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A quasilinearization technique for the solution of singularly perturbed delay differential equation

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Abstract

This study deals with the singularly perturbed initial value problems for a quasilinear first-order delay differential equation. A quasilinearization technique for the appropriate delay differential problem theoretically and experimentally analyzed. The parameter uniform convergence is confirmed by numerical computations.

Keywords: Delay differential equation, singular perturbation, finite difference scheme, piecewise-uniform mesh, quasilinearization technique.

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1. Introduction

Consider the following singularly perturbed quasilinear delay differential problem in the interval $\bar{I} = [0, T]$:

$$\varepsilon u'(t) + f(t, u(t), u(t-r)) = 0, \quad t \in I, \tag{1.1}$$

$$u(t) = \varphi(t), \quad t \in I_0, \tag{1.2}$$

where $I = (0, T] = \bigcup_{p=1}^{m} I_p$, $I_p = \left\{t : r_{p-1} < t \leqslant r_p\right\}$, $1 \leqslant p \leqslant m$ and $r_s = sr$, for $0 \leqslant s \leqslant m-1$ and $r_m = T(0 < T - r_{m-1} \leqslant r)$, $I_0 = (-r, 0]$. $0 < \varepsilon \leqslant 1$ is the perturbation parameter, $r > 0$ is a constant delay, $\varphi(t)$ and $f(t, u, v)$ are given sufficiently smooth functions satisfying certain regularity conditions in $\bar{I}$ and $\bar{I} \times \mathbb{R}^2$, respectively, and moreover

$$0 < \alpha \leqslant \frac{\partial f}{\partial u} \leqslant M_1, \qquad \left|\frac{\partial f}{\partial v}\right| \leqslant M_1^*.$$

Delay differential equations are used to model a large variety of practical phenomena in the biosciences, engineering and control theory and in many other areas of science and technology, in which

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the time evolution depends not only on present states but also on states at or near a given time in the past (see, e.g., [5, 6, 8, 11]). If we restrict the class of delay differential equations to a class in which the highest derivative is multiplied by a small parameter, then it is said to be a singularly perturbed delay differential equation. Such problems arise in the mathematical modeling of various practical phenomena, for example, in population dynamics [11], the study of bistable devices [7], description of the human pupil-light reflex [12], variational problems in control theory [13]. In the direction of numerical study of singularly perturbed delay differential equation, several can be seen in [2–4, 9, 10, 14–18].

In the present paper we discretize (1.1)-(1.2) using a numerical method, which is composed of an implicit finite difference scheme on piecewise-uniform S-meshes on each time-subinterval. In Section 2, we describe the finite difference discretization and introduce the piecewise uniform grid and followed by the quasilinearization technique for solving this problem is presented. Numerical example in comparison with their exact solution is being presented in Section 3. The technique to construct discrete problem and error analysis for approximate solution is similar to those ones from [3, 4, 9, 10] and [1].

Throughout the paper, C denotes a generic positive constant independent of ε and the mesh parameter. A subscripted such constant is also independent of ε and mesh, but whose value is fixed.

2. The continuous problem

Here we show some properties of the solution of (1.1)-(1.2), which are needed in later sections for the analysis of appropriate numerical solution. For any continuous function $g(t)$, $\|g\|_\infty$ denotes a continuous maximum norm on the corresponding closed interval, in particular we shall use

$$\|g\|_{\infty,p} = \max_{\bar{I}_p} |g(x)|, \quad 0 \leqslant p \leqslant m.$$

Lemma 2.1. *The solution* $u(t)$ *of the problem* (1.1)-(1.2) *satisfies the following estimates*

$$\|u(t)\|_{\infty,p} \leqslant C_p, \quad 1 \leqslant p \leqslant m,$$

where

$$C_p = \|\varphi\|_{\infty,0} (1 + \alpha^{-1} M_1^*)^p + \alpha^{-1} \sum_{s=1}^{p} (1 + \alpha^{-1} M_1^*)^{p-s} \|F\|_{\infty,p}, \quad p = 1, 2, \cdots, m,$$

$$F(t) = f(t, 0, 0),$$

and

$$|u'(t)| \leqslant C \left\{ 1 + \frac{(t - r_{p-1})^{p-1}}{\varepsilon^p} \exp\left(-\frac{\alpha(t - r_{p-1})}{\varepsilon} \right) \right\}, \quad t \in I_p, \; 1 \leqslant p \leqslant m,$$

provided

$$|\partial f / \partial t| \leqslant C \quad \textit{for} \quad t \in \bar{I} \quad \textit{and} \quad |u|, \; |v| \leqslant C_0,$$

where

$$C_0 = \|\varphi\|_{\infty,0} (1 + \alpha^{-1} M_1^*)^m + (M_1^*)^{-1} \|F\|_{\infty,\bar{I}} \{(1 + \alpha^{-1} M_1^*)^m - 1\}.$$

