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The non-linear Dodson diffusion equation: Approximate solutions and beyond with formalistic fractionalization

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Abstract

The Dodson mass diffusion equation with exponentially diffusivity is analyzed through approximate integral solutions. Integral-balance solutions were developed to integer-order versions as well as to formally fractionalized models. The formal fractionalization considers replacement of the time derivative with a fractional version with either singular (Riemann-Liouville or Caputo) or non-singular fading memory. The solutions developed allow seeing a new side of the Dodson equation and to separate the formal fractional model with Caputo-Fabrizio time derivative with an integral-balance allowing relating the fractional order to the physical relaxation time as adequate to the phenomena behind.

Keywords: Non-linear diffusion, singular fading memory, non-singular fading memory, formal fractionalization, Caputo-Fabrizio derivative, integral balance approach.
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1. Introduction

1.1. Dodson diffusion equation

The Dodson diffusion equation (1.1) is related to diffusion in minerals [11, 12] related to cooling history in geology

$$\frac{\partial C(x,t)}{\partial t} = D_0 \exp(-\beta t)\frac{\partial^2 C(x,t)}{\partial x^2}, \qquad \beta = 1/\tau, \tag{1.1}$$

where the diffusion in solids is thermally promoted process [15, 41]. The same equation is relevant to other solid diffusion process in modern materials [30, 38].

Moreover, this is a common approach to model diffusion in solids [30, 38]. In solid diffusion, the dependence of the diffusion coefficient on the absolute temperature T can be simply expressed by the Arrhenius equation (1.2)

$$D(T) = D_0 \exp\left(-\frac{E}{RT}\right). \tag{1.2}$$

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The extreme value D_0 of the diffusion coefficient is related to processes at very high temperatures where the exponent in Arrhenius law approached unity; R is the universal gas constant and E is the diffusion activation energy.

According to original work of Dodson [11] over a certain temperature range, the inverse temperature $1/T$ [11, 12, 15, 33, 35, 41] varies linearly in time. With this empirically derived relationship the Arrhenius can be expressed as

$$D = D_0 \exp\left(-\frac{E}{RT_0} - \frac{t}{\tau}\right) = D_0 e^{-\frac{t}{\tau}}, \quad \frac{d}{dt}\left(\frac{E}{RT}\right) = \frac{1}{\tau}, \quad \beta = \frac{1}{\tau},$$

with initial conditions

$$D_0 = D(t=0), \quad T_0 = T(t=0).$$

With this construction of the diffusion coefficient hence, the diffusion equation can be expressed as [11, 12, 14, 15, 33, 35, 41] as (1.1).

Introducing the diffusional time of the systems as $t_D = a^2/D_0$ and the ratio $\tau D_0/a^2 = \tau/t_D$ the nondimensalization of (1.1) results in [11]

$$\frac{\partial(c)}{\partial\theta} = -\frac{\tau D_0}{a^2} e^{-\theta} \frac{\partial^2 c}{\partial \bar{x}^2}(c), \tag{1.3}$$

with $c = C/C_0$, $\theta = t/\tau$ and $\bar{x} = x/a$.

In the original work [11] $C_0 = C(x.0)$ is the initial concentration at $t = 0$. In the Dirichlet problem considered further at this article we will accept zero initial condition $C_0 = C(x.0) = 0$. In (1.3) the parameter a is the length scale of the finite area at issue. It was demonstrated in [27] that the ratio τ/t_D is the Deborah number of the diffusion process defined as $De = \tau D_0/a^2$.
Hence, the alternative form of (1.3)

$$\frac{\partial}{\partial\theta}(c) = -De\left(e^{-\theta}\frac{\partial}{\partial\bar{x}^2}(c)\right). \tag{1.4}$$

However, the exponential term remains and the equation is still non-linear.

Further, with the nonlinear transformations [10]

$$q = \left(1 - e^{-\theta} - c\right), \quad u = \int_0^t D(z) = M\left(1 - e^{-\theta}\right),$$

Equation (1.4) can be transformed in a homogeneous form as [11]

$$\frac{\partial q}{\partial u} = \frac{\partial^2 q}{\partial x^2}, \quad q_s = 1 - \left(1 - \frac{u}{M}\right)^{\lambda\tau}, \tag{1.5}$$

with initial condition $q = 0$ at $\theta = 0$. Moreover, for $\theta = 0$ we have $u = 0$ and for $\theta \to \infty$ we have $u \to M$. The parameter λ depends on geometry of the system at issue.

For the sake of clarity of the above equations in the original work of Dodson [11] the symbol $M = \tau D_0/a^2$ was used instead the Deborah number De introduced in [27].

The solution of (1.5) developed by Dodson [11] is based on the results of Carslaw and Jaeger [9] (especially the example at page 104, Equation 3) where for 1-D rectangular system it is

$$q = 2\sum_{i=1}^{\infty}(-1)^{i+1}\left(\frac{\cos(i-1/2)\pi x}{(i-1/2)\pi}\right)\left(1 - \frac{\Gamma(\lambda\tau+1)}{(i-1/2)^2\pi^2 M}\right)^{\lambda\tau}. \tag{1.6}$$

Certainly, the solution (1.6) is useful for computer calculations but quite inconvenient for engineering applications due to large number of terms of the series required. Moreover, the post-solution analyzes of the diffusion phenomena modelled by it are practically impossible since the terms of (1.6) have no physical meaning.

The Dodson equation is attractive for solutions due to its exponential non-linearity vanishing in time. In the book of Crank [10] which is a basic reference source for many scientists, however, this problem is mentioned briefly as Example 7.1 at page 104 (Chapter 7) without details beyond citing the results (1.5) and (1.6).

To complete this section it is noteworthy to mention that the Dodson equation (1.1) is a special (integer-order) form a time-fractional diffusion equation [27] expressed in terms of Caputo-Fabrizio fractional derivatives, when the product βt obeys the condition $\beta t << 1$.

1.2. *Problem formulation*

The present study focuses on approximate solutions and relevant analyzes of a generalized form of the Dodson equation

$$\frac{\partial^{\mu} C(x,t)}{\partial t^{\mu}} = D_0 e^{-\beta t} \frac{\partial^2 C(x,t)}{\partial x^2}, \qquad \beta = 1/\tau. \tag{1.7}$$

In the general form of expression (1.7) the time derivative $\partial^{\mu} C(x,t)/\partial t^{\mu}$ could be a left-sided time-fractional derivative of Riemann-Liouville, Caputo type with singular kernels [39] or Caputo-Fabrizio derivative with exponential non-singular kernel [7, 8] as well as the integer time-derivative when $\mu = 1$. Before any further analyzes, we have to stress the attention that the form (1.7) is outcome of the so-called formalistic fractionalization where simply the time derivatives of well-known integer-order models are mechanistically replaced by time-fractional derivative, a common approach in the existing literature [3, 16, 17, 31, 32, 40].

