu(t) \leqslant \alpha^\nu t$ for each $\nu \in \mathbb{N}$. Let $x \in ces(B^\mu_\Lambda, q)$. Using inequality $|\lambda|^{q_i} \leqslant \max(1, |\lambda|^H)$, we have

$$\sum_{n=1}^{\infty}\left[f_i^\nu\Big(\frac{1}{n}\sum_{i=1}^{n}|B^\mu_\Lambda x_i|\Big)\right]^{q_n} \leqslant \max(1, \alpha^{\nu H}\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{i=1}^{n}|B^\mu_\Lambda x_i|\Big)^{q_n},$$

and hence $x \in ces(B^\mu_\Lambda, F^\nu, q)$. □

Theorem 3.5. *Let* $m, \nu \in \mathbb{N}$ *be such that* $m < \nu$. *If there exists a positive constant* α *such that* $f(t) \leqslant \alpha t$ *for all* $t \geqslant 0$, *then*

$$ces(B^\mu_\Lambda, q) \subseteq ces(B^\mu_\Lambda, F^m, q) \subseteq ces(B^\mu_\Lambda, F^\nu, q).$$

Proof. Let $r = \nu - m$. Since $f_i(t) \leqslant \alpha t$, we have $f^\nu(t) < M^r f_k^m(t) < M^\nu t$, where $M = 1 + [\alpha]$. Let $x \in ces(B^\mu_\Lambda, q)$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left[f_k^\nu\Big(\frac{1}{n}\sum_{i=1}^{n}|B^\mu_\Lambda x_i|\Big)\right]^{q_n} &< M^{rH}\sum_{n=1}^{\infty}\left[f_i^m\Big(\frac{1}{n}\sum_{i=1}^{n}|B^\mu_\Lambda x_i|\Big)\right]^{q_n}\\ &< M^{\nu H}\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{i=1}^{n}|B^\mu_\Lambda x_i|\Big)^{q_n},\end{aligned}$$

and the required inclusion follows. This completes the proof. □

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