Available online at www.isr-publications.com/mns
Math. Nat. Sci., 3 (2018), 1–11
Research Article

Online: ISSN 2600-7665
Mathematics in Natural Science
An International Journal
Journal Homepage: www.isr-publications.com/mns

A fixed point theorem in ordered G-metric spaces with its application via new functions

Stojan Radenović[a,b], Arslan Hojat Ansari[c,d], Ali Turab[e,*], Muaadh Almahalebi[f]

[a] Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam.
[b] Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam.
[c] Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.
[d] Department of Mathematics, Payame Noor University, P. O. Box 19395-3697, Tehran, Iran.
[e] Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, Pathum Thani 12121, Thailand.
[f] Department of Mathematics, Ibn Tofail University, Kenitra, Morocco.

Abstract

In this paper, we will investigate a fixed point theorem for (ψ, φ)-weak contraction via new functions in generalized ordered metric spaces. Furthermore, we present an illustrative application in integral equations.

Keywords: Ω-distance, fixed point, G-metric space, $(\psi; \varphi)$-weak contraction.

2010 MSC: 47H10, 54H25.

1. Introduction and preliminaries

The notion of metric space, introduced by Fréchet in 1906, is one of the useful topic not only in mathematics but also in several quantitative sciences. Due to its importance and application potential, this notion has been extended, improved and generalized in many ways. An incomplete list of such attempts are following: symmetric space, b-metric space, partial metric space, partially ordered metric spaces, quasi-metric space, fuzzy metric space, dislocated metric space, dislocated quasi-metric space, right and left dislocated metric spaces, etc..

In 1922, Banach proved that a contraction mapping on a complete metric space possesses a unique fixed point. Banach contraction theorem is one of the pivotal results of functional analysis. It has many applications in various fields of mathematics such as functional equations, differential equations, integral equations, etc.. After Banach contraction theorem number of fixed point theorems have been established

*Corresponding author
Email addresses: radens@beotel.net (Stojan Radenović), mathanalsisamir4@gmail.com (Arslan Hojat Ansari), taurusnoor@yahoo.com (Ali Turab), muaadh1979@hotmail.fr (Muaadh Almahalebi)

doi: 10.22436/mns.03.01.01

Received: 2017-11-29 *Revised:* 2018-04-01 *Accepted:* 2018-04-11

by various authors and they made different generalizations of this theorem [3–13, 20, 22, 29]. Some new results for contractions in partially ordered metric spaces have been given by Nieto and Rodriguez-Lopez [20], Ran and Reurings [25] and Petrusel and Rus [23]. The main idea in [21, 25] involves combining the ideas of an iterative technique in the contraction mapping principle with those in the monotone technique. In [8], Dutta presented the concept of (ψ,ϕ)-weak contraction which includes the generalizations Theorem (1.2) in [15] and Theorem (1.4) in [26]. Also, Mustafa and Sims [18] introduced the concept of G-metric. Some authors [2, 17, 19, 28] have proved some fixed point theorems in these spaces. Aage [1], proved a fixed point theorem for weak contraction in G-metric space. Recently, Saadati et al. [27], used the concept of G-metric, defined an Ω-distance on complete G-metric space and generalized the concept of ω-distance due to Kada et al. [14].

In 2013, Gholizadeh [10] considered the concept of Ω-distance on a complete, partially ordered G-metric space and prove a fixed point theorem for (ψ,ϕ)-weak contraction in generalized partially ordered metric spaces. We intend through this work to bring and share queries about a fixed point theorem for (ψ,ϕ)-weak contraction via a new function in generalized partially ordered metric spaces and an illustrative application in integral equations.

At first we recall some definitions and results. For more information see [2, 8, 16–18, 24].

Definition 1.1 ([18]). Let X be a non-empty set. A function $G : X \times X \times X \longrightarrow [0,\infty)$ is called a G-metric if the following conditions are satisfied:

(i) $G(x,y,z) = 0$ if $x = y = z$ (coincidence);
(ii) $G(x,x,y) > 0$ for all $x,y \in X$, where $x \neq y$;
(iii) $G(x,x,z) \leqslant G(x,y,z)$ for all $x,y,z \in X$, with $z \neq y$;
(iv) $G(x,y,z) = G(p\{x,y,z\})$, where p is a permutation of x,y,z (symmetry);
(v) $G(x,y,z) \leqslant G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

A G-metric is said to be symmetric if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.

Definition 1.2 ([18]). Let (X, G) be a G-metric space,

(1) a sequence $\{x_n\}$ in X is said to be G-Cauchy sequence if for each $\varepsilon > 0$ there exists a positive integer n_0 such that for all $m,n,l \geqslant n_0$, $G(x_n,x_m,x_l) < \varepsilon$;
(2) a sequence $\{x_n\}$ in X is said to be G-convergent to a point $x \in X$ if for each $\varepsilon > 0$ there exists a positive integer n_0 such that for all $m,n, \geqslant n_0$, $G(x_m,x_n,x) < \varepsilon$.

Definition 1.3 ([27]). Let (X, G) be a G-metric space. Then a function $\Omega : X^3 \longrightarrow [0,\infty)$ is called an Ω-distance on X if the following conditions are satisfied:

(a) $\Omega(x,y,z) \leqslant \Omega(x,a,a) + \Omega(a,y,z)$ for all $x,y,z,a \in X$;
(b) for any $x,y \in X, \Omega(x,y,.), \Omega(x,.,y) : X \to [0,\infty)$ are lower semi-continuous;
(c) for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\Omega(x,a,a) \leqslant \delta$ and $\Omega(a,y,z) \leqslant \delta$ imply $G(x,y,z) \leqslant \varepsilon$.