2. Mathematical preliminaries

2.1. *Time-fractional integral and derivatives*

In accordance with the Riemann-Liouville approach the fractional integral of order $\mu > 0$ is a natural result of the Cauchy formula reducing calculations of the m-fold primitive of a function $f(t)$ resulting in a single integral of convolution type [20] for arbitrary positive number $\mu > 0$, namely

$${}_0I^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-z)^{\mu-1} f(z) dz, \quad t > 0, \quad n \in \mathbf{R}^+.$$

For sake of convenience, we will also use the notation ${}_0D^{-\mu} f(t)$ for ${}_0I^{\mu} f(t)$. Further, the law of exponents for fractional integrals means ${}_0D^{-\mu}{}_0D^{-\gamma} f(t) =_0 D^{-\mu-\gamma} f(t) =_0 D^{-\gamma}{}_0D^{-\mu} f(t)$.

The Laplace transform of the fractional integral is defined by the convolution theorem as

$$L\left[{}_0D_t^{\mu} f(t)\right] = L\left[\frac{t^{\mu}}{\Gamma(\mu)}\right] L\left[f(t); s\right] = s^{-\mu} F(s),$$

where $\Re(s) > 0$, $\Re(s) > 0$ and $F(s)$ is the Laplace transform of $f(t)$.

2.1.1. *Riemann-Liouville time-fractional derivative*

Therefore, we may define the fractional derivative $D^{\mu} f(t)$ with by the relations [20] as: ${}_0D^{\mu}{}_0I^{\mu} = I$ but ${}_0I^{\mu}{}_0D^{\mu} \neq I$. Therefore, D^{μ} is left-inverse, but not right-inverse, to the integral operator I^{μ}. Hence, introducing a positive integer m such that $m-1 < \mu < m$, the natural definition of the Riemann-Liouville (left-sided) fractional derivative of order $\mu > 0$ is

$${}_0D^{\mu} f(t) =_0 D^m{}_0I^{m-\mu} f(t) = \frac{1}{\Gamma(m-\mu)} \frac{d^m}{dt^m} \int_0^t \frac{f(z)}{(t-z)^{\mu+1-m}} dz, \quad m-1 < \mu < m, \quad m \in \mathbf{N}.$$

Consequently, it follows that $D^0 = I^0 = I$, that is, $D^{\mu} I^{\mu} = I$ for $\mu \geqslant 0$.

In addition, the fractional derivative of power-law function and a constant, frequently used in this section are

$$ {}_0D^{\mu}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\mu)}t^{\beta-\mu}, \qquad {}_0D^{\mu}C = C\frac{t^{-\mu}}{\Gamma(1-\mu)}, \quad \mu>0, \quad \beta>-1, \quad t>0. $$

Similarly, the fractional integrals of the power-law function and a constant are

$$ {}_0D^{-\mu}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\mu)}t^{\mu+\beta}, \qquad {}_0D^{-\mu}C = \frac{C}{\Gamma(1+\mu)}t^{\mu}, \quad \mu \neq 1,2,\cdots. $$

The Laplace transform of the Riemann-Liouville fractional derivative for $m \in \mathbb{N}$ is

$$ L\left[\frac{d^m}{dt^m}f(t);s\right] = s^mF(s) - \sum_{k=0}^{m-1} s^{m-k-1}f^{(m)}(0+) = s^mF(s) - \sum_{k=1}^{m} s^{k-1}D_t^{\mu-k}f(0+), $$

where $\Re(s)>0$, $\Re(s)>0$ and $m-1<\mu<m$.

2.1.2. *Caputo time-fractional derivative*

The Caputo derivative of a casual function $f(t)$, i.e., $f(t)=0$ for $t<0$, is defined as [20]

$$ {}_CD_t^{\mu}f(t) = {}_0I^{m-\mu}\frac{d^m}{t^m}f(t) = {}_0D_t^{-(m-\mu)}f^{(m)} = \frac{1}{\Gamma(m-\mu)}\int_0^t \frac{f^{(m)}(z)}{(t-z)^{\mu+1-m}}dz, $$

where $m-1<\mu<m$.

The Laplace transform of Caputo derivative is

$$ L\left[{}_CD_t^{\mu}f(t);s\right] = s^{\mu}F(s) - \sum_{k=0}^{m-1} f^{(m)}(0)s^{\mu-k-1}. $$

Caputo derivative of a constant is zero, i.e., ${}_CD_t^{\mu}C=0$ that matches the common knowledge we have from the integer order calculus and for this reason the Caputo fractional derivative is the preferred one among engineers [39].

If $f(0)=f'(0)=f''(0)=\cdots=f^{(n)}(0)=0$, then both Riemann-Liouville and Caputo derivatives coincide. In particular for $\mu \in (0,1)$ and $f(0)=0$ one has ${}_CD_t^{\mu}f(t) =_{RL} D_t^{\mu}f(t)$. Further in this article we will use the notations ${}_CD_t^{\mu}f(t)$ and ${}_{RL}D_t^{\mu}f(t)$ to discriminate the effects on the solutions when both derivatives are used. In addition in some situations we will also use the notation $\partial^{\mu}C(x,t)/\partial t^{\mu}$ meaning a time-fractional derivative without specification of the type.

2.1.3. *Caputo-Fabrizio time-fractional derivative*

Caputo and Fabrizio [7] constituted a time-fractional derivative with a non-singular stretched- exponential kernel defined as

$$ {}_{CF}D_t^{\alpha}f(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t \exp\left[\frac{-\alpha(t-s)}{1-\alpha}\right]\frac{df(t)}{dt}ds = \frac{1}{1-\alpha}\int_0^t \exp\left[\frac{-\alpha(t-s)}{1-\alpha}\right]\frac{df(t)}{dt}ds. \tag{2.1} $$

In (2.1) $M(\alpha)$ such that $M(0)=M(1)=1$ is normalizing function. From the definition (2.1) it follows that if $f(t)=C=\text{const.}$, then ${}_{CF}D_t^{\alpha}=0$ as in the classical Caputo derivative [39]. An alternative definition of ${}_{CF}D_t^{\alpha}$ is given by [34]

$$ {}_{CF}D_t^{\alpha} = \frac{\alpha}{(1-\alpha)^2}\int_a^t [f(t)-f(a)]\exp\left[-\frac{\alpha}{1-\alpha}(t-s)\right]ds, \quad t>0. $$

The Laplace transform of ${}_{CF}D_t^{\alpha}$ with $a=0$ has the following Laplace transform given with variable [14]

$$ L\left[{}_{CF}D_t^{\alpha}f(t)\right] = \frac{sL\left[f(t)-f(0)\right]}{s+\alpha(1-s)}. \tag{2.2} $$

Applications of the Caputo-Fabrizio time-fractional derivative form a hot contemporary area of research which is very attractive for the scientists in the last 2 years after the seminal works [7, 8]. The basic idea was followed by numerous articles [2–4, 6, 16, 31, 32, 34, 40], initiated new computational techniques [17, 28], new definition of transient [26] and steady-state diffusion [25] equations, and boosted the creation of new fractional derivatives with non-singular kernels [1, 5].

