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On some particular regular Diophantine 3-tuples

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Abstract

Diophantine n-tuple where n=3 is called as a Diophantine triple. It means that Diophantine triple is a set of three positive integers satisfying special condition. For example, $\{a, b, c\}$ is called a $D(k)$-Diophantine triple if multiplying of any two different of them plus k is a perfect square integer where k is an integer.

In this work, we take in consideration some kind of regular $D(\pm 3^3)$-Diophantine triples. We demonstrate that such sets can not be extendible to $D(\pm 3^3)$-Diophantine quadruple by using algebraic methods such as classical Pell equations solutions, solutions of $ux^2 + vy^2 = w$ Diophantine equations where $u, v, w \in \mathbb{Z}$, factorization in the set of integers, and so on. Besides, we obtain some notable characteristic properties for such sets.

Keywords: Diophantine Triple, Pell equations, Diophantine equations, modular arithmetic, reciprocity theorem, Legendre symbol.

2010 MSC: 11Dxx, 11C08, 11G99.

1. Introduction

Number theory is one of the most significant fields of mathematics, especially, primes, prime factorization and diophantine equations have a great importance in number theory. A set of m distinct positive integers $\{\delta_1, \delta_2, \ldots, \delta_m\}$ is called a Diophantine m-tuple with k and represented as $D(k)$ or P_k if $\delta_i\delta_j + k(i \neq j, i, j = 1, 2, , m)$ is a perfect square integer. Although the topic of Diophantine m-tuple is very ancient problem, still many authors have been working on it with different techniques.

Bashmakova [2] gave the definition of Diophantine m-tuple as the statement: If we choose k=1 in the above mentioned definition (Diophantine m-tuple with k), we get the set of m positive integers which is called a Diophantine m-tuple while the product of any two of its distinct elements increased by 1 is a perfect square integer.

Baker and Davenport [1] considered the general solutions of each separated equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$ by using algebraic number theory. Brown [4] proved some general results for P_k while $k = 2(\text{mod } 4)$ and demonstrated that the $P_{(-1)}$ set $\{1, 2, 5\}$ can not be extendable. The book [6]

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written by Dickson, has many crucial results on number theory and group theory. Deshpande [5] created Diophantine triples considering a recurrence relation in the terms of a special sequence. Dujella and Jurasic [7] defined the some types of regular Diophantine m-tuples and obtained significant results for them.

Fermat [8] worked on the problem over integers considering the quadruple set $\{1, 3, 8, 120\}$. Gopalan et al. ([12–14]) constructed different types of interesting triple sets by creating new Dio triple set definition. Also, the author Gopalan and Özer [15] prepared a book on the varied types of Pell equations. Kedlaya [16] determined a new elementary method to solve special systems of Diophantine equation Özer [19, 22] proved some varied types of Diophantine triples using different algebraic methods.

Readers can get many significant and valuable information on number theory in the lecture notes of Goldmakher [11], Kurur and Saptharishi [17], and the books of Mollin [18] and Roberts [23]. Besides, one may refer [3, 9, 10] for an extensive review of various problems on Diophantine m-tuples.

In this paper, we consider several types of $D(\pm 3^3)$-Diophantine triples. Firstly, we demonstrate that they are regular Diophantine triples. Secondly, we prove that they can not extendable to $D(\pm 3^3)$-Diophantine quadruples. Lastly, we give some results on the characterization of the elements of $D(\pm 3^3)$-Diophantine triples using algebraic structures in algebraic and elementary number theory.

2. Preliminaries

Definition 2.1 ([11, Quadratic residue]). Let p be an odd prime, $\alpha \equiv 0 (\mathrm{mod}\ p)$. We say that α is a quadratic residue mod p if α is a square mod p (it is a quadratic non-residue otherwise).

Lemma 2.2 ([11]). *Let $\alpha \equiv 0(\mathrm{mod}\ p)$. Then α is a quadratic residue* mod p *if and only if* $\alpha^{(p-1)/2} \equiv 1(\mathrm{mod}\ p)$.

