



Proposing a Developed GramSchmidt Algorithm to Construct Orthogonal Unit Vectors

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Abstract

The GramSchmidt method is among the most well-known approaches for the orthogonalization of vectors; however, the accuracy of this algorithm might decline when it is implemented on large-scale vectors. This paper proposes a Developed GramSchmidt Algorithm (DGSA). The Schmidt vectors obtained from the proposed algorithm are prone to a lower error rate than those resulting from the GramSchmidt algorithm.

Keywords: Mathematics, Gram-Schmidt Algorithm, Orthogonalization of Vectors.

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1. Introduction

In diverse areas of applied mathematics use is made of orthogonal sets of vectors, or one needs to make a vector orthogonal to the linear span of a given set of vectors. The Gram-Schmidt algorithm is instrumental for these purposes[1]. In fact, the Gram-Schmidt algorithm should immediately be used whenever it is necessary to develop orthogonal and unit vectors out of several vectors.

2. Gram-Schmidt (GS) Method

The Gram-Schmidt (GS) method is a mathematical approach that can be employed to develop an orthogonal unit basis out of an arbitrary basis. Known as a transform in the vector space, the GS method is mostly adopted to reduce the correlation of datasets. It can also be employed to identify unimportant vectors and delete them from the calculation process if necessary. This method was introduced by a scientist named Gram Schmidt [2].

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3. Gram-Schmidt (GS) Algorithm

The following steps should be taken to execute the GS algorithm on the data matrix (i.e $X = [X_1, X_2, \dots, X_n]_{m \times n}$) [3] :

1. A vector (e.g. X_1) is considered equal to Z_1 and used as the first Schmidt vector:

$$Z_1 = X_1 \quad (3.1)$$

2. The second Schmidt vector is developed through the following equation:

$$Z_2 = X_2 - \frac{X_2 Z_1}{Z_1 Z_1} Z_1 \quad (3.2)$$

3. The third Schmidt vector is constructed as below:

$$Z_3 = X_3 - \frac{X_3 Z_1}{Z_1 Z_1} Z_1 - \frac{X_3 Z_2}{Z_2 Z_2} Z_2 \quad (3.3)$$

4. The above procedure can be continued to easily calculate all Schmidt vectors through the following equation:

$$Z_j = X_j - \sum_{k=1}^{j-1} \frac{X_j Z_k}{Z_k Z_k} Z_k \quad (3.4)$$

5. End.

The following equation can be employed to unitize Schmidt vectors:

$$sv_{i,j} = \frac{z_{i,j}}{\sqrt{\sum_{p=1}^m z_{p,j}^2}}; i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n \quad (3.5)$$

4. Developed Gram-Schmidt (DGS) Method

This method consists of the following steps:

1. Put $k = 1$.
2. Out of the vectors in matrix X , find the vector that has the highest correlation with the other vectors and put it into vector Z_k .
3. Delete that vector with the highest correlation from matrix X .
4. Put $k = k + 1$.
5. Use the following equation to orthogonalize all the remaining vectors in X on vector Z_{k-1} :

$$X_j = X_j - \frac{X_j Z_{k-1}}{Z_{k-1} Z_{k-1}} Z_{k-1}; j = 1, 2, \dots, n - k + 1 \quad (4.1)$$
6. Out of the vectors in matrix X , find the vector with the highest variance and put it into vector Z_k .
7. Delete that vector with the highest variance from matrix X .
8. If $k \leq (n - 1)$, repeat Step (4); otherwise, go to Step (9).
9. End.

The following equation can be adopted to unitize Schmidt vectors:

$$sv_{i,j} = \frac{z_{i,j}}{\sqrt{\sum_{p=1}^m z_{p,j}^2}}; i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n \quad (4.2)$$

5. Schmidt Vector Error Calculation

If matrix $SV = (sv_{i,j})_{m \times n}$ contains all Schmidt vectors, the calculation error of this matrix can be determined through the following equation:

$$\text{Error} = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \left| \sum_{k=1}^m (sv_{k,i} sv_{k,j}) \right| \quad (5.1)$$

It should be noted that Equation (5.1) calculates the absolute summation of the dot product Schmidt vector pairs. The closer this value to zero, the more accurate Schmidt vectors.