With (2.2) the Caputo-Fabrizio derivative of exponential function $\exp(-\beta t)$ is at issue in the present work. Here we take into account that by definition ${}_{CF}D_t^{\alpha} f(t)$ is a convolution of $f(t)$ and the exponential kernel as well the factor $1/(1-\alpha)$. Hence, the Laplace transform $L\left[{}_{CF}D_t^{\alpha}\exp(\beta t)\right]$ for $\beta > 0$ is

$$L\left[{}_{CF}D_t^{\alpha}\exp(\beta t)\right] = \left(\frac{1}{1-\alpha}\right)\frac{\beta}{(s-\beta)}\frac{1}{(s+A)}, \quad A = \frac{\alpha}{1-\alpha}. \tag{2.3}$$

The inverse Laplace transform of (2.3) yields

$${}_{CF}D_t^{\alpha}\exp(\beta t) = \frac{\beta\exp(\beta t) - \exp(-At)}{\beta + \alpha(1-\beta)}. \tag{2.4}$$

For $\alpha \to 1$ the second exponential term in the nominator of (2.4) goes to zero and

$$\lim_{\alpha\to 1} {}_{CF}D_t^{\alpha}\exp(\beta t) = \beta\exp(\beta t).$$

For $\beta < 0$ the Laplace transform of ${}_{CF}D_t^{\alpha}\exp(-\beta t)$ is

$$L\left[{}_{CF}D_t^{\alpha}\exp(-\beta t)\right] = \left(\frac{1}{1-\alpha}\right)\frac{-\beta}{(s+\beta)}\frac{1}{(s+A)}. \tag{2.5}$$

Then, the inverse Laplace transform of (2.5) is

$${}_{CF}D_t^{\alpha}\exp(-\beta t) = \frac{-\beta\exp(-\beta t) + \exp(-At)}{A-\beta}. \tag{2.6}$$

For $\alpha \to 1$ the second term in the nominator of (2.6) goes to zero and

$$\lim_{\alpha\to 1} {}_{CF}D_t^{\alpha}\exp(-\beta t) = -\beta\exp(-\beta t).$$

2.2. *The integral-balance method*

The method used the concept that the diffusant penetrates the medium at a final depth δ and therefore the speed of the flux, thus this concept corrects the unphysical infinite speed of the flux inherent of the parabolic models. The introduction of the finite penetration depth δ transforms the problem with boundary conditions at infinity $C(\infty, t) = 0$ and $\partial C(\infty, t)/\partial x = 0$ as a two-point problem with boundary conditions at the front $\delta(t)$

$$C(\delta, t) = 0, \quad \frac{\partial C(\delta, t)}{\partial x} = 0. \tag{2.7}$$

The conditions (2.7) define a sharp-front movement $\delta(t)$ of the boundary between disturbed and undisturbed medium when an appropriate boundary condition at $x = 0$ is applied. The position of $\delta(t)$ is unknown and should be determined trough the solution. When the diffusivity is concentration and time-independent, the integration of, the integration of (1.1) for $\beta = 0$, over a finite penetration depth δ yields (2.7)

$$\int_0^{\delta}\frac{\partial C(x,t)}{\partial t}dx = \int_0^{\delta}\frac{\partial^2 C(x,t)}{\partial x^2}dx. \tag{2.8}$$

Applying the Leibniz rule to left-side of (2.8) we get

$$\frac{d}{dt}\int_0^{\delta} C(x,t)dx = -D_0\frac{\partial C(0,t)}{\partial x}. \tag{2.9}$$

Equation (2.9) is the principle relationship of the simplest version of the single-integration method [21] known as Heat-balance Integral Method (HBIM) [18, 19].

After this first step, replacing $C(x,t)$ by an assumed profile C_a (expressed as a function of the relative space co-ordinate x/δ) the integration in (2.9) results in an ordinary differential equation about δ [18, 19, 21, 36]. The principle problem emerging in application of (2.9) is that its right-side depends on the type of the assumed profile.The principle problem emerging in application of (2.9) is that its right-side depends on the type of the assumed profile [18, 19, 21, 36, 37].

The principle drawback of HBIM can be avoided by application of the double integration approach (DIM) [13, 22, 23, 36, 37]. In accordance with the technique of DIM the first is integration of the diffusion equations is from 0 to x and then the resulting equation is integrated again from 0 to δ. for the sake of simplicity, in explanation of the method, we assume that the diffusion coefficient is constant D_0, the principle relationship of DIM is [24, 36, 37]

$$\frac{d}{dt}\int_0^{\delta} xC(x,t)dx = D_0C(0,t).$$

An alternative expression of the DIM principle relationship applicable to fractional in time diffusion equations can be easily derived, too [22, 23], namely

$$\int_0^{\delta}\left(\int_x^{\delta}\frac{\partial C(x,t)}{\partial t}dx\right)dx = D_0C(0,t), \tag{2.10}$$

$$\frac{d}{dt}\int_0^{\delta}\left(\int_x^{\delta} C(x,t)dx\right)dx = D_0C(0,t). \tag{2.11}$$

When time-dependent term is expressed though a time-fractional derivative of order μ, then we have [22, 23]

$$\int_0^{\delta}\left(\int_x^{\delta}\frac{\partial^{\mu} C(x,t)}{\partial t^{\mu}}dx\right)dx = D_0C(0,t). \tag{2.12}$$

Since the Leibniz rule is not applicable yet to left-hand side of the integral relations.

The integral relations presented by (2.10) and (2.12) will be used further in this work in the development of the problems at issue.

3. Integer-order Dodson diffusion equation

3.1. Approximate solutions

3.1.1. HBIM solution

If the concept of finite penetration depth of the diffusant $\delta(t)$, which evolves in time, is assumed then the infinite speed of the flux propagation, which is inherent for the parabolic model (1.1), could be ad hoc avoided. The finite penetration depth is the basic concept of the integral balance method of Goodman [13, 18, 19, 21, 22] which is a simple mass balance over the depth of the diffusion layer with a thickness δ, namely

$$\int_0^{\delta}\frac{\partial C(x,t)}{\partial t}dx = \int_0^{\delta} D(t)\frac{\partial^2 C(x,t)}{\partial x^2}dx \Rightarrow \frac{d}{dt}\int_0^{\delta} C(x,t)dx = D(t)\frac{\partial C(0,t)}{\partial x}, \tag{3.1}$$

$$D(t) = D_0 e^{-\beta t}, \quad \beta = 1/\tau,$$

with conditions (2.7) at the front $\delta(t)$.

The second version of (3.1) is a result of application of the Leibniz rule for integration under the integral sign as was demonstrated by (2.9). The thickness of the diffusion layer $\delta(t)$ can be determined as a step of the solution by using assumed profile expressed as a function of the dimensionless space variable x/δ.