Definition 2.3 ([11, Legendre symbol]). Legendre symbol introduces the following notation for prime p:

$$(\alpha/p) = \begin{cases} 0, & \text{if } p/\alpha, \\ 1, & \text{if } x^2 = a(\mathrm{mod}\ p) \text{ has a nonzero solution}, \\ -1, & \text{if } x^2 \equiv a(\mathrm{mod}\ p) \text{has no solution}. \end{cases} \tag{2.1}$$

Lemma 2.4. *Let $(\frac{\cdot}{\cdot})$ be Legendre symbol and p is prime. Then, followings are satisfied.*

$$(\frac{-1}{p}) = \begin{cases} 1, & \text{if } p \equiv 1 \ (\mathrm{mod}\ 4), \\ -1, & \text{if } p \equiv -1 \ (\mathrm{mod}\ 4), \end{cases} \tag{2.2}$$

$$(\frac{3}{p}) = \begin{cases} 1, & \text{if } p \equiv \pm 1 \ (\mathrm{mod}\ 12), \\ -1, & \text{if } p \equiv \pm 5 \ (\mathrm{mod}\ 12), \end{cases} \tag{2.3}$$

$$(\frac{2}{p}) = \begin{cases} 1, & \text{if } p \equiv \pm 1 \ (\mathrm{mod}\ 8), \\ -1, & \text{if } p \equiv \pm 3 \ (\mathrm{mod}\ 8), \end{cases} \tag{2.4}$$

$$(\frac{5}{p}) = \begin{cases} 1, & \text{if } p \equiv \pm 1 \ (\mathrm{mod}\ 5), \\ -1, & \text{if } p \equiv \pm 2 \ (\mathrm{mod}\ 5). \end{cases} \tag{2.5}$$

Definition 2.5 ([11, Jacobi symbol]). It is convenient to extend the Legendre symbol $(\frac{\alpha}{p})$ to a symbol $(\frac{\alpha}{m})$, where m is an arbitrary odd integer; this generalization is called the Jacobi symbol. Whenever m is an odd prime, we take $(\frac{\alpha}{m})$ to be the Legendre symbol. We now extend this by multiplicativity to all positive odd integers m, i.e., if $m = p_1^{s_1} \cdots p_k^{s_k}$ where the p_i are odd primes, set

$$(\frac{\alpha}{m}) = (\frac{\alpha}{p_1})^{s_1} \cdots \frac{\alpha}{p_k})^{s_k}.$$

As usual with empty products, we set $(\frac{\alpha}{1}) = 1$.

Theorem 2.6 ([17, Reciprocity theorem]). *If* $p \neq q$ *are odd primes, then*

$$(\frac{p}{q})(\frac{p}{q}) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} = \begin{cases} 1, & \text{if } p \text{ or } q \equiv \pm 1 (\text{mod } 4), \\ -1, & \text{otherwise}. \end{cases} \tag{2.6}$$

Theorem 2.7 ([17, Quadratic reciprocity law]). *If* m, n *are odd numbers such that* $(m, n) = 1$*, then*

$$\begin{aligned} (\frac{2}{n}) &= (-1)^{\frac{n^2-1}{8}}, \\ (\frac{m}{n})(\frac{n}{m}) &= (-1)^{\frac{(m-1)}{2}\frac{(n-1)}{2}}. \end{aligned} \tag{2.7}$$

Definition 2.8 ([7]). A $D(m)$-triple $\{u, v, w\}$ is called regular if it satisfies the condition

$$(w - v - u)^2 = 4(u.v + m). \tag{2.8}$$

3. Theorems and results

Theorem 3.1. *A set* $D(3^3) = \{1, 9, 22\}$ *is regular Diophantine triple but can not be extended to* $D(3^3)$*-Diophantine quadruple.*

Proof. If we consider Definition 2.4 and apply (2.10) to $\{1, 9, 22\}$, it is seen that $D(3^3) = \{1, 9, 22\}$ is a regular Diophantine set. We suppose that $\{1, 9, 22\}$ can be extended to quadruple for any positive integer β. Then, $\{1, 9, 22, \beta\}$ is a $D(27)$ Diophantine set. So, there exist a, b, c integers such that;

$$\beta + 27 = a^2, \tag{3.1}$$

$$9\beta + 27 = b^2, \tag{3.2}$$

$$22\beta + 27 = c^2. \tag{3.3}$$

By eliminating β between (3.1) and (3.2), we have

$$9a^2 - b^2 = 216. \tag{3.4}$$

In (3.4), the left side can be written as difference of two squares since 9 is a perfect square, so, we have $(3a - b)(3a + b) = 216$. Factorizing 216 as finitely, we have Table 1.

Table 1: Solutions of $9a^2 - b^2 = 216$.