5.1. Numerical Example

Table (1) presents the data of matrix $X_{10 \times 10}$.

Table 1: Data of matrix $X_{10 \times 10}$

44	−36	−81	1	77	−35	46	−9	10	−53
2	−20	17	−1	6	33	−61	−49	83	87
−35	76	70	73	−82	0	−9	7	11	41
−9	91	−54	14	−3	72	4	−82	72	35
−41	−100	−61	−66	−36	50	−71	12	−9	−21
37	20	−93	82	9	−21	57	71	−68	30
85	57	66	33	92	3	−67	46	−53	−10
−42	28	−95	1	−27	17	−1	8	22	98
−5	−77	−27	65	−95	2	−48	34	−13	−82
−22	42	−65	−43	48	47	24	72	22	−6

After the GS algorithm is executed, the matrix contains Schmidt vectors as shown in Table (2).

Table 2: The matrix containing Schmidt vectors through the baseline approach

0.350	−0.228	−0.417	−0.044	0.399	−0.078	0.013	−0.399	−0.009	−0.570
0.016	−0.105	0.097	0.022	0.336	0.684	0.027	−0.452	0.012	0.439
−0.278	0.426	0.317	0.489	0.321	0.138	−0.074	0.016	−0.321	−0.413
−0.072	0.479	−0.330	−0.031	−0.592	0.272	0.246	−0.372	0.005	−0.177
−0.326	−0.479	−0.154	−0.195	−0.233	0.341	−0.209	0.172	−0.562	−0.208
0.294	0.069	−0.515	0.466	0.064	−0.037	0.182	0.259	−0.425	0.373
0.676	0.215	0.159	−0.087	−0.163	0.333	−0.503	0.240	−0.014	−0.120
−0.334	0.184	−0.430	0.073	0.088	−0.128	−0.745	−0.133	0.184	0.183
−0.040	−0.394	−0.056	0.620	−0.233	0.255	0.000	0.153	0.527	−0.188
−0.175	0.238	−0.323	−0.325	0.357	0.350	0.219	0.549	0.297	−0.123

Now if the proposed GS algorithm is executed, the matrix will contain Schmidt vectors as shown in Table (3)

Table 3: The matrix containing Schmidt vectors through the novel approach

−0.308	−0.473	0.333	−0.116	−0.151	−0.343	0.176	0.334	−0.250	−0.460
0.290	0.158	0.087	−0.242	0.631	−0.064	0.461	0.427	0.168	0.003
0.000	0.340	−0.414	0.466	−0.012	0.157	0.083	0.216	−0.097	−0.639

0.633	−0.097	0.060	0.389	−0.247	−0.530	0.240	−0.180	0.054	0.030
0.439	−0.181	−0.154	−0.561	0.009	0.187	−0.076	−0.388	−0.078	−0.488
−0.185	−0.499	−0.023	0.290	0.131	0.357	0.422	−0.299	0.463	−0.070
0.026	0.327	0.536	0.066	−0.122	0.370	0.424	−0.254	−0.452	0.005
0.149	−0.422	−0.167	0.290	0.465	0.109	−0.156	−0.002	−0.633	0.188
0.018	−0.126	−0.527	−0.244	−0.443	0.138	0.475	0.248	−0.209	0.316
0.413	−0.208	0.298	0.115	−0.270	0.493	−0.281	0.511	0.169	0.017

The error rate of Schmidt vectors was reported 1.36×10^{-14} in the baseline approach, whereas it was 5.62×10^{-15} reported in the novel approach. Therefore, the novel approach yields a lower error rate. It can then be concluded that Schmidt vectors are more accurate in the novel approach.

5.2. Numerical Example

Table(4) presents the data of matrix $X_{30 \times 10}$.