With the transform $u = 1 - e^{-\beta t}$ leading to $\partial C/\partial t = (\partial C/\partial u)(\partial u/\partial t)(\beta e^{-\beta t})$ we may express (1.1) as

$$\frac{\partial C(x,u)}{\partial u} = \frac{D_0}{\beta}\frac{\partial^2 C(x,u)}{\partial x^2}. \tag{3.2}$$

Then the integral-balance equation (2.9) of HBIM is

$$\frac{d}{dt}\int_0^{\delta} C(x,u(t))dx = -\frac{D_0}{\beta}\frac{\partial C(0,u(t))}{\partial x}.$$

Suggesting the assumed profile as a parabolic profile with unspecified exponent $C_a(x,u(t)) = C_s(1-x/\delta)^n$ [13, 21, 24, 26, 36, 37] and for the sake of simplify considering the Dirichlet problem with $C_s = 1$. Then, the replacement of $C(x,t)$ in (3.2) by C_a leads to [27]

$$\frac{1}{n+1}\frac{d\delta}{du} = \frac{D_0}{\beta}\frac{n}{\delta} \Rightarrow \delta^2 = \frac{D_0}{\beta}u[2n(n+1)].$$

In terms of the original variables the penetration depth $\delta(t)$ is

$$\delta = \sqrt{\frac{D_0}{\beta}}\sqrt{\left(1-e^{-\beta t}\right)[2n(n+1)]}. \tag{3.3}$$

The ratio D_0/β has dimension of $[m^2]$, while the non-linear time function $1-e^{-\beta t}$, growing in time, saturates rapidly to 1 as $1/e^{\beta t} \to 0$ for large times, i.e., $t \to \infty$. In this extreme case $\delta \equiv \sqrt{D_0/\beta} = \text{const.}$, that is the penetration depth stops to evolve in time and the diffusion process ceases.

3.1.2. DIM solution

Applying the integral relationship (2.11) to the transformed equation (3.2) we get

$$\frac{d}{dt}\int_0^{\delta}\left(\int_x^{\delta} C(x,u)\right)dx = \frac{D_0}{\beta}C(0,u). \tag{3.4}$$

With assumed profile $C_a(x,u(t)) = C_s(1-x/\delta)^n$ and $C(0,t) = C(0,u) = C_s = 1$ the integral relation (3.4) results in

$$\frac{1}{(n+1)(n+2)}\frac{d}{dt}\delta^2 = \frac{D_0}{\beta} \Rightarrow \delta = \sqrt{\frac{D_0}{\beta}}\sqrt{\left(1-e^{-\beta t}\right)(n+1)(n+2)}.$$

The ratio $\Delta_{DIM} = \delta/\sqrt{D_0/\beta}$ is dimensionless and actually defines the numerical factor depending on the exponent n and, the more important issue, the relaxation function $\left(1-e^{-\beta t}\right)$.

3.1.3. Approximate integral-balance solutions

From the HBIM and DIM solutions we get two approximate solutions

$$C(x,t)_{HBIM} = \left(1 - \frac{x}{\sqrt{\frac{D_0}{\beta}}\sqrt{\left(1-e^{-\beta t}\right)[2n(n+1)]}}\right)^n, \tag{3.5}$$

$$C(x,t)_{DIM} = \left(1 - \frac{x}{\sqrt{\frac{D_0}{\beta}}\sqrt{(1-e^{-\beta t})\,[(n+1)(n+2)]}}\right)^n. \tag{3.6}$$

For large times, i.e., $t \to \infty$, when $1/e^{\beta t} \to 0$ the steady-state profiles defined by the integral solutions are

$$C(x,\infty)_{HBIM} = \left(1 - \frac{x}{\sqrt{\frac{D_0}{\beta}}\sqrt{2n(n+2)}}\right)^n, \tag{3.7}$$

$$C(x,\infty)_{DIM} = \left(1 - \frac{x}{\sqrt{\frac{D_0}{\beta}}\sqrt{(n+1)(n+2)}}\right)^n. \tag{3.8}$$

The approximate profiles (3.5) and (3.6) define a non-conventional similarity variable defined as

$$\eta_u = \frac{x}{\sqrt{\frac{D_0}{\beta}u}} = \frac{x}{\sqrt{\frac{D_0}{\beta}\left(1-e^{-\beta t}\right)}}. \tag{3.9}$$

In the space (x,u) (3.5) and (3.6) have known optimal exponents [36, 37], namely: for HBIM solution $n_{opt(HBIM)} \approx 2.233$ and $n_{opt(DIM)} \approx 2.218$ for DIM solution. This allows us to complete the integral-balance solutions

3.2. *Exact similarity solution*

With the nonlinear change of variables $u = 1 - \exp(-\beta t)$ the original equation (1.1) can be expressed in the (x,u) space as (3.2). Using the well-known technology of the similarity solution through the Boltzmann variable we define (inspired by the integral-balance solutions and the homogenous form of (3.2) in the (x,u) space), a similarity variable

$$\eta_u = \frac{x}{\sqrt{D_\beta u}}, \qquad D_\beta = \frac{D_0}{\beta}. \tag{3.10}$$

Even though the approximate solutions are at the focus of this study, we stress the attention on a possibility to develop a solution with a technique well-known from the textbooks. However, now we will use the unconventional similarity variable η_u defined by (3.10).

Looking for a solution in the form $C(x,t) = u^k g(\eta_u)$, where exponent k and the function $g(\eta_u)$ have to be determined through the solution. With the new similarity variable transforming (3.2) we have

$$u^{k-1}\left(kg - 1\frac{1}{2}\eta_u\frac{dg}{du} - \frac{d^2g}{du^2}\right) = 0.$$

Hence, the equation that should be solved is

$$\frac{dg}{d^2u^2} + \frac{1}{2}\eta_u\frac{dg}{du} = kg. \tag{3.11}$$

Now, let us consider the special case with $k = 0$. From the definition $C(x,t) = u^k g(\eta_u)$, we immediately get the Boltzmann similarity variable. For $k = 0$, equation (3.11) reduces to

$$\frac{dg}{d^2u^2} + \frac{1}{2}\eta_u\frac{dg}{du} = 0 \Rightarrow \frac{dg}{du} = B\exp\left(-\frac{1}{4}\eta_u^2\right) \Rightarrow g(\eta_u) = B\int_{-\infty}^{\eta_u}\exp\left(-\frac{1}{4}\eta_u^2\right)d\eta_u,$$

where B should be defined through the solution.

Therefore, we get the well-known solution expressed through the Gauss error-function $1-\mathrm{erf}(\eta_u/2)$.