Solution	Class 1 solutions 1	Class 2 solutions
a	± 7	± 5
b	± 15	± 3

By dropping β between (3.1) and (3.3), then we have

$$22a^2 - c^2 = 567. \tag{3.5}$$

Considering solution 1, substituting $a^2 = 49$ into (3.5) we get $c^2 = 511$ where c is not an integer solution. In the same way, substituting another solution $a^2 = 25$ into the (3.5), we obtain $c^2 = -17$ which is a contradiction. It is shown that any value of c is not integer for the solution of (3.5). So, there is no such $\beta \in \mathbb{Z}^+$ and the set $P_{+27} = \{1, 9, 22\}$ can not be extended. □

Theorem 3.2. *A* $P_{+27} = \{1, 142, 169\}$ *set is both regular and non-extendible to* P_{+27}*-Diophantine quadruple.*

Proof. We can easily see that $\{1, 142, 169\}$ is a regular P_{+27} triple set from (2.8) in the Definition 2.8. Suppose that there exists a positive integer d such that $\{1, 142, 169, d\}$ is a P_{+27} set. Then the following equations have integral solutions x, y, z in the set of integers.

$$d + 27 = u^2, \tag{3.6}$$

$$142d + 27 = v^2, \tag{3.7}$$

$$169d + 27 = w^2. \tag{3.8}$$

From (3.6) and (3.8), we obtain

$$169u^2 - w^2 = 4536. \tag{3.9}$$

By factorizing the left side of (3.9), we have

$$(13u - w)(13u + w) = 4536. \tag{3.10}$$

If we search the solutions of the (3.10), we get Table 2.

Table 2: Solutions of $169u^2 - w^2 = 4536$.

Solution (u,w)	Class 1 solutions $(\pm 15, \pm 183)$	Class 2 solutions $(\pm 13, \pm 155)$

From (3.6) and (3.7), we have

$$142u^2 - v^2 = 3807. \tag{3.11}$$

If we substituting $u^2 = 225$ or $u^2 = 169$ into the (3.11), we obtain $v^2 = 28143$ or $v^2 = 20191$ which they are not integer solutions of (3.11) in return. Hence, $P_{+27} = \{1, 142, 169\}$ is non-extendible to P_{+27}-Diophantine quadruple. □

Theorem 3.3. *A* $P_{+27} = \{1, 169, 198\}$ *set is not only regular triple but also non-extendible.*

Proof. Let consider the set $\{1, 169, 198\}$. By applying (2.8) condition into the elements of the set, it is seen that $P_{+27} = \{1, 169, 198\}$ is a regular Diophantine triple. Assume that there exists a positive integer $\mathcal{V}$ such that $\{1, 169, 198, \mathcal{V}\}$ is a P_{27} quadruple. Then there are x, y, z integers such that

$$\mathcal{V} + 27 = x^2, \tag{3.12}$$

$$169\mathcal{V} + 27 = y^2, \tag{3.13}$$

$$198\mathcal{V} + 27 = z^2. \tag{3.14}$$

By dropping $\mathcal{V}$ between (3.12) and (3.13), we obtain

$$169x^2 - y^2 = 4536, \tag{3.15}$$

and from (3.12) and (3.14) we get

$$198x^2 - z^2 = 5319. \tag{3.16}$$

In (3.15), if we use factorization of integers, we write left side as $(13x - y)(13x + y) = 4536$. Also, 4536 can be factorized as finitely. So, integer solutions of (3.15) are as Table 2 for (x, y). If we substitute $x^2 = 225$ or $x^2 = 169$ into the (3.16), we have $z^2 = 39231$ or $z^2 = 28143$, where they are not integer solutions of (3.16), respectively.

Thus, there is no such $\mathcal{V} \in \mathbb{Z}^+$ and the $\{1, 169, 198\}$ cannot be extended to P_{27} quadruple. □

Remark 3.4. There are many different types of regular P_{+27}-Diophantine triples such as $\{2,11,27\}$, $\{2,47,71\}$, $\{3,18,39\}$, $\{3,66,99\}$, $\{6,9,33\}$, $\{9,13,46\}$, $\{11,18,59\}$, and so on. In here, we just prove some of them by applying factorization method in the set of integers.

Theorem 3.5. *There isn't any set* P_{+27} *including elements divided by* $4,5,7$, *or* 17.

Proof.