Proof. See [3]. □

3. Difference algorithm and quasilinearization

Let $\bar{\omega}_{N_0}$ be any non-uniform mesh on $\bar{I}$:

$$\bar{\omega}_{N_0} = \{0 = t_0 < t_1 < \cdots < t_{N_0} = T, \quad \tau_i = t_i - t_{i-1}\},$$

which contains by N mesh points at each subinterval $I_p (1 \leqslant p \leqslant m-1)$ and $[(T - r_{m-1}) N / r]$ points at

I_m:

$$\omega_{N,p} = \{t_i : (p-1)N+1 \leqslant i \leqslant pN\}, \quad 1 \leqslant p \leqslant m-1,$$

$$\omega_{N,m} = \{t_i : (m-1)N+1 \leqslant i \leqslant N_0\},$$

and consequently

$$\omega_{N_0} = \bigcup_{p=1}^{m} \omega_{N,p}.$$

To simplify the notation we set $g_i = g(t_i)$ for any function $g(t)$, and moreover y_i denotes an approximation of $u(t)$ at t_i. For any mesh function $\{w_i\}$ defined on $\bar{\omega}_{N_0}$ we use

$$w_{\bar{t},i} = (w_i - w_{i-1})/\tau_i,$$

$$\|w\|_{\infty,N,p} = \|w\|_{\infty,\omega_{N,p}} := \max_{1\leqslant i\leqslant N} |w_i|.$$

For the difference approximation to (1.1), we integrate (1.1) over (t_{i-1}, t_i):

$$\varepsilon u_{\bar{t},i} + \tau^{-1} \int_{t_{i-1}}^{t_i} f(t, u(t), u(t-r))dt = 0,$$

which yields the relation

$$\varepsilon u_{\bar{t},i} + f(t_i, u_i, u_{i-N}) + R_i = 0, \quad 1 \leqslant i \leqslant N_0, \tag{3.1}$$

with the local truncation error

$$R_i = -\tau_i^{-1} \int_{t_{i-1}}^{t_i} \left\{ (t-t_{i-1}) \frac{d}{dt} f(t, u(t), u(t-r)) \right\} dt.$$

As a consequence of (3.1), we propose the following difference scheme for approximation (1.1)-(1.2)

$$\varepsilon y_{\bar{t},i} + f(t_i, y_i, y_{i-N}) = 0, \quad 1 \leqslant i \leqslant N_0, \tag{3.2}$$

$$y_i = \varphi_i, \quad -N \leqslant i \leqslant 0. \tag{3.3}$$

The difference scheme (3.2)-(3.3), in order to be ε-uniform convergent, we will use the Shishkin mesh. For the even number N, the piecewise uniform mesh $\omega_{N,p}$ divides each of the interval $[r_{p-1},\ \sigma_p]$ and $[\sigma_p,\ r_p]$ into $N/2$ equidistant subintervals, where the transition point σ_p, which separates the fine and coarse portions of the mesh is obtained by

$$\sigma_p = r_{p-1} + \min\left\{r/2,\ \alpha^{-1}\theta_p \varepsilon \ln N\right\}, \quad \text{for } 1 \leqslant p \leqslant m-1,$$

$$\sigma_m = r_{m-1} + \min\left\{(T-r_{m-1})/2,\ \alpha^{-1}\theta_m \varepsilon \ln \tilde{N}\right\},$$

where $\theta_1 \geqslant 1$ and $\theta_p > 1$ $(2 \leqslant p \leqslant m)$ are some constants, $\tilde{N} = [(T-r_{m-1})\,N/r]$ (if $\tilde{N}$ is odd then we take $\tilde{N} = [(T-r_{m-1})\,N/r]+1$). Hence, denoting by $\tau_p^{(1)}$and $\tau_p^{(2)}$the stepsizes in $[r_{p-1},\ \sigma_p]$ and $[\sigma_p,\ r_p]$ respectively, we have

$$\tau_p^{(1)} = 2(\sigma_p - r_{p-1})N^{-1}, \quad \tau_p^{(2)} = 2(r_p - \sigma_p)N^{-1}, \quad 1 \leqslant p \leqslant m-1,$$

$$\tau_m^{(1)} = 2(\sigma_m - r_{m-1})\tilde{N}^{-1}, \quad \tau_m^{(2)} = 2(T - \sigma_m)\tilde{N}^{-1},$$

so

$$\bar{\omega}_{N,p} = \begin{cases} t_i = r_{p-1} + (i-(p-1)N)\tau_p^{(1)}, & i = (p-1)N, \ldots, (p-1/2)N, \\ t_i = \sigma_p + (i-(p-1/2)N)\tau_p^{(2)}, & i = (p-1/2)N+1, \ldots, pN, \end{cases} \quad 1 \leqslant p \leqslant m-1.$$

Analogous formula is being written for $\bar{\omega}_{N,m}$ via $\tilde{N}$. In the rest of the paper we only consider this type mesh.