However, this solution would face problems in calculations since the backward transform to the original variables in the (x,t) space is non-linear and hard to be handled. Because of that, denoting $Q = g\left[\exp(\eta_u^2/4)\right]$ we may transform (3.11) as

$$\frac{d}{du^2}Q - \frac{1}{2}\eta_u \frac{d}{du}Q = \left(k + \frac{1}{2}\right)Q,$$

The trivial solution for $k = -1/2$ is $Q = b = \text{const}$. From this point of view we have

$$g(\eta_u) = b\left[\exp\left(-\frac{1}{4}\eta_u^2\right)\right].$$

Therefore, in the (x,u) space the solution has a simple form as

$$C(x,u) = b\frac{1}{\sqrt{u}}\exp\left(-\frac{x^2}{4D_\beta u}\right).$$

Further, in terms of the original variables (x,t) we have

$$C(x,t) = b\frac{1}{\sqrt{1-\exp(-\beta t)}}\exp\left(-\frac{x^2}{4D_\beta[1-\exp(-\beta t)}]\right).$$

With the boundary condition $C(0,t) = C_s$ we may denote $b = C_s\sqrt{1-\exp(-\beta t)}$. Therefore the solution (assuming $C_s = 1$ for the Dirichlet problem) is

$$C(x,t) = \exp\left(-\frac{x^2}{4D_\beta[1-\exp(-\beta t)]}\right). \tag{3.12}$$

At large time when $t \to \infty$ from (3.12) we get a steady-state profile

$$C(x,\infty) = \exp\left(-\frac{x^2}{4D_\beta}\right). \tag{3.13}$$

The steady-state solution (3.13) automatically defines the similarity variable $x/\sqrt{D_\beta}$, which we will see as naturally appearing dimensionless group in the approximate solutions when the Caputo-Fabrizio time derivative is involved in the fractional version of (1.1). From (3.13) it is clear that it is impossible to determine the depth the diffusion layer (the boundary between the disturbed $C(x,t) \neq 0$ and the undisturbed area $C(x,t) = 0$ of the medium). If the result (3.13) is used, we may define, for convenience, and mainly from practical point of view, that the front is defined by point where $C(x,\infty) = C(x_{final}) = \epsilon C(0) = \epsilon \neq 0$, where $\epsilon << 1$. Then $x_{final} = \sqrt{(D_0/\beta)}\sqrt{(4\ln\epsilon)}$. This result differs from (3.7) and (3.8) where the parabolic profile defines $x_{final(HBIM)} = \sqrt{D_0/\beta}\sqrt{2n(n+1)}$ and $x_{final(DIM)} = \sqrt{D_0/\beta}\sqrt{(n+1)(n+2)}$.

As final comments, it is noteworthy that we used the classical diffusion equation in the space (x,u). However, the initial governing equation (1.1) in the (x,t) space is non-linear. The change of variables $t \to u$ which is linearizing the governing equation is a result of the non-linear transform $u = \int_0^t e^{-\beta z}dz = \left(1 - e^{-\beta t}\right)$ (see [10, Chapter 7]).

4. Formal fractional models

4.1. *Integral-balance solution by DIM*

From now we will continue with DIM technology in the solution of the formal fractional versions of the Dodson equation, presented by (1.7).

Applying the integral relation (2.12) we have in the left side the double integral of the fractional derivative of the assumed profile $C_a = (1 - x/\delta)^n$, namely

$$\int_0^\delta \left(\int_x^\delta \frac{\partial^\mu}{\partial t^\mu} C_a(x,t) dx \right) dx = J_2 = \int_0^\delta \left(\int_x^\delta \frac{\partial^\mu}{\partial t^\mu} \left(1 - \frac{x}{\delta}\right)^n dx \right) dx. \tag{4.1}$$

As it was demonstrated in previous works [22, 23] the result of the integration in (4.1), irrespective of the type of fractional derivative used (Riemann-Liouville or Caputo) is

$$J_2 = \frac{\partial^\mu}{\partial t^\mu} \left[\frac{\delta^2}{(n+1)(n+2)} \right], \tag{4.2}$$

because we have a physically defined condition $\delta(t=0)=0$, that there is no diffusion at $t=0$. However, for the fractional derivatives in (4.2) this means that we have zero initial condition that makes $\frac{\partial^\mu}{\partial t^\mu}\delta^2(t) = _{RL} D_t^\mu \delta^2 = _C D_t^\mu \delta^2$ [39].

Therefore, the fractional ordinary equation with respect to $\delta(t)$ relevant to the problem at issue can be expressed as

$$\frac{\partial^\mu}{\partial t^\mu} \left[\frac{\delta^2}{(n+1)(n+2)} \right] = D_0 e^{-\beta t} C(0,t).$$

With $C(0,t)=1$ corresponding to the assumed profile $C_a = (1-x/\delta)^n$ we get

$$\frac{\partial^\mu}{\partial t^\mu}\delta^2 = D_0 e^{-\beta t} N, \qquad N = (n+1)(n+2). \tag{4.3}$$

For the special case with $\beta = 0$ we get [22, 23] from (4.3)

$$\delta^2 = D_0 N \frac{1}{\Gamma(1+\mu)} t^\mu, \tag{4.4}$$

and $\delta \equiv \sqrt{D_0 t^\mu}$. However, now, we have to solve (4.3) with $\beta \neq 0$.

4.1.1. Fractional models with Riemann-Liouville and Caputo derivatives

With Riemann-Liouville derivative applying the equation (4.3) simply yields

$$_{RL}D_t^\mu \delta^2 = D_0 e^{-\beta t} N. \tag{4.5}$$

Equation (4.5) can be re-arranged as

$$\frac{d}{dt}\delta^2 = _{RL} D_t^{1-\mu}(D_0 e^{-\beta t} N). \tag{4.6}$$

The exponential function $e^{\beta t}$ can be expressed as an infinite series through the Mittag-Leffler function

$$e^{\beta t} = E_{1,1}(\beta t) = \sum_{i=0}^{\infty} \frac{(\beta t)^i}{\Gamma(i+1)} = \sum_{i=0}^{\infty} \frac{\beta^i t^i}{\Gamma(i+1)}.$$

Hence, we may present $e^{-\beta t}$ as

$$e^{-\beta t} = \sum_{i=0}^{\infty} \frac{(-1)^i \beta^i t^i}{\Gamma(i+1)}. \tag{4.7}$$

Now, with (4.7) we have

$$_{RL}D_t^{1-\mu} e^{-\beta t} = _{RL} D_t^{1-\mu} \sum_{i=0}^{\infty} \frac{(-1)^i \beta^i t^i}{\Gamma(i+1)}. \tag{4.8}$$

With the simple rule $_{RL}D_t^{1-\mu}(t^i) = \frac{\Gamma(i+1)}{\Gamma(i+\mu)} t^{i+\mu-1}$ we may express (4.8) as

$$_{RL}D_t^{1-\mu}e^{-\beta t} = \sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(i+\mu)}\beta^i t^{i+\mu-1} = t^{\mu}\sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(i+\mu)}\beta^i t^{i-1}.$$

Now, from (4.6) and (4.8) we have

$$\frac{d}{dt}\delta^2 = (D_0 N)\sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(i+\mu)}\beta^i t^{i+\mu-1}. \tag{4.9}$$

Then, the integration in (4.9) yields

$$\delta^2 = (D_0 N)\sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(i+\mu)}\beta^i \frac{t^{i+\mu}}{i+\mu}\delta^2.$$