Table 4: Data of matrix $X_{30 \times 10}$

71	78	73	10	31	77	4	16	39	54
32	43	34	53	11	64	40	33	71	72
71	14	33	11	54	39	40	80	54	78
5	23	25	23	57	56	69	7	43	75
32	6	24	38	71	58	90	9	4	85
33	19	90	65	67	80	24	19	17	4
12	42	5	79	72	90	80	56	84	13
89	42	61	74	59	66	5	25	54	66
38	32	46	21	83	46	30	68	46	79
37	40	11	32	55	5	66	28	20	34
71	30	42	87	67	10	45	69	86	20
39	52	65	28	65	78	50	23	33	52
52	90	87	14	63	83	21	61	42	3
59	65	48	67	83	61	12	12	75	4
3	33	85	26	88	42	44	85	72	85
37	52	90	30	30	87	79	41	40	37
87	35	42	1	9	8	58	33	57	71
54	34	33	51	73	35	41	16	86	22
84	39	9	40	62	2	13	45	60	55
85	67	76	15	16	54	13	41	24	81
69	33	38	64	44	52	89	6	66	57
49	57	69	49	85	2	88	80	51	84
75	77	50	23	13	54	39	82	88	90
1	11	43	36	60	33	69	71	28	14
68	16	24	23	18	3	51	43	45	3
10	86	13	53	57	89	69	72	26	83
5	88	7	5	56	12	88	79	7	14
17	23	77	55	40	50	4	61	11	9
43	26	31	4	60	78	32	7	7	65
1	76	61	78	41	62	43	12	90	16

After the GS algorithm is executed, the matrix contains Schmidt vectors as shown in Table (5).

Table 5: The matrix containing Schmidt vectors through the baseline approach

0.247	0.148	0.019	-0.284	-0.109	0.130	-0.184	-0.167	0.015	0.012
0.111	0.109	-0.013	0.165	-0.286	0.162	0.098	0.131	0.165	0.247
0.247	-0.200	-0.004	-0.091	0.218	0.138	0.045	0.373	0.121	-0.024
0.017	0.106	0.064	0.054	0.211	0.170	0.179	-0.192	0.215	0.208
0.111	-0.092	0.057	0.152	0.264	0.223	0.275	-0.187	-0.245	0.203
0.115	-0.025	0.426	0.174	-0.016	0.093	-0.096	-0.088	-0.259	-0.155
0.042	0.182	-0.147	0.408	0.084	0.321	0.076	0.223	0.176	-0.278
0.310	-0.118	0.031	0.189	-0.112	0.069	-0.272	-0.025	-0.179	0.183
0.132	0.026	0.089	-0.049	0.334	0.019	-0.141	0.173	0.096	0.150
0.129	0.074	-0.159	0.075	0.147	-0.180	0.129	-0.165	-0.188	0.010
0.247	-0.113	-0.004	0.347	-0.052	-0.288	-0.037	0.160	-0.005	-0.111
0.136	0.131	0.135	-0.076	0.115	0.139	0.011	-0.175	-0.023	-0.017
0.181	0.287	0.107	-0.286	0.056	0.040	-0.197	0.011	-0.002	-0.397
0.205	0.124	-0.065	0.161	0.079	-0.029	-0.357	-0.250	0.099	-0.181
0.010	0.168	0.412	-0.057	0.288	-0.184	-0.024	0.171	0.361	0.250
0.129	0.139	0.298	-0.109	-0.177	0.152	0.335	-0.038	-0.012	-0.194
0.303	-0.148	-0.059	-0.225	-0.104	-0.095	0.301	-0.066	0.154	0.013
0.188	-0.025	-0.037	0.146	0.155	-0.051	-0.080	-0.199	0.302	-0.149
0.292	-0.115	-0.275	0.082	0.151	-0.165	-0.234	0.047	-0.002	0.092
0.296	0.034	0.045	-0.254	-0.225	0.006	-0.050	0.014	-0.211	0.191
0.240	-0.089	-0.036	0.206	-0.110	0.078	0.305	-0.251	0.015	-0.003
0.170	0.119	0.119	0.030	0.146	-0.478	0.188	-0.065	-0.123	0.240
0.261	0.127	-0.132	-0.160	-0.245	0.039	0.066	0.313	0.234	0.121
0.003	0.056	0.230	0.127	0.165	-0.048	0.211	0.230	-0.045	-0.182
0.236	-0.178	-0.061	0.002	-0.065	-0.087	0.227	0.099	0.004	-0.331
0.035	0.429	-0.251	0.154	0.006	0.224	0.009	0.234	-0.278	0.257
0.017	0.459	-0.284	-0.138	0.209	-0.235	0.191	0.042	-0.236	-0.197
0.059	0.059	0.366	0.145	-0.137	-0.065	-0.129	0.271	-0.301	-0.017
0.150	-0.026	0.004	-0.123	0.282	0.369	-0.057	-0.114	-0.073	0.034
0.003	0.410	0.110	0.247	-0.275	-0.132	-0.030	-0.250	0.275	0.073