(a) We suppose that u is an element of set P_{+27}. If 4α is also an element of set P_{+27} for $\alpha \in \mathbb{Z}$, then we have

$$4\alpha u + 27 = A^2 \tag{3.17}$$

must be satisfied for some integer A. Applying (mod 4) on the both sides of (3.17), we obtain following inequality

$$A^2 \equiv 3 (\text{mod } 4). \tag{3.18}$$

If A is odd integer then we get $A^2 \equiv 1(\text{mod } 4)$ and also $A^2 \equiv 0(\text{mod } 4)$ for even integer A. So, (3.18) can not solvability. This is a contradiction. Therefore, 4α can not be an element of P_{+27} for any $\alpha \in \mathbb{Z}$.

(b) We suppose that v and $5\beta, (\beta \in Z)$ are elements of the set P_{+27}, then

$$5\beta v + 27 = B^2 \tag{3.19}$$

is satisfied for integer B. Applying (mod 5) to (3.19), we get

$$B^2 \equiv 2(\text{mod } 5) \tag{3.20}$$

has solutions if and only if Legendre symbol is $(\frac{2}{5}) = 1$. If we consider Lemma 2.4 and applying (2.4), we obtain $(\frac{2}{5}) = -1$ which implies that (3.20) has no solution. This is a contradiction. Consequently, there is no set P_{+27} involving elements with 5.

(c) In a similar way of (a) or (b), assume that w is an element of set P_{+27}. If $7\gamma, (\gamma \in \mathbb{Z})$ is also an element of set P_{+27}, then

$$7\gamma w + 27 = C^2$$

is obtained for integer C. If we apply (mod 7), we have

$$C^2 \equiv -1(\text{mod } 7). \tag{3.21}$$

Using (2.2) from Lemma 2.4, we have

$$(\frac{-1}{7}) = (-1)^{\frac{7-1}{2}} = -1.$$

This shows that equation (3.21) is unsolvable. Hence, 7γ can not be an element of P_{+27} for $\gamma \in \mathbb{Z}$.

(d) In the same manner, suppose that r and $17\theta, (\theta \in \mathbb{Z})$ are elements of set P_{+27}. Then

$$17\theta r + 27 = D^2 \tag{3.22}$$

has to get solution for integer D. Applying (mod 17) to (3.22), we get

$$D^2 \equiv 10(\text{mod } 17). \tag{3.23}$$

Using (2.7) of Theorem 2.7, then we obtain

$$(\frac{10}{17}) = (\frac{2}{17})(\frac{5}{17}). \tag{3.24}$$

Applying (2.4) and (2.5) of Lemma 2.4 into the (3.24), we have $(\frac{2}{17}) = +1$ and $(\frac{5}{17}) = -1$. These imply that $(\frac{10}{17}) = -1$ and the equation (3.23) isnt solvable. Therefore, 17θ can not be an element of P_{+27} for $\theta \in \mathbb{Z}$. □

Theorem 3.6. *A* $D(-27) = \{1, 31, 36\}$ *set can not be extended to* $D(-27)$ *quadruple but it is regular* $D(-27)$ *Diophantine triple.*

Proof. The set $\{1, 31, 36\}$ has the property of $D(-27)$ Diophantine set. From (2.8) of Definition 2.8, it is clear that $\{1, 31, 36\}$ is a regular $D(-27)$ Diophantine triple. Let show that the set is non-extendible. Let μ be any other positive integer in $\{1, 31, 36, \mu\}$. Then following equations hold for some X, Y, Z integers.

$$\mu - 27 = X^2, \tag{3.25}$$
$$31\mu - 27 = Y^2, \tag{3.26}$$
$$36\mu - 27 = Z^2. \tag{3.27}$$

Elimination of μ between (3.25) and (3.27) as well as between (3.25) and (3.26), we obtain following equations, respectively.

$$Z^2 - 36^2 = 945, \tag{3.28}$$
$$Y^2 - 31X^2 = 810. \tag{3.29}$$

By using factorization, solutions of (3.28) are obtained as Table 3. Putting solutions from Table 3 into the

Table 3: Solutions of $Z^2 - 36X^2 = 945$.