Therefore,

$$_{RL}\delta_{\mu} = \sqrt{(D_0 N)}\left(\sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(1+i+\mu)}\beta^i t^{i+\mu}\right)^{\frac{1}{2}}. \tag{4.10}$$

Extracting the $i = 0$ term in (4.10) we have

$$_{RL}\delta_{\mu} = \sqrt{(D_0 t^{\mu})}\sqrt{N}\left(\frac{1}{\Gamma(1+\mu)} - \sum_{i=1}^{\infty} \frac{(-1)^i}{\Gamma(1+i+\mu)}(\beta t)^i\right)^{\frac{1}{2}}. \tag{4.11}$$

Neglecting the sum in (4.11) we get $\delta_{\mu} \approx \sqrt{D_0 t^{\mu}}\sqrt{N/\Gamma(1+\mu)}$. The reason for this approximation is the possibility to express $e^{-\beta t}$ as a series $e^{-\beta t} \approx 1 - \beta t + (\beta t)^2/2 - (\beta t)^3/6 + O((\beta t)^4)$ and neglecting all terms containing β, or actually accepting $\beta = 0$, which means $D_0 e^{-\beta t} \approx D_0$ corresponding to the case with constant diffusion coefficient [21, 24]. Therefore, the result (4.11) reduces to (4.4) for $\beta = 0$. For $\beta \neq 0$ we may express approximately the penetration depth as

$$\delta_{\mu} = \sqrt{(D_0 t^{\mu})}\sqrt{N}\left(\frac{1}{\Gamma(1+\mu)} - \frac{\beta t}{\Gamma(2+\mu)} + \frac{\beta^2 t^2}{\Gamma(3+\mu)} - \frac{\beta^3 t^3}{\Gamma(4+\mu)} + O(\beta^4 t^4)\right).$$

Expressing

$$_{RL}\delta_{\mu} = \sqrt{\left(\frac{D_0}{\beta}t^{\mu}\right)}\sqrt{N}\left(\frac{\beta}{\Gamma(1+\mu)} - \frac{\beta^2 t}{\Gamma(2+\mu} + \frac{\beta^2 t^2}{\Gamma(3+\mu)} - \frac{\beta^4 t^3}{\Gamma(4+\mu)} + O(\beta^5 t^4)\right), \tag{4.12}$$

we try to extract the pre-factor $\sqrt{(\frac{D_0}{\beta}t^{\mu})}$ as in the integer order solution. Then, from (4.10) and (4.12) we have

$$\delta_{\mu} = \sqrt{\frac{D_0}{\beta}}\left(\sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(1+i+\mu)}\beta^{i+1} t^{i+\mu}\right)^{\frac{1}{2}} \sqrt{(n+1)(n+2)}. \tag{4.13}$$

The ratio $\Delta_{\mu} = \delta_{\mu}/\sqrt{D_0/\beta}$ is the scaled penetration depth. It is dimensionless and actually defines the numerical factor depending on the exponent n, and the more important issue, the relaxation function $\left(\sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(1+i+\mu)}\beta^{i+1}t^{i+\mu}\right)^{\frac{1}{2}}$ which for $\mu \to 1$ approaches $\sqrt{(1-e^{-\beta t})}$.

The result (4.13), but more clearly (4.10), shows how $\delta_{\mu}(t)$ evolves in time and allows to see physically how the additional relaxation effect introduced by the fractional derivative $\partial^{\mu}C(x,t)/\partial t^{\mu}$, formally replacing $\partial C(x,t)/\partial t$ in the diffusion equation, alters the solution.

4.1.2. Fractional models with Caputo derivative

Representing the decaying exponential term of the diffusion coefficient as (see (4.7))

$$e^{-\beta t}=\sum_{i=0}^{\infty}\frac{(-1)^i\beta^i t^i}{\Gamma(i+1)}=1+\sum_{i=1}^{\infty}\frac{(-1)^i\beta^i t^i}{\Gamma(i+1)}.$$

It is easy to see that

$${}_C D_t^{1-\mu}e^{-\beta t}={}_C\, D_t^{1-\mu}\left[1+\sum_{i=1}^{\infty}\frac{(-1)^i}{\Gamma(i+\mu)}\beta^i t^{i+\mu-1}\right]=\sum_{i=1}^{\infty}\frac{(-1)^i}{\Gamma(i+\mu)}\beta^i t^{i+\mu-1}=t^{\mu}\sum_{i=1}^{\infty}\frac{(-1)^i}{\Gamma(i+\mu)}\beta^i t^{i-1}.$$

Then, the solution is practically the same as that performed with the Riemann-Liouville derivative. The result about the penetration depth is

$${}_C\delta_\mu=\sqrt{\frac{D_0}{\beta}}\left(\sum_{i=1}^{\infty}\frac{(-1)^i}{\Gamma(1+i+\mu)}\beta^{i+1}t^{i+\mu}\right)^{\frac{1}{2}}\sqrt{(n+1)(n+2)}. \tag{4.14}$$

4.2. Fractional models with Caputo-Fabrizio derivative

Now, with the Caputo-Fabrizio derivative the equation about $\delta(t)$ is (we omit the prefix and use as symbol of the fractional order to distinguish the solution from the cases with singular fractional derivatives)

$$D_t^{\alpha}\delta^2=D_0Ne^{-\beta t}. \tag{4.15}$$

Here we will apply a different strategy for the solution. The Laplace transform of (4.15), taking into account that $\delta(0)=0$, is

$$\frac{s\delta^2(s)}{s+\alpha(1-s)}=D_0N\left(\frac{1}{s+\beta}\right)\Rightarrow\delta^2(s)=\left(\frac{1-\alpha}{s+\beta}\right)+\frac{\alpha}{s}\left(\frac{1}{s+\beta}\right). \tag{4.16}$$

The inverse Laplace transform of (4.16) is

$$\delta^2=\frac{D_0}{\beta}N\left[\alpha\left(1-e^{-\beta t}\right)+\beta(1-\alpha)e^{-\beta t}\right]. \tag{4.17}$$

Now, expressing (4.17) in a form similar to the solution of the integer-order problem, we get

$${}_{CF}\delta_\alpha=\sqrt{\frac{D_0}{\beta}}\sqrt{N}\sqrt{(1-e^{-\beta t})}\sqrt{\alpha+(\alpha-1)\frac{\beta}{(e^{-\beta t}-1)}}. \tag{4.18}$$

For $\alpha=1$, (4.18) reduces to the integer order solution (see (3.2))

$${}_{CF}\delta_\alpha=\sqrt{\frac{D_0}{\beta}}\sqrt{N}\sqrt{(1-e^{-\beta t})}\sqrt{(n+1)(n+2)}.$$

The scaled penetration depth $\Delta_\alpha=\delta_\alpha/\sqrt{D_0/\beta}$ actually defines the same relation function $\left(1-e^{-\beta t}\right)$ as in the case of $\alpha=1$. From this point of view, more realistic formal fractionalization could be obtained by replacement of the time derivative in the Dodson equation by the Caputo-Fabrizio time derivative.