Example 1.4. Let (X,d) be a metric space and $G : X^3 \longrightarrow [0,\infty)$ defined by

$$G(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\}$$

for all $x,y,z \in X$. Then $\Omega = G$ is an Ω-distance on X.

Example 1.5. Let $X = \mathbb{R}$ and consider the G-metric

$$G(x,y,z) = \frac{1}{3}(\mid x-y \mid + \mid y-z \mid + \mid x-z \mid)$$

for all $x,y,z \in \mathbb{R}$. Then $\Omega : \mathbb{R}^3 \longrightarrow [0,\infty)$ defined by

$$\Omega(x,y,z) = \frac{1}{3}(\mid x-y \mid + \mid z-x \mid)$$

for all $x,y,z \in \mathbb{R}$ is an Ω-distance on $\mathbb{R}$.

Lemma 1.6 ([27]). *Let (X, G) be a $G-$metric space and Ω be an Ω-distance on X. Let x_n, y_n be sequences in X, α_n, β_n be sequences in $[0, \infty)$ converging to zero and let $x, y, z, a \in X$. Then we have the following.*

(1) if $\Omega(y, x_n, x_n) \leqslant \alpha_n$ and $\Omega(x_n, y, z) \leqslant \beta_n$ for $n \in \mathbb{N}$, then $G(y, y, z) < \varepsilon$ and hence $y = z$;
(2) if $\Omega(y_n, x_n, x_n) \leqslant \alpha_n$ and $\Omega(x_n, y_m, z) \leqslant \beta_n$ for $m > n$, then $G(y_n, y_m, z) \to 0$ and hence $y_n \to z$;
(3) if $\Omega(x_n, x_m, x_l) \leqslant \alpha_n$ for any $l, m, n \in \mathbb{N}$ with $n \leqslant m \leqslant l$, then x_n is a G-Cauchy sequence;
(4) if $\Omega(x_n, a, a) \leqslant \alpha_n$ for any $n \in \mathbb{N}$, then x_n is a G-Cauchy sequence.

Definition 1.7. A function $\varphi : [0, \infty) \longrightarrow [0, \infty)$ is called an ultra-altering distance function if φ is continuous, and $\varphi(t) > 0$, for all $t > 0$.

In 2014, the concept of C-class functions was introduced by Ansari [4] as the following.

Definition 1.8 ([4]). A mapping $f : [0, \infty)^2 \to \mathbb{R}$ is called C-*class* function if it is continuous and satisfies following axioms:

(1) $f(s, t) \leqslant s$;
(2) $f(s, t) = s$ implies that either $s = 0$ or $t = 0$ for all $s, t \in [0, \infty)$.

Note that for some f we have $f(0, 0) = 0$. We denote C-class functions as $\mathcal{C}$.

Example 1.9 ([4]). The following functions $f : [0, \infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$:

(1) $f(s, t) = s - t$, $f(s, t) = s \Rightarrow t = 0$;
(2) $f(s, t) = ms$, $0<m<1$, $f(s, t) = s \Rightarrow s = 0$;
(3) $f(s, t) = \frac{s}{(1+t)^r}$; $r \in (0, \infty)$, $f(s, t) = s \Rightarrow s = 0$ or $t = 0$;
(4) $f(s, t) = \log(t + a^s)/(1 + t)$, $a > 1$, $f(s, t) = s \Rightarrow s = 0$ or $t = 0$;
(5) $f(s, t) = \ln(1 + a^s)/2$, $a > e$, $f(s, t) = s \Rightarrow s = 0$;
(6) $f(s, t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1, r \in (0, \infty)$, $f(s, t) = s \Rightarrow t = 0$;
(7) $f(s, t) = s \log_{t+a} a$, $a > 1$, $f(s, t) = s \Rightarrow s = 0$ or $t = 0$;
(8) $f(s, t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$, $f(s, t) = s \Rightarrow t = 0$;
(9) $f(s, t) = s\beta(s)$, $\beta : [0, \infty) \to [0, 1)$ and is continuous, $f(s, t) = s \Rightarrow s = 0$;
(10) $f(s, t) = s - \frac{t}{k+t}$, $f(s, t) = s \Rightarrow t = 0$;
(11) $f(s, t) = s - \varphi(s)$, $f(s, t) = s \Rightarrow s = 0$, here $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;
(12) $f(s, t) = sh(s, t)$, $f(s, t) = s \Rightarrow s = 0$, here $h : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous function such that $h(t, s) < 1$ for all $t, s > 0$;
(13) $f(s, t) = s - (\frac{2+t}{1+t})t$, $f(s, t) = s \Rightarrow t = 0$;
(14) $f(s, t) = \sqrt[n]{\ln(1 + s^n)}$, $f(s, t) = s \Rightarrow s = 0$.
(15) $f(s, t) = \phi(s)$, $f(s, t) = s \Rightarrow s = 0$, here $\phi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$;
(16) $f(s, t) = \frac{s}{(1+s)^r}$; $r \in (0, \infty)$, $f(s, t) = s \Rightarrow s = 0$.

2. Main results

Definition 2.1. Suppose that $(X, \preceq)$ is a partially ordered space and $T : X \to X$. We say that T is non-decreasing if for $x, y \in X$,

$$x \preceq y \Rightarrow T(x) \preceq T(y).$$

We denote $\Psi = \{\psi | \psi : [0, \infty) \to [0, \infty)\}$ as the set of continuous, non-decreasing functions with $\psi^{-1}(0) = 0$ and $\Phi = \{\phi | \phi : [0, \infty) \to [0, \infty)\}$ as the set ultra-altering distance functions.