In [3] the authors proved that

$$\|y-u\|_{\infty,\bar{\omega}_{N,p}} \leqslant CN^{-1}\ln N, \quad 1 \leqslant p \leqslant m.$$

Now we propose the following quasilinearization algorithm for the solving the nonlinear difference problem (3.2)-(3.3)

$$\varepsilon y^{(n)}_{\bar{t},i} + f(t_i, y_i^{(n-1)}, y_{i-N}^{(n)}) + \frac{\partial f}{\partial u}(t_i, y_i^{(n-1)}, y_{i-N}^{(n)})[y_i - y_i^{(n-1)}] = 0, \quad i = 1, 2, \cdots, N, \quad n = 1, 2, \cdots. \tag{3.4}$$

$$y_i^{(n)} = \varphi_i, \quad -N \leqslant i \leqslant 0, \tag{3.5}$$

$$y_i^{(0)} \text{ given}, \quad 1 \leqslant i \leqslant N.$$

To estimate error of the iterative process (3.4)-(3.5), we write the following relation for the exact solution of the problem (3.2)-(3.3)

$$\varepsilon y_{\bar{t},i} + f(t_i, y_i^{(n-1)}, y_{i-N}^{(n)}) + \frac{\partial f}{\partial u}(t_i, y_i^{(n-1)}, y_{i-N}^{(n)})[y_i - y_i^{(n-1)}] + R_i^{(n)} = 0, \tag{3.6}$$

with

$$\begin{aligned} R_i^{(n)} &= (y_{i-N} - y_{i-N}^{(n)})\frac{\partial f}{\partial v}(t_i, y_i, \tilde{y}_{i-N}^{(n)}) \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial u^2}(t_i, \tilde{y}_i^{(n-1)}, y_{i-N}^{(n)})[y_i - y_i^{(n-1)}]^2 = 0, \end{aligned} \tag{3.7}$$

where the tilde indicates that the partial derivative is evaluated at an intermediate point. If

$$|\frac{\partial^2 f}{\partial u^2}| \leqslant M_2,$$

for all $t \in \bar{I}$ and all real u, v, it then from (3.7) follows that

$$|R_i^{(n)}| \leqslant M_1^*|y_{i-N} - y_{i-N}^{(n)}| + \frac{1}{2}M_2|y_i - y_i^{(n-1)}|^2. \tag{3.8}$$

Let $\omega_i^{(n)} = y_i^{(n)} - y_i$. Then, in view of (3.4) and (3.6), we have

$$\varepsilon \omega^{(n)}_{\bar{t},i} + \frac{\partial f}{\partial u}(t_i, y_i^{(n-1)}, y_{i-N}^{(n)})\omega_i^{(n)} = R_i^{(n)}, \quad 1 \leqslant i \leqslant N, \tag{3.9}$$

$$\omega_i^{(n)}, \quad -N \leqslant i \leqslant 0.$$

From (3.9), by using (3.8) and maximum principle we obtain

$$\begin{aligned} \|\omega^{(n)}\|_{\infty,p} &\leqslant \alpha^{-1}\|R^{(n)}\|_{\infty,p} \\ &\leqslant \alpha^{-1}\{M_1\|\omega^{(n)}\|_{\infty,p-1} + \frac{1}{2}M_2\|\omega^{(n-1)}\|^2_{\infty,p}\}. \end{aligned}$$

Resolving this in respect to linear part, we get

$$\|\omega^{(n)}\|_{\infty,p} \leqslant \frac{1}{2}\alpha^{-1}M_2\sum_{s=1}^{p}(\alpha M_1)^{p-s}\|\omega^{(n-1)}\|^2_{\infty,p}.$$

Thereby

$$\|\omega^{(n)}\|_{\infty,\bar{I}} \leqslant q\|\omega^{(n-1)}\|^2_{\infty,\bar{I}},$$

where

$$q = \frac{1}{2}\alpha^{-1}M_2(\alpha M_1 - 1)^{-1}(\alpha M_1)^m - 1.$$

Thus for sufficiently good initial guess the iterative process (3.4)-(3.5) converges quadratically.