The ratio $\delta_\alpha/\delta_{\alpha=1}$ is a measure of the deviation of the penetration depth from the case with $\alpha=1$. Hence we have

$$\frac{{}_{CF}\delta_\alpha}{\delta_\alpha}=\sqrt{\alpha-(1-\alpha)\frac{\beta}{1-e^{-\beta t}}}. \tag{4.19}$$

For $\alpha = 1$ we get immediately $\delta_\alpha/\delta_{\alpha=1} = 1$. Since from the analysis done in [27] the Dodson equation corresponds to the case when $\beta t << 1$, the denominator in the second term, i.e., $1 - e^{-\beta t}$ approximates as $1 - e^{-\beta t} \approx \beta t$. Then, (4.19) can be approximated as

$$\left(\frac{{}_{CF}\delta_\alpha}{\delta_\alpha}\right)_{(\beta t << 1)} = \sqrt{\alpha - (1-\alpha)\frac{1}{t}}, \tag{4.20}$$

or alternatively as

$$\left(\frac{{}_{CF}\delta_\alpha}{\delta_\alpha}\right)_{(\beta t << 1)} \approx \sqrt{\alpha(1-\beta) + \beta}. \tag{4.21}$$

Now, recall that, from the analysis done in [27] the Dodson equation (see also the Introduction) corresponds to the case when $\beta << 1$ then when the product $\beta t << 1$, the time-dependent denominator in the second term of (4.18), i.e., can be approximated as $1 - e^{-\beta t} \approx 1 - (1 - \beta t + O(\beta^2 t^2) \approx \beta t$. Then, for large times (4.20) approximates as

$$\left(\frac{{}_{CF}\delta_\alpha}{\delta_\alpha}\right)_{(\beta t << 1)} \approx \sqrt{\alpha},$$

and demonstrates explicitly what the effect of fractional order α is the same results can be obtained from (4.21) and (4.21) if β is neglected with respect α.

Hence, physically the decrease in α reduces the penetration depth and the diffusion ceases rapidly in contrast to cases with large values of α. That is, the distance from the surface $x = 0$ at which the diffusant stops to penetrate in depth depends on the fractional order α, in fact decreases with decrease in α and vice versa.

Moreover, as it was developed in [27] the rate coefficient is represented as $\beta = (1-\alpha)/\alpha$ and then for $\alpha = 0.5$, for example we get $\beta = 1$, while for $\alpha = 0.25$ we have $\beta \approx 3$. Therefore, the range of application of the Dodson equation, for $\beta < 1$ corresponds to the range $0.5 < \alpha < 1$ and more realistically when $\alpha \longrightarrow 1$, since physically β is defined as inverse of the relaxation time, i.e., $\beta = 1/\tau$.

Finally, the approximate solution in case of Caputo-Fabrizio time derivative is

$$C^\alpha(x,t)_{DIM} = \left(1 - \frac{x}{\sqrt{\frac{D_0}{\beta}}\sqrt{N}\sqrt{(1-e^{-\beta t})}\sqrt{\alpha + (\alpha-1)\frac{\beta}{(e^{-\beta t}-1)}}}\right)^n, \tag{4.22}$$

and for large times the steady-state profile can be approximated as

$$C^\alpha(x,\infty)_{DIM} = \left(1 - \frac{x}{\sqrt{\frac{D_0}{\beta}}\sqrt{N}\sqrt{\alpha(1-\beta)+\beta}}\right)^n. \tag{4.23}$$

When $\beta t << 1$ from (4.22) (as well as from (4.23) neglecting β) we get

$$C^\alpha(x,\infty)_{DIM} = \left(1 - \frac{x}{\sqrt{\frac{D_0}{\beta}}\sqrt{N}\sqrt{\alpha}}\right)^n. \tag{4.24}$$

The final concentration profiles $C^\alpha(x,\infty,\alpha)_{DIM}$ are presented in Figure 1 for stipulated exponent $n = 3$ (see the comments in the next point). The two-dimensional profiles (Figure 1 a) clearly demonstrate the retardation effect of the fractional order α. More generalized presentation by the 3-D profile in Figure 1 b shows the evolution of the penetration depth $\alpha(\alpha)$ in the front plane. It is clear that the profile for $\alpha = 0.1$ goes rapidly to zero while at the same time (the same value of the similarity variable η_u the profile with $\alpha = 1$ (no retardation effects) reaches about 0.2 time of the surface concentration (at $x = 0$).

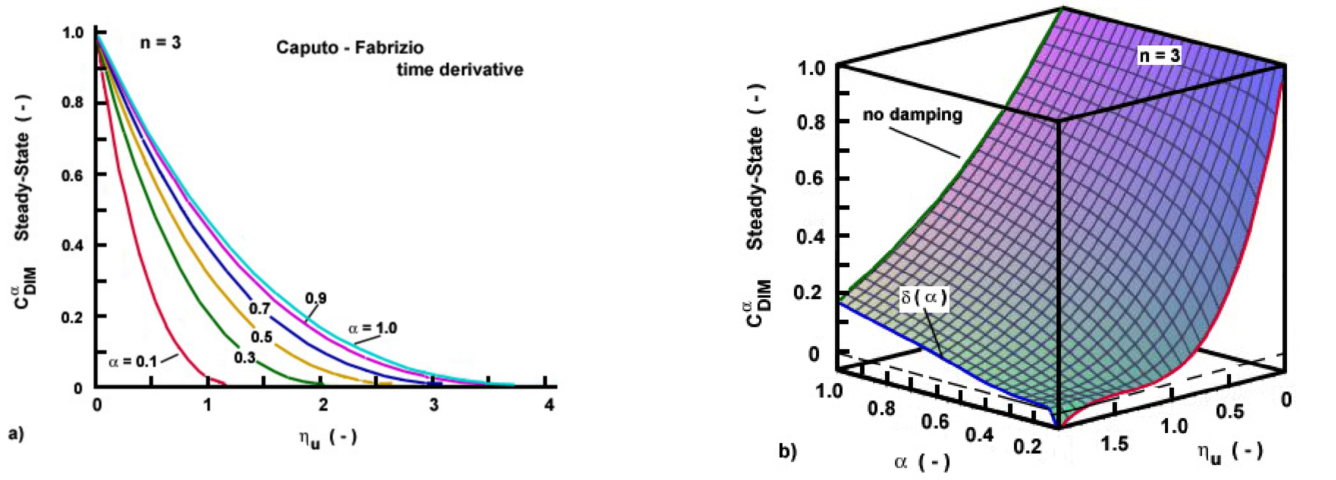

Figure 1: Steady-state concentration profiles: a) Two-dimensional; b) Three-dimensional plot. In both cases $n = 3$.