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered space and f be a C-class function. Suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space. Also Ω is an Ω-distance on X and T is a non-decreasing mapping from X into itself. Suppose that*

$$\psi(\Omega(Tx, Ty, Tz)) \leqslant f(\psi(\Omega(x, y, z)), \phi(\Omega(x, y, z))) \quad \text{for all } x, y, z \in X \text{ with } x \preceq y \preceq z,$$

where $\phi \in \Phi, \psi \in \Psi$ and $f \in \mathcal{C}$. Also, for every $x \in X$

$$\inf\{\Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, Tx, y) : x \preceq Tx\} > 0,$$

for every $y \in X$ with $y \neq Ty$. If there exists an $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point. Moreover, if $\nu = T\nu$, then $\Omega(\nu, \nu, \nu) = 0$.

Proof. If $x_0 = Tx_0$, then the result is proved. Hence, we suppose $x_0 \neq Tx_0$. Since $x_0 \preceq Tx_0$ and T is non-decreasing, we obtain

$$x_0 \preceq Tx_0 \preceq T^2x_0 \preceq \cdots \preceq T^{n+1}x_0 \preceq \cdots.$$

Now if for some $n \in \mathbb{N}$, $\Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0) = 0$, then

$$\psi(\Omega(T^{n+1}x_0, T^{n+2}x_0, T^{n+2}x_0)) \leqslant f\left(\psi(\Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0)), \phi(\Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0))\right),$$

therefore $\Omega(T^{n+1}x_0, T^{n+2}x_0, T^{n+2}x_0) = 0$, and by part (c) of Definition 1.3, $G(T^nx_0, T^{n+2}x_0, T^{n+2}x_0) = 0$ and consequently $T^nx_0 = T^{n+2}x_0$, which implies T^nx_0 is a fixed point of T. If n is even, and T^2x_0 is a fixed point of T. If n is odd, then proof is complete.

Otherwise $\Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0) > 0$, for all $n \in \mathbb{N}$ and we have

$$\psi(\Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0)) \leqslant f\left(\psi(\Omega(T^{n-1}x_0, T^nx_0, T^nx_0)), \phi(\Omega(T^{n-1}x_0, T^nx_0, T^nx_0))\right). \tag{2.1}$$

Then,

$$\psi(\Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0)) \leqslant \psi(\Omega(T^{n-1}x_0, T^nx_0, T^nx_0)).$$

Similarly,

$$\psi(\Omega(T^{n-1}x_0, T^nx_0, T^nx_0)) \leqslant \psi(\Omega(T^{n-2}x_0, T^{n-1}x_0, T^{n-1}x_0)).$$

This shows that $\{\Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0)\}$ is non-increasing. Then, there exists $r \geqslant 0$ such that

$$\lim_{n\to\infty} \Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0) = r.$$

If $r > 0$, then $\phi(r) > 0$ and by taking $n \to \infty$ on (2.1), we obtain

$$\psi(r) \leqslant f(\psi(r), \phi(r)),$$

which is a contraction. So,

$$\lim_{n\to\infty} \Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0) = 0.$$

We claim that $\{T^nx_0\}$ is a G-Cauchy sequence. Suppose $\{T^nx_0\}$ is not a G-Cauchy sequence. Then, there exists $\varepsilon > 0$ and subsequences $\{T^{n_k}x_0\}$ and $\{T^{m_k}x_0\}$ such that n_k is the smallest integer with $n_k > m_k > k$ and

$$\Omega(T^{m_k}x_0, T^{n_k}x_0, T^{n_k}x_0) > \varepsilon.$$

Then,

$$\Omega(T^{m_k}x_0, T^{n_k-1}x_0, T^{n_k-1}x_0) \leqslant \varepsilon.$$

By part (a) of Definition 1.3, we obtain

$$\varepsilon < \Omega(T^{m_k}x_0, T^{n_k}x_0, T^{n_k}x_0) \leqslant \Omega(T^{m_k}x_0, T^{n_k-1}x_0, T^{n_k-1}x_0) + \Omega(T^{n_k-1}x_0, T^{n_k}x_0, T^{n_k}x_0)$$

$$\leqslant \varepsilon + \Omega(T^{n_k-1}x_0, T^{n_k}x_0, T^{n_k}x_0).$$

Thus,

$$\lim_{k\to\infty} \Omega(T^{m_k}x_0, T^{n_k}x_0, T^{n_k}x_0) = \varepsilon.$$

Since,

$$\Omega(T^{m_k-1}x_0, T^{n_k-1}x_0, T^{n_k-1}x_0) \leqslant \Omega(T^{m_k-1}x_0, T^{m_k}x_0, T^{m_k}x_0) + \Omega(T^{m_k}x_0, T^{n_k-1}x_0, T^{n_k-1}x_0),$$

and,

$$\begin{aligned}
\psi(\varepsilon) &< \psi(\Omega(T^{m_k}x_0, T^{n_k}x_0, T^{n_k}x_0)) \\
&\leqslant f\left(\psi(\Omega(T^{m_k-1}x_0, T^{n_k-1}x_0, T^{n_k-1}x_0)), \phi(\Omega(T^{m_k-1}x_0, T^{n_k-1}x_0, T^{n_k-1}x_0))\right) \\
&< \psi(\Omega(T^{m_k-1}x_0, T^{n_k-1}x_0, T^{n_k-1}x_0)),
\end{aligned}$$

then, we obtain

$$\lim_{k\to\infty} \Omega(T^{m_k-1}x_0, T^{n_k-1}x_0, T^{n_k-1}x_0) = \varepsilon.$$