Now if the proposed GS algorithm is executed, the matrix will contain Schmidt vectors as shown in Table (6).

Table 6: The matrix containing Schmidt vectors through the novel approach

0.247	0.112	-0.270	-0.082	0.049	-0.064	-0.060	0.085	0.245	-0.158
0.111	0.177	-0.004	-0.006	0.190	0.233	-0.311	-0.053	-0.049	0.196
0.247	-0.052	0.005	0.231	0.106	-0.055	-0.033	-0.199	-0.332	-0.223
0.017	0.226	0.197	-0.199	0.264	0.138	0.090	0.014	-0.027	-0.151
0.111	0.151	0.260	-0.297	0.218	-0.255	0.175	-0.051	-0.174	0.107
0.115	0.243	-0.126	-0.057	-0.220	-0.168	0.185	0.251	-0.210	0.183
0.042	0.351	0.157	0.040	-0.268	0.216	-0.074	-0.431	-0.190	-0.007
0.310	0.008	-0.261	-0.054	0.082	-0.010	0.127	-0.050	-0.050	0.330
0.132	0.080	-0.017	0.240	0.219	0.020	0.241	-0.066	-0.087	-0.150

0.129	-0.093	0.267	-0.087	-0.035	-0.110	0.155	-0.035	0.185	0.096
0.247	-0.177	0.103	0.165	-0.225	0.190	0.090	-0.018	-0.140	0.272
0.136	0.215	0.001	-0.117	0.022	-0.087	0.097	0.093	0.055	-0.128
0.181	0.197	-0.174	0.177	-0.321	-0.127	0.024	0.078	0.207	-0.330
0.205	0.080	-0.173	-0.100	-0.231	0.217	0.304	-0.079	0.198	-0.024
0.010	0.172	0.110	0.368	0.303	0.274	0.206	0.313	-0.108	-0.211
0.129	0.260	0.126	-0.092	-0.154	-0.128	-0.297	0.287	-0.101	-0.112
0.303	-0.235	0.151	-0.122	0.083	0.013	-0.248	0.133	-0.017	-0.187
0.188	-0.017	0.044	-0.136	-0.130	0.310	0.209	-0.040	-0.007	-0.174
0.292	-0.252	-0.054	0.092	0.063	0.088	0.227	-0.245	0.091	0.073
0.296	-0.031	-0.186	0.035	0.168	-0.223	-0.175	0.148	0.152	0.109
0.240	0.010	0.220	-0.374	-0.040	0.064	-0.104	0.028	-0.087	0.113
0.170	-0.143	0.368	0.151	0.157	-0.020	0.191	0.311	0.164	0.166
0.261	0.000	-0.043	0.231	0.154	0.156	-0.415	-0.088	0.063	-0.070
0.003	0.139	0.260	0.217	-0.163	-0.073	0.033	0.081	-0.246	-0.021
0.236	-0.198	0.155	-0.004	-0.297	-0.053	-0.167	-0.018	-0.168	-0.083
0.035	0.353	0.107	0.175	0.186	-0.156	-0.105	-0.368	0.240	0.300
0.017	0.036	0.402	0.216	-0.192	-0.246	0.016	-0.123	0.460	-0.094
0.059	0.163	-0.129	0.307	-0.125	-0.158	0.023	0.208	-0.164	0.338
0.150	0.203	-0.094	-0.171	0.147	-0.221	0.167	-0.163	-0.066	-0.215
0.003	0.264	0.058	-0.089	-0.068	0.468	-0.065	0.236	0.294	0.166

The error rate of Schmidt vectors was reported 7.84×10^{14} in the baseline approach, whereas it was 3.86×10^{15} reported in the novel approach. Therefore, the novel approach yields a lower error rate. It can then be concluded that Schmidt vectors are more accurate in the novel approach.