To this end, we have to mention that similar retardation effect through the fractional order μ can be observed when the time-fractional derivative has singular fading memory. To be clear, look at (4.11) where if the sum is neglected due to the small value of β as in the case presented by (4.24), then the penetration depth scales as ${}_{RL}\delta_{\mu} = \sqrt{D_0 t^{\mu}}\sqrt{N}/\Gamma(1+\mu)$ which is the result obtained in [23] and [22]. In this case the sum in β is not neglected this causes a problem in definition of the similarity variable as well as difficulties in definition how many terms of the sum should be taking in account in the calculations. From this point of view, the use of the Caputo-Fabrizio time-derivative leads to a formal fractional models which is mimicking the expected real-world behavior, that is the retardation effect. A serious advantage of the model with the Caputo-Fabrizio time-derivative is the possibility to relate the fractional order α and the rate constant β, which is task without success when the Riemann-Liouville and Caputo derivative with power-law memory are used.

4.3. *The exponent of the approximate profile*

This study demonstrates the principle solution of the formally fractionalized versions of the Dodson equation. The exponent of the profile could be either stipulated or determined by optimization procedure minimizing the residual function, precisely the mean-squared error of approximation over the penetration length δ. In this case, the principle problem emerging in evaluation of the residual function is the time-fractional derivative of composite function since the assumed profile is $[1 - x/\delta(t)]^n$. In the simple case when $\beta = 0$ and the derivatives were with singular kernels [23], the problem was resolved by expressing the profile as a power-law series of the similarity variable $\eta_u = x/\sqrt{D_0 t^{\mu}}$. However, in the case of the Dodson equation we have an additional nonlinearity due to the exponential term in the diffusion coefficient. If the fractional order is $\mu = 1$, then we obtain the integer order model and the solution is with known optimal exponents if the similarity variable is defined by (3.9) (see the comments about (3.9)). Moreover, if the Caputo-Fabrizio fractional derivative is taken as time derivative, then the same problem emerge and should be resolved.

The development of approximate fractional derivative of the approximate profiles is beyond the scope of the present work but we will mark some restrictions that should be obeyed by the exponent n. If the residual function is presented generally as

$$R = \left[\frac{\partial t^{\mu}}{\partial t^{\mu}}\left(1 - \frac{x}{\delta}\right)^{n} - D_0 e^{-\beta t}\frac{n(n-1)}{\delta^2}\left(1 - \frac{x}{\delta}\right)^{n-2}\right].$$

Then when at the vicinity of the front, the condition for a positive solution is as in the case of integer-order models [13, 22–24, 36, 37] the general condition is $n > 2$. Hence, the exponent n could be stipulated and the approximate solutions could be straightforwardly developed as it was demonstrated in Figure 1. In this case the error of approximation is predetermined. The alternative way with optimal exponent is still undeveloped. Despite the incomplete solutions in this direction we may comment and compare the results of the basis of the functional relationships of the front $\delta(t)$ and this the problem discussed in the next point.

4.4. Comments on the integral-balance of the formal fractional models

The approximate integral-balance solutions (either HBI or DIM) ((3.3) or (3.7)) of the integer-order model (1.1) provide that front propagates as $\delta(t) \equiv \sqrt{(D_0/\beta)\,(1-e-\beta t)}$ and when the diffusion process ceases we have $\delta_{final} \equiv \sqrt{(D_0/\beta)}$. The same result follows from the exact solution (3.13) where $x_{final} \equiv \sqrt{(D_0/\beta)}$.

The formal fractionalization implements ad-hoc a relaxation through the time-fractional derivative. When the time fractional derivative is with a singular kernel (either Riemann-Liouville or Caputo) we have that $\delta_{\mu}(t) \equiv \sqrt{D_0 t^{\mu}}$ and explicit effect the rate constant $\beta = 1/\tau$ disappears. Moreover, these solutions (see (4.11) and (4.14)) involve infinite series that makes them hard to be implemented in practical calculations. A principle difficulty emerging in this type of fractionalization is that the fractional order μ and the rate constant β are interrelated, that is, there are no physical reasons to create a functional relationship between them.

However, when the Caputo-Fabrizio time derivative is ad-hoc used the front propagate as in the case of the integer-order model, i.e., ${}_{CF}\delta(t) \equiv \sqrt{(D_0/\beta)\,(1-e-\beta t)}$ and the retardation factor is approximately proportional to $\sqrt{\alpha}$. Then, the final depth is ${}_{CF}\delta(\infty) \equiv \sqrt{(D_0/\beta)\alpha}$. This behaviour is more physically adequate in contrast to the other approaches to fractionalize the model since the retardation effect is obvious. Moreover, as commented earlier and established in [27] we have $\beta = \alpha/(1-\alpha)$ and $0 < \alpha = \beta/(1+\beta) \leqslant 1$. Then, the solution can be expressed through the fractional order α or through the rate constant β.

To recapitulate, the formal fractionalization with the Caputo-Fabrizio derivative results in an approximate solution where the formal fractionalization does not change the law of the time evolution of the front, in contrast to the case when time-derivatives with singular kernels are used. Moreover, the solution about the penetration front explicitly demonstrates the retardation effect expressed through the fractional order. As a support of the model with the Caputo-Fabrizio derivative we may refer to the results in [25, 26] and [27] where the correct fractionalization through definition of an exponential damping function of the diffusion flux does not affect the time derivative of the starting parabolic model (1.1).

5. Conclusions

This work developed approximate integral-balance solutions of the non-linear Dodson equation in its integer-order form and formally fractionalized versions. As a good example, it was demonstrated that an exact similarity solution is also available after an initial non-linear transformation. In fact, these are the first developed solutions where the time evolution of the front of the solution can be explicitly presented.

As intermediate results relevant to solution developed it was presented how the time-fractional Caputo-Fabrizio derivative works with exponential function $\exp(\pm\beta t)$. This was missing information in articles published on problems involving Caputo-Fabrizio derivative.

The comparative analysis indicates that the use of the Caputo-Fabrizio time derivative allows creating formally fractionalized models with an integral-balance solution which can be easily analyzed. Moreover, this solution directly generates a non-Boltzmann similarity variable incorporating the diffusion coefficient and the rate constant inverse of the physically defined relaxation time τ.

Since the Dodson equation was originally developed to model the final penetration depth of the diffusion and recovering from that the age of the geological formation, the results developed for the steady-state regime when the Caputo-Fabrizio derivative is used are explicit and physically adequate. Moreover, this solution allows directly relating the fractional order of the fractionalized equation to the physically defined rate constant of the diffusion coefficient (inverse to the relaxation time). Similar results developed with the time-fractional derivatives with singular kernels (Riemann-Liouville and Caputo) also incorporate retardation effects due to the fractional order but two principle problems emerge: the solutions about the penetration depths contain infinite series that hinders definition of a similarity variable, and the relationship of the fractional order μ and the physically defined relaxation time τ could not be straightforwardly established. These are problems that might be resolved in other studies beyond the scope of this article, but at the moment the physically sound solution appears when the formal fractionalization of the Dodson equation involves the Caputo-Fabrizio time derivative.

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