Again, we have

$$\begin{aligned}
\psi(\varepsilon) &< \psi(\Omega(T^{m_k}x_0, T^{n_k}x_0, T^{n_k}x_0)) \\
&\leqslant f\left(\psi(\Omega(T^{m_k-1}x_0, T^{n_k-1}x_0, T^{n_k-1}x_0)), \phi(\Omega(T^{m_k-1}x_0, T^{n_k-1}x_0, T^{n_k-1}x_0))\right).
\end{aligned}$$

So, $\psi(\varepsilon) \leqslant f(\psi(\varepsilon), \phi(\varepsilon))$, which is a contradiction. Therefore $\{T^n x_0\}$ is a G-Cauchy sequence. Since X is G-complete, $\{T^n x_0\}$ converges to a point $u \in X$. Now, for $\varepsilon > 0$ and by lower semi-continuity of Ω,

$$\Omega(T^n x_0, T^m x_0, u) \leqslant \liminf_{p\to\infty} \Omega(T^n x_0, T^m x_0, T^p x_0) \leqslant \varepsilon, \qquad m \geqslant n$$

and,

$$\Omega(T^n x_0, u, T^l x_0) \leqslant \liminf_{p\to\infty} \Omega(T^n x_0, T^p x_0, T^l x_0) \leqslant \varepsilon, \qquad l \geqslant n.$$

Assume that $u \neq Tu$. Since $T^n x_0 \preceq T^{n+1}x_0$,

$$0 < \inf\{\Omega(T^n x_0, u, T^n x_0) + \Omega(T^n x_0, u, T^{n+1}x_0) + \Omega(T^n x_0, T^{n+1}x_0, u) : n \in \mathbb{N}\} \leqslant 3\varepsilon,$$

which is a contraction. Therefore, we have $u = Tu$.

Now, if $v = Tv$, we have,

$$\psi(\Omega(v,v,v)) = \psi(\Omega(Tv,Tv,Tv)) \leqslant f(\psi(\Omega(v,v,v)), \phi(\Omega(v,v,v))).$$

So, $\Omega(v,v,v) = 0$. □

The following corollaries are direct results of Theorem 2.2. As, if we put $f(s;t) = ks$ with $0 < k < 1$, $f(s;t) = s - t$, $f(s;t) = \frac{s}{1+t}$, $f(s;t) = s\log_{a+t} a$, and $f(s;t) = \sqrt[n]{\ln(1+s^n)}, n \in \mathbb{N}$, in Theorem 2.2 then we get the following corollaries 2.3, 2.4, 2.5, 2.6, and 2.7 respectively.

Corollary 2.3. *Let $(X, \preceq)$ be a partially ordered space and f be a C-class function. Suppose there exists a G-metric on X such that (X, G) is a complete G-metric space and Ω is an Ω-distance on X and T is a non-decreasing mapping from X into itself. Suppose that*

$$\psi(\Omega(Tx, Ty, Tz)) \leqslant k\psi(\Omega(x,y,z)), \textit{ for all } x,y,z \in X \textit{ with } x \preceq y \preceq z,$$

where $\psi \in \Psi$. Also, for every $x \in X$

$$\inf\{\Omega(x,y,x) + \Omega(x,y,Tx) + \Omega(x,Tx,y) : x \preceq Tx\} > 0,$$

for every $y \in X$ with $y \neq Ty$. If there exists an $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point. Moreover, if $v = Tv$, then $\Omega(v,v,v) = 0$.

Corollary 2.4. *Let $(X, \preceq)$ be a partially ordered space. Suppose there exists a G-metric on X such that (X, G) is a complete G-metric space and Ω is an Ω-distance on X and T is a non-decreasing mapping from X into itself. Suppose that*

$$\psi(\Omega(Tx, Ty, Tz)) \leqslant \psi(\Omega(x, y, z)) - \phi(\Omega(x, y, z)), \text{ for all } x, y, z \in X \text{ with } x \preceq y \preceq z,$$

where $\phi \in \Phi$ and $\psi \in \Psi$. Also, for every $x \in X$

$$\inf\{\Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, Tx, y) : x \preceq Tx\} > 0,$$

for every $y \in X$ with $y \neq Ty$. If there exists an $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point. Moreover, if $\nu = T\nu$, then $\Omega(\nu, \nu, \nu) = 0$.

Corollary 2.5. *Let $(X, \preceq)$ be a partially ordered space and f be a C-class function. Suppose there exists a G-metric on X such that (X, G) is a complete G-metric space and Ω is an Ω-distance on X and T is a non-decreasing mapping from X into itself. Suppose that*

$$\psi(\Omega(Tx, Ty, Tz)) \leqslant \frac{\psi(\Omega(x, y, z))}{1 + \phi(\Omega(x, y, z))}, \text{ for all } x, y, \in X \text{ with } x \preceq y \preceq z,$$

where $\phi \in \Phi$ and $\psi \in \Psi$. Also, for every $x \in X$

$$\inf\{\Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, Tx, y) : x \preceq Tx\} > 0,$$

for every $y \in X$ with $y \neq Ty$. If there exists an $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point. Moreover, if $\nu = T\nu$, then $\Omega(\nu, \nu, \nu) = 0$.