At last we note that, the relation (3.4) can be rewritten as

$$y_i^{(n)} = y_i^{(n-1)} - \frac{(y_i^{(n-1)} - y_{i-1}^{(n)})\rho_i^{-1} + f(t_i, y_i^{(n-1)}, y_{i-N}^{(n)})}{\frac{\partial f}{\partial u}(t_i, y_i^{(n-1)}, y_{i-N}^{(n)}) + \rho_i^{-1}}, \quad i = 1, 2, \cdots, N-1, \quad n = 1, 2, \cdots.$$

4. Numerical results

We now look at computational results a particular problem

$$\varepsilon u'(t) + 2u(t) = u(t) * u(t-1), \quad t \in (0, \infty), \quad u(t) = 1, \quad -1 \leqslant t \leqslant 0.$$

The exact solution for $0 \leqslant t \leqslant 2$ is given by

$$u(t) = \begin{cases} e^{-t/\varepsilon}, & t \in (0,1], \\ e^{1-e^{(1-t)/\varepsilon}} * e^{(1-2t)/\varepsilon}, & t \in (1,2]. \end{cases}$$

We define the exact error $e_\varepsilon^{N,p}$ and the computed parameter-uniform maximum pointwise error $e^{N,p}$ as follows:

$$e_\varepsilon^{N,p} = \|y - u\|_{\infty, \omega_{N,p}}, \quad p = 1, 2,$$

$$e^{N,p} = \max_\varepsilon e_\varepsilon^{N,p}, \quad p = 1, 2,$$

where y is the numerical approximation to u for various values of $N, \varepsilon, \theta_1, \theta_2$. We also define the computed parameter-uniform rate of convergence to be

$$r^{N,p} = \ln\left(e^{N,p}/e^{2N,p}\right)/\ln 2, \quad p = 1, 2.$$

The values of ε for which we solve the test problem are $\varepsilon = 2^{-i}, i = 1, 2, 4, \cdots, 12$.

Table 1: Exact errors $e_\varepsilon^{N,1}$, computed ε-uniform errors $e^{N,1}$ and convergence rates $r^{N,1}$.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	0.00582415	0.00289292	0.00144173	0.00071968	0.00035955
	1.00	1.00	1.00	1.00	
2^{-2}	0.0118053	0.00582415	0.00289292	0.00144173	0.00071968
	1.01	1.00	1.00	1.00	
2^{-4}	0.0242705	0.0118056	0.00582415	0.00289292	0.00144173
	1.03	1.00	1.00	1.00	
2^{-6}	0.0252618	0.0144056	0.00811589	0.00452842	0.00250043
	0.81	0.82	0.84	0.85	
2^{-8}	0.0252618	0.0144056	0.00811589	0.00452842	0.00250043
	0.81	0.82	0.84	0.85	
2^{-10}	0.0252618	0.0144056	0.00811589	0.00452842	0.00250043
	0.81	0.82	0.84	0.85	
2^{-12}	0.0252618	0.0144056	0.00811589	0.00452842	0.00250043
	0.81	0.82	0.84	0.85	
$e^{N,1}$	0.0252618	0.0144056	0.00811589	0.00452842	0.00250043
$r^{N,1}$	0.81	0.82	0.84	0.85	

Table 2: Exact errors $e_{\varepsilon}^{N,2}$, computed ε-uniform errors $e^{N,2}$ and convergence rates $r^{N,2}$.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	0.00425125	0.00212012	0.00105868	0.000528998	0.00026441
	1.00	1.00	1.00	1.00	
2^{-2}	0.00224024	0.00113262	0.00056935	0.000285434	0.00014290
	0.98	0.99	1.00	1.00	
2^{-4}	0.00141144	0.00077044	0.00040206	0.000205322	0.00010374
	0.87	0.93	0.97	0.99	
2^{-6}	0.00141144	0.00077044	0.00040206	0.000205322	0.00010374
	0.87	0.93	0.97	0.99	
2^{-8}	0.00141144	0.00077044	0.00040206	0.000205322	0.00010374
	0.87	0.93	0.97	0.99	
2^{-10}	0.00141144	0.00077044	0.00040206	0.000205322	0.00010374
	0.87	0.93	0.97	0.99	
2^{-12}	0.00141144	0.00077044	0.00040206	0.000205322	0.00010374
	0.87	0.93	0.97	0.99	
$e^{N,1}$	0.00141144	0.00077044	0.00040206	0.000205322	0.00010374
$r^{N,1}$	0.87	0.93	0.97	0.99	

Tables 1 and 2 verify the ε-uniform convergence of the numerical solution on both subintervals and computed rates are essentially in agreement with our theoretical analysis.

5. Conclusion

A delay differential problem for a quasilinear singularly perturbed first-order differential equation is considered. A quasilinearization technique for the appropriate delay difference problem, which has earlier been proposed by authors, theoretically and experimentally analyzed. This problem is solved by employing standard backward difference operators on a non-uniform mesh which consists of the special piecewise uniform meshes on each time subinterval. It is shown that the method displays uniform convergence with respect to the perturbation parameter. The parameter uniform convergence is confirmed by numerical computations.

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