5.3. Numerical Example

In this section, 20 random examples were made. Table (7) presents the relevant data.

Table 7: 20 random examples

Example	X		Error	
	Number of Rows(m)	Number of Columns(n)	Old Method	New Method
1	8	4	3.580E-15	7.086E-16
2	8	6	3.793E-15	1.096E-15
3	9	8	2.563E-14	4.727E-15
4	5	5	1.441E-14	2.755E-15
5	13	9	1.258E-14	2.521E-15
6	9	5	2.489E-15	7.459E-16
7	4	4	1.998E-15	1.360E-15
8	4	3	7.633E-16	1.665E-16
9	10	10	6.442E-14	4.303E-15
10	5	5	4.316E-15	2.366E-15
11	6	4	1.166E-15	7.685E-16
12	8	3	7.355E-16	4.996E-16
13	8	5	5.676E-15	1.429E-15
14	11	8	9.963E-15	3.777E-15
15	10	5	3.779E-15	1.645E-15

16	13	8	1.434E-14	2.344E-15
17	8	7	3.953E-14	3.556E-15
18	10	7	5.721E-15	2.221E-15
19	10	5	2.007E-15	1.135E-15
20	8	7	6.529E-15	2.592E-15
Sum			2.234E-13	4.072E-14

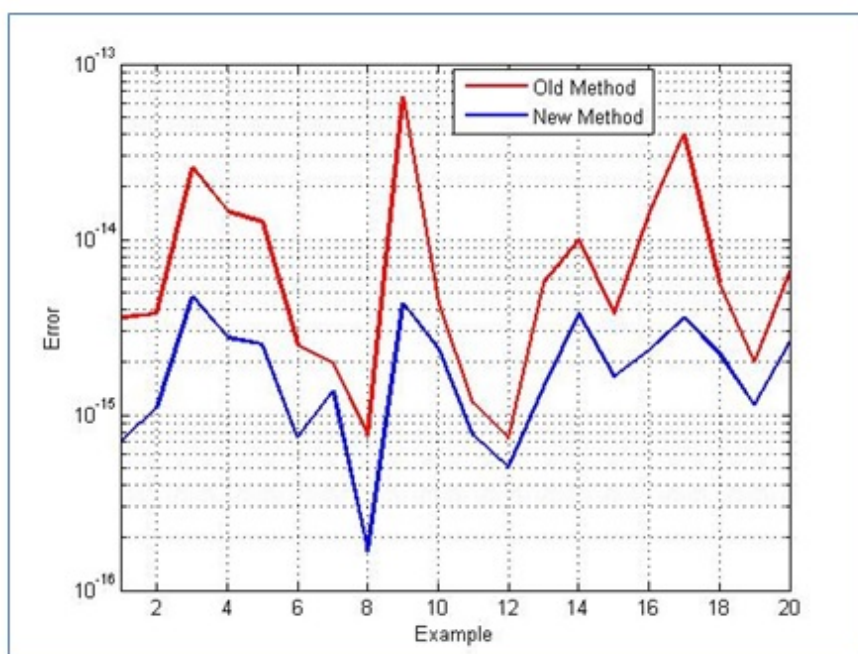


Figure 1: Error rates of both methods

Figure 1 shows the error rates of both methods in a semi-logarithmic plot. Accordingly, the novel method has a lower error rate than the baseline method.

6. Conclusion

This paper proposes a developed GramSchmidt algorithm (DGSA). In the baseline GS algorithm, vectors are selected randomly; however, this process is performed more sensitively in the proposed approach. A few numerical examples were employed to show the superiority of the proposed approach to the baseline method; hence, researchers are recommended to make further use of the proposed approach in their studies.

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