Corollary 2.6. *Let $(X, \preceq)$ be a partially ordered space and f be a C-class function. Suppose there exists a G-metric on X such that (X, G) is a complete G-metric space and Ω is an Ω-distance on X and T is a non-decreasing mapping from X into itself. Suppose that*

$$\psi(\Omega(Tx, Ty, Tz)) \leqslant \psi(\Omega(x, y, z)) \log_{a+\phi(\Omega(x,y,z))} a, \text{ for all } x, y, \in X \text{ with } x \preceq y \preceq z,$$

where $\phi \in \Phi$ and $\psi \in \Psi$. Also, for every $x \in X$

$$\inf\{\Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, Tx, y) : x \preceq Tx\} > 0,$$

for every $y \in X$ with $y \neq Ty$. If there exists an $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point. Moreover, if $\nu = T\nu$, then $\Omega(\nu, \nu, \nu) = 0$.

Corollary 2.7. *Let $(X, \preceq)$ be a partially ordered space and f be a C-class function. Suppose there exists a G-metric on X such that (X, G) is a complete G-metric space and Ω is an Ω-distance on X and T is a non-decreasing mapping from X into itself. Suppose that*

$$\psi(\Omega(Tx, Ty, Tz)) \leqslant \sqrt[n]{\log(1 + \psi(\Omega(x, y, z))^n)}, \ n \in \mathbb{N}, \text{ for all } x, y, z \in X \text{ with } x \preceq y \preceq z,$$

where $\psi \in \Psi$. Also, for every $x \in X$

$$\inf\{\Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, Tx, y) : x \preceq Tx\} > 0,$$

for every $y \in X$ with $y \neq Ty$. If there exists an $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point. Moreover, if $\nu = T\nu$, then $\Omega(\nu, \nu, \nu) = 0$.

Example 2.8. Let $X = [0, 1]$, the order is usual and $G(x, y, z) = \frac{1}{3}(\mid x - y \mid + \mid y - z \mid + \mid x - z \mid)$. Then (X, G) is a complete G-metric space. Suppose $\Omega(x, y, z) = \frac{1}{3}(\mid x - y \mid + \mid z - x \mid)$, $T(x) = \frac{x}{4}$, $\psi(t) = t$ and $f(s, t) = ks$ where $0 < k < 1$, then,

$$\begin{aligned}\psi(\Omega(Tx, Ty, Tz)) &= \psi(\frac{1}{3}(\mid Tx - Ty \mid + \mid Tz - Tx \mid)) \\ &= \psi(\frac{1}{3}(\mid \frac{x}{4} - \frac{y}{4} \mid + \mid \frac{z}{4} - \frac{x}{4} \mid)) = \frac{1}{12} \mid x - y \mid + \mid z - x \mid \leqslant \frac{1}{3}\psi(\Omega(x, y, z)).\end{aligned}$$

Also, for every $x \in X$

$$\inf\{\Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, Tx, y) : x \preceq Tx\} > 0,$$

for every $y \in X$ with $y \neq Ty$. So, by Theorem 2.2, T has a fixed point that is 0.

Example 2.9. Let $X = [0, 1]$, the order is usual and $G(x, y, z) = \frac{1}{3}(\mid x - y \mid + \mid y - z \mid + \mid x - z \mid)$. Then (X, G) is a complete G-metric space. Suppose $\Omega(x, y, z) = \frac{1}{3}(\mid x - y \mid + \mid z - x \mid)$, $T(x) = \frac{x}{5}$, $\phi(t) = 1$, $\psi(t) = 15t$ and $f(s, t) = \frac{s}{1+t}$. Then,

$$\begin{aligned}\psi(\Omega(Tx, Ty, Tz)) &= \psi(\frac{1}{3}(\mid Tx - Ty \mid + \mid Tz - Tx \mid)) \\ &= \psi(\frac{1}{3}(\mid \frac{x}{5} - \frac{y}{5} \mid + \mid \frac{z}{5} - \frac{x}{5} \mid)) \\ &= \mid x - y \mid + \mid z - x \mid \leqslant f\left(\psi(\Omega(x, y, z)), \phi(\Omega(x, y, z))\right).\end{aligned}$$

Also, for every $x \in X$

$$\inf\{\Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, Tx, y) : x \preceq Tx\} > 0,$$

for every $y \in X$ with $y \neq Ty$. So, by Theorem 2.2, T has a fixed point that is 0.

Example 2.10. Let $X = [0, 1]$, the order is usual and $G(x, y, z) = \frac{1}{3}(\mid x - y \mid + \mid y - z \mid + \mid x - z \mid)$. Then (X, G) is a complete G-metric space. Suppose $\Omega(x, y, z) = \frac{1}{3}(\mid x - y \mid + \mid z - x \mid)$, $T(x) = \frac{2x}{1+x}$, $\psi(t) = t$ and $f(s, t) = ks$. Then,

$$\begin{aligned}\psi(\Omega(Tx, Ty, Tz)) &= \psi(\frac{1}{3}(\mid Tx - Ty \mid + \mid Tz - Tx \mid)) \\ &= \psi(\frac{1}{3}(\mid \frac{2x}{(1+x)} - \frac{2y}{(1+y)} \mid + \mid \frac{2z}{(1+z)} - \frac{2x}{(1+x)} \mid)) \\ &= \frac{2}{3} \mid \frac{x}{1+x} - \frac{y}{1+y} \mid + \mid \frac{z}{1+z} - \frac{x}{1+x} \mid \leqslant \frac{4}{5}\psi(\Omega(x, y, z)).\end{aligned}$$

Also, for every $x \in X$

$$\inf\{\Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, Tx, y) : x \preceq Tx\} > 0,$$

for every $y \in X$ with $y \neq Ty$. So, by Theorem 2.2, T have two fixed points 0 and 1.

We denote Λ by the set all functions $\lambda : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

(i) λ is a Lebesgue-integrable mapping on each compact subset of $[0, +\infty)$;

(ii) for every $\varepsilon > 0$, we have $\int_0^\varepsilon \lambda(s)ds > 0$;

(iii) $\|\lambda\| < 1$, where $\|\lambda\|$ denotes to the norm of λ.

Corollary 2.11. *Let $(X, \preceq)$ be a partially ordered space and f be a C-class function. Suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space and Ω is an Ω-distance on X and T is a non-decreasing mapping from X into itself. Suppose that for all $x, y, z \in X$ with $x \preceq y \preceq z$ holds*

$$\int_0^{\psi(\Omega(Tx,Ty,Tz))} \lambda(s)ds \leqslant f\left(\int_0^{\psi(\Omega(x,y,z))} \lambda(s)ds, \int_0^{\phi(\Omega(x,y,z))} \lambda(s)ds\right), \tag{2.2}$$

where $\lambda \in \Lambda$, $\phi \in \Phi$, $\psi \in \Psi$, *and* $f \in \mathcal{C}$. *Also, for every* $x \in X$

$$\inf\{\Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, Tx, y) : x \preceq Tx\} > 0,$$

for every $y \in X$ *with* $y \neq Ty$. *If there exists an* $x_0 \in X$ *with* $x_0 \preceq Tx_0$, *then* T *has a fixed point.*

Proof. Define $\gamma : [0, +\infty) \to [0, +\infty)$ by $\gamma(t) = \int_0^t \lambda(s)ds$, then from inequality (2.2), we have

$$\gamma(\psi(\Omega(Tx, Ty, Tz))) \leqslant f\left(\gamma(\psi(\Omega(x, y, z))), \gamma(\phi(\Omega(x, y, z)))\right),$$

which can be written as

$$\psi_1(\Omega(Tx, Ty, Tz)) \leqslant f\left(\psi_1(\Omega(x, y, z)), \phi_1(\Omega(x, y, z))\right),$$

where $\psi_1 = \gamma \circ \psi$ and $\phi_1 = \gamma \circ \phi$. Since the functions ψ_1 and ϕ_1 satisfy the properties of ψ and ϕ, by Theorem 2.2, T has a fixed point. □

3. Application

In this section, we give an existence theorem for a solution of the following integral equation:

$$x(t) = \int_0^1 K(t, s, x(s))ds + g(t), \qquad t \in [0, 1]. \tag{3.1}$$

Let $X = C([0, 1])$ be the set all continuous functions defined on $[0, 1]$. Define $G : X \times X \times X \to \mathbb{R}$ by

$$G(x, y, z) = \|x - y\| + \|y - z\| + \|z - x\|,$$

where $\|x\| = \sup\{| x(t) | : t \in [0, 1]\}$. Then (X, G) is a complete G-metric space. Let $\Omega = G$. Then Ω is an Ω-distance on X. Define an ordered relation $\preceq$ on X by

$$x \preceq y \quad \text{if and only if} \quad x(t) \leqslant y(t) \quad \text{for all} \quad t \in [0, 1].$$

Then $(X, \preceq)$ is a partially ordered set. Now, we prove the following result.

Theorem 3.1. *Suppose the following hypotheses hold:*

(1) $K : [0, 1] \times [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *and* $g : [0, 1] \to \mathbb{R}$ *are continuous mappings;*
(2) K *is nondecreasing in its first variable and* g *is nondecreasing;*
(3) *There exists a continuous function* $F : [0, 1] \times [0, 1] \to [0, +\infty)$ *such that*

$$| K(t, s, u) - K(t, s, v) | \leqslant F(t, s) | u - v |$$

for every comparable $u, v \in \mathbb{R}^+$ *and* $s, t \in [0, 1]$ *with* $\sup_{t \in [0,1]} \int_0^1 F(t, s)ds \leqslant \frac{1}{2}$;
(4) *Let* $\phi \in \Phi$, $\psi \in \Psi$ *and* $f \in \mathcal{C}$, *then* $\psi(r) \leqslant f(\psi(2r), \phi(2r))$ *for all* $r \in [0, \infty)$.

Then the integral equation (3.1) *has a solution in* $C([0, 1])$.

Proof. Define $Tx(t) = \int_0^1 K(t, s, x(s))ds + g(t)$. By hypothesis (2), we have that T is nondecreasing. Now, if

$$\inf\{\Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, Tx, y) : x \preceq Tx\} = 0,$$

for every $y \in X$ with $y \neq Ty$, then for each $n \in \mathbb{N}$, there exists $x_n \in C([0, 1])$ with $x_n \preceq Tx_n$ such that

$$\Omega(x_n, y, x_n) + \Omega(x_n, y, Tx_n) + \Omega(x_n, Tx_n, y) \leqslant \frac{1}{n}.$$

Then, we have

$$\Omega(x_n, y, Tx_n) = \sup_{t\in[0,1]} \mid x_n - y \mid + \sup_{t\in[0,1]} \mid y - Tx_n \mid + \sup_{t\in[0,1]} \mid Tx_n - x_n \mid \leqslant \frac{1}{n}.$$

Thus,

$$\lim_{n\to\infty} x_n(t) = y(t), \quad \lim_{n\to\infty} Tx_n(t) = y(t).$$

By the continuity of K, we have

$$y(t) = \lim_{n\to\infty} Tx_n(t) = \int_0^1 K(t, s, \lim_{n\to\infty} x_n(s))ds + g(t) = \int_0^1 K(t, s, y(s))ds + g(t) = Ty(t).$$

Which is a contradiction. Therefore,

$$\inf\{\Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, Tx, y) : x \preceq Tx\} > 0.$$

Now, for $x, y, z \in X$ with $x \preceq y$, we have

$$\begin{aligned}
\psi(\Omega(Tx, Ty, Tz)) &= \psi(\sup_{t\in[0,1]} \mid Tx(t) - Ty(t) \mid + \sup_{t\in[0,1]} \mid Ty(t) - Tz(t) \mid + \sup_{t\in[0,1]} \mid Tz(t) - Tx(t) \mid) \\
&\leqslant \psi(\sup_{t\in[0,1]} \int_0^1 \mid K(t, s, x(s)) - K(t, s, y(s)) \mid ds + \sup_{t\in[0,1]} \int_0^1 \mid K(t, s, y(s)) - K(t, s, z(s)) \mid ds \\
&\quad + \sup_{t\in[0,1]} \int_0^1 \mid K(t, s, z(s)) - K(t, s, x(s)) \mid ds) \\
&\leqslant \psi(\sup_{t\in[0,1]} (\int_0^1 F(t, s) \mid x(s) - y(s) \mid ds) + \sup_{t\in[0,1]} (\int_0^1 F(t, s) \mid y(s) - z(s) \mid ds) \\
&\quad + \sup_{t\in[0,1]} (\int_0^1 F(t, s) \mid z(s) - x(s) \mid ds)) \\
&\leqslant \psi(\sup_{t\in[0,1]} (\mid x(t) - y(t) \mid) \sup_{t\in[0,1]} \int_0^1 F(t, s)ds + \sup_{t\in[0,1]} (\mid y(t) - z(t) \mid) \sup_{t\in[0,1]} \int_0^1 F(t, s)ds \\
&\quad + \sup_{t\in[0,1]} (\mid z(t) - x(t) \mid) \sup_{t\in[0,1]} \int_0^1 F(t, s)ds) \\
&\leqslant \psi(\frac{1}{2} \sup_{t\in[0,1]} (\mid x(t) - y(t) \mid) + \frac{1}{2} \sup_{t\in[0,1]} (\mid y(t) - z(t) \mid) + \frac{1}{2} \sup_{t\in[0,1]} (\mid z(t) - x(t) \mid)) \\
&\leqslant \psi(\frac{1}{2}\Omega(x, y, z)) \leqslant f(\psi(\Omega(x, y, z)), \phi(\Omega(x, y, z))).
\end{aligned}$$

Thus, by Theorem 2.2, there exists a solution $u \in C[0, 1]$ of the integral equation (3.1). □

Now, we present the following corollaries as an example of our result.

Corollary 3.2. *Suppose the following hypotheses hold:*

(1) $K : [0, 1] \times [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *and* $g : [0, 1] \to \mathbb{R}$ *are continuous mappings;*
(2) K *is nondecreasing in its first variable and* g *is nondecreasing;*
(3) *there exists a continuous function* $F : [0, 1] \times [0, 1] \to [0, +\infty)$ *such that*

$$\mid K(t, s, u) - K(t, s, v) \mid \leqslant F(t, s) \mid u - v \mid$$

for every $u, v \in \mathbb{R}^+$ *and* $s, t \in [0, 1]$ *with* $\sup_{t\in[0,1]} \int_0^1 F(t, s)ds \leqslant \frac{1}{2}$*;*

(4) *there exist continuous, non-decreasing function* $\psi : [0, \infty) \to (0, \infty)$ *with* $\psi^{-1}(0) = 0$ *and* $\psi(r) \leqslant k\psi(2r), 0 < k < 1$ *for all* $r \in [0, \infty)$.

Then the integral equation (3.1) *has a solution in* $C([0, 1])$.

Corollary 3.3. *Suppose the following hypotheses hold:*

(1) $K : [0, 1] \times [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *and* $g : [0, 1] \to \mathbb{R}$ *are continuous mappings;*
(2) K *is nondecreasing in its first coordinate and* g *is nondecreasing;*
(3) *there exists a continuous function* $F : [0, 1] \times [0, 1] \to [0, +\infty)$ *such that*

$$| K(t, s, u) - K(t, s, v) | \leqslant F(t, s) | u - v |$$

for every $u, v \in \mathbb{R}^+$ *and* $s, t \in [0, 1]$ *with* $\sup_{t\in[0,1]} \int_0^1 F(t, s)ds \leqslant \frac{1}{2}$;
(4) *let* $\phi \in \Phi, \psi \in \Psi$, *then* $\psi(r) \leqslant \frac{\psi(2r)}{1+\phi(2r)}$, *for all* $r \in [0, \infty)$.

Then the integral equation (3.1) *has a solution in* $C([0, 1])$.

Corollary 3.4. *Suppose the following hypotheses hold:*

(1) $K : [0, 1] \times [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *and* $g : [0, 1] \to \mathbb{R}$ *are continuous mappings;*
(2) K *is nondecreasing in its first coordinate and* g *is nondecreasing;*
(3) *there exists a continuous function* $F : [0, 1] \times [0, 1] \to [0, +\infty)$ *such that*

$$| K(t, s, u) - K(t, s, v) | \leqslant F(t, s) | u - v |$$

for every $u, v \in \mathbb{R}^+$ *and* $s, t \in [0, 1]$ *with* $\sup_{t\in[0,1]} \int_0^1 F(t, s)ds \leqslant \frac{1}{2}$;
(4) *let* $\phi \in \Phi$, $\psi \in \Psi$, *then* $\psi(r) \leqslant \psi(2r) \log_{a+\phi(2r)} a$, *for all* $r \in [0, \infty)$.

Then the integral equation (3.1) *has a solution in* $C([0, 1])$.

Corollary 3.5. *Suppose the following hypotheses hold:*

(1) $K : [0, 1] \times [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *and* $g : [0, 1] \to \mathbb{R}$ *are continuous mappings;*
(2) K *is nondecreasing in its first coordinate and* g *is nondecreasing;*
(3) *there exists a continuous function* $F : [0, 1] \times [0, 1] \to [0, +\infty)$ *such that*

$$| K(t, s, u) - K(t, s, v) | \leqslant F(t, s) | u - v |$$

for every $u, v \in \mathbb{R}^+$ *and* $s, t \in [0, 1]$ *with* $\sup_{t\in[0,1]} \int_0^1 F(t, s)ds \leqslant \frac{1}{2}$;
(4) *there exist continuous, non-decreasing function* $\psi : [0, \infty) \to (0, \infty)$ *with* $\psi^{-1}(0) = 0$ *and*

$$\psi(r) \leqslant \sqrt[n]{\log(1 + \psi(2r)^n)}, n \in \mathbb{N},$$

for all $r \in [0, \infty)$.

Then the integral equation (3.1) *has a solution in* $C([0, 1])$.

References

[1] C. T. Aage, J. N. Salunke, *Fixed point for weak contractions in G-metric spaces*, Appl. Math. E-Notes, **12** (2012), 23–28. 1

[2] M. Abbas, B. E. Rhoades, *Common fixed point results for non-commuting mappings without continuity in generalized metric spaces*, Appl. Math. Comput., **215** (2009), 262–269. 1

[3] R. P. Agarwal, M. A. El-Gebeily, D. O'Regan, *Generalized in partially ordered metric space*, Appl. Anal., **87** (2008), 109–116. 1

[4] A. H. Ansari, *Note on φ-ψ-contractive type mappings and related fixed point*, The 2nd Regional Conference on Mathematics and Applications (Payame Noor University), **2014** (2014), 377–380. 1, 1.8, 1.9

[5] L. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45** (1974), 267–273.

[6] L. B. Ćirić, *Coincidence and fixed points for maps on topological spaces*, Topology Appl., **154** (2007), 3100–3106.

[7] L. B. Ćirić, S. Ješić, M. M. Milovanović, J. S. Ume, *On the steepest descent approximation method for the zeros of generalized accretive operators*, Nonlinear Anal., **69** (2008), 763–769.

[8] P. N. Dutta, B. S. Choudhury, *A generalisation of contraction principle in metric spaces*, Fixed Point Theory Appl., **2008** (2008), 8 pages. 1

[9] J.-X. Fang, Y. Gao, *Common fixed point theorems under strict contractive conditions in Menger space*, Nonlinear Anal., **70** (2009), 184–193.

[10] L. Gholizadeh, *A fixed point theorems in generalized ordered metric spaces with application*, J. Nonlinear Sci. Appl., **6** (2013), 244–251. 1

[11] T. Gnana Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65** (2006), 1379–1393.

[12] T. Gnana Bhaskar, V. Lakshmikantham, J. Vasundhara Devi, *Monotone iterative technique for functional differential equations with retardation and anticipation*, Nonlinear Anal., **66** (2007), 2237–2242.

[13] N. Hussain, *Common fixed point in best approximation for Banach operator pairs with Ćirić type I-contractions*, J. Math. Anal. Appl., **338** (2008), 1351–1363. 1

[14] O. Kada, T. Suzuki, W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric space*, Math. Japon., **44** (1996), 381–391. 1

[15] M. S. Khan, M. Swaleh, S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Austral. Math. Soc., **30** (1984), 1–9. 1

[16] H. Kumar Nashine, *Coupled common fixed point results in ordered G-metric spaces*, J. Nonlinear Sci. Appl., **5** (2012), 1–13. 1

[17] Z. Mustafa, H. Obiedat, F. Awawdeh, *Some fixed point theorems for mapping on complete* G*-metric space*, Fixed Point Theory Appl., **2008** (2008), 12 pages. 1

[18] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7** (2006), 289–297. 1, 1.1, 1.2

[19] Z. Mustafa, B. Sims, *Fixed point theorems for contractive mappings in complete G-metric spaces*, Fixed Point Theory Appl., **2009** (2009), 10 pages. 1

[20] J. J. Nieto, R. Rodriguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22** (2005), 223–239. 1

[21] J. J. Nieto, R. Rodriguez-López, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sin. (Engl. Ser.), **23** (2007), 2205–2212. 1

[22] D. O'Regan, R. Saadati, *Nonlinear contraction theorems in probabilistic spaces*, Appl. Math. Comput., **195** (2008), 86–93. 1

[23] A. Petrusel, I. A. Rus, *Fixed point theorems in ordered* L*-spaces*, Proc. Amer. Math. Soc., **134** (2006), 411–418. 1

[24] V. Popa, A.-M. Patriciu, *A general fixed point theorem for pairs of weakly compatible mappings in* G*-metric spaces*, J. Nonlinear Sci. Appl., **5** (2012), 151–160. 1

[25] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132** (2004), 1435–1443. 1

[26] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal., **47** (2001), 2683–2693. 1

[27] R. Saadati, S. M. Vaezpour, P. Vetro, B. E. Rhoades, *Fixed point theorems in generalized partially ordered G-metric spaces*, Math. Comput. Modelling, **52** (2010), 797–801. 1, 1.3, 1.6

[28] W. Shatanawi, *Fixed point theory for contractive mappings satisfying Φ-maps in* G*-metric spaces*, Fixed Point Theory Appl., **2010** (2010), 9 pages. 1

[29] W. Shatanawi, H. Kumar Nashine, *A generalization of Banach's contraction principle for nonlinear contraction in a partial metric space*, J. Nonlinear Sci. Appl., **5** (2012), 37–43. 1