Solutions (X,Z)	Class 1 solutions $(\pm 26, \pm 159)$	Class 1 solutions $(\pm 8, \pm 57)$	Class 2 solutions $(\pm 4, \pm 39)$	Class 3 solutions $(\pm 2, \pm 33)$

(3.29), we have $Y^2 = 21766$, $Y^2 = 2794$, $Y^2 = 1306$ or $Y^2 = 934$ for $X^2 = 676$, $X^2 = 64$, $X^2 = 16$ or $X^2 = 4$, respectively. These imply that they (values of Y) are not integer solution of (3.29). Thus, there is no such $\mu \in \mathbb{Z}^+$ and the $D(-27) = \{1, 31, 36\}$ set is non-extendible to Diophantine $D(-3^3)$ quadruple. □

Theorem 3.7. *A* $D(-3^3) = \{1, 36, 43\}$ *is regular triple set however it can not be extended.*

Proof. Diophantine triple $D(-3^3) = \{1, 36, 43\}$ is regular triple set since it satisfies (2.8) condition of Definition 2.8. Assume that $\{1, 36, 43, \varphi\}$ is a $D(-3^3)$ Diophantine quadruple. So, there are A, B, C integers such that

$$\varphi - 27 = A^2, \tag{3.30}$$
$$36\varphi - 27 = B^2, \tag{3.31}$$
$$43\varphi - 27 = C^2. \tag{3.32}$$

Dropping φ between (3.30) and (3.31), we obtain

$$B^2 - 36A^2 = 945 \tag{3.33}$$

and similarly from (3.30) and (3.32), we have

$$C^2 - 43A^2 = 1134. \tag{3.34}$$

The Table 3 gives the solutions of (3.33) if we take (A, B) instead of (X, Z) in such table. Putting values of A^2 from Table 3 into the (3.34), respectively, we obtain $C^2 = 30202$, $C^2 = 3886$, $C^2 = 1822$, or $C^2 = 1306$. It shows that C is not integer yields (3.34). So, there is no positive φ integer and the $D(-3^3) = \{1, 36, 43\}$ is not extendible to $D(-3^3)$ Diophantine quadruple. □

Theorem 3.8. *A Diophantine triple* $P_{(-3^3)} = \{2, 18, 26\}$ *is regular and non-extendible to* $D(-3^3)$ *Diophantine quadruple.*

Proof. From (2.8) in Definition 2.8, it is clear that $\{2, 18, 26\}$ is regular Diophantine triple. Assume that there exists a positive integer k such that $\{2, 18, 26, k\}$ is a $P_{(-3^3)}$ quadruple. Then there exist x, y, z integers such that

$$2k - 27 = x^2, \quad (3.35)$$
$$18k - 27 = y^2, \quad (3.36)$$
$$26k - 27 = z^2. \quad (3.37)$$

Dropping k from (3.35) and (3.36), we obtain $y^2 - 9x^2 = 216$. Using factorization, we have the solutions of $y^2 - 9x^2 = 216$ as Table 4.

Table 4: Solutions of $y^2 - 9x^2 = 216$.

Solutions	Class 1 solutions	Class 2 solutions
(x,y)	$(\pm5, \pm21)$	$(\pm1, \pm15)$

From (3.35) and (3.37), we get

$$z^2 - 13x^2 = 324. \quad (3.38)$$

Substituting solutions $x^2 = 25$ or $x^2 = 1$ into the (3.38), we have $z^2 = 649$ or $z^2 = 337$ which are not integer solutions of (3.38), respectively. Thus, there is no such $k \in \mathbb{Z}^+$ and the $P_{-3^3} = \{2, 18, 26\}$ set can not be extended to P_{-3^3} Diophantine quadruple. □

Theorem 3.9. *A $P_{-27} = \{3, 9, 12\}$ set can not be extended to P_{-3^3} Diophantine quadruple however it is regular P_{-27}-Diophantine triple.*

Proof. It is trivial that $\{3, 9, 12\}$ triple is regular from (2.8). Let us determine whether or not there is any other positive integer in this set. Assume that $\{3, 9, 12, d\}$ is P_{-27} Diophantine quadruple. Then following equations are satisfied for some a, b, c integers.

$$3d - 27 = a^2, \quad (3.39)$$
$$9d - 27 = b^2, \quad (3.40)$$
$$12d - 27 = c^2. \quad (3.41)$$

Elimination of d between (3.39) and (3.41) as well as (3.39) and (3.40), we have

$$c^2 - 4a^2 = 81, \quad (3.42)$$
$$b^2 - 3a^2 = 54. \quad (3.43)$$

Solutions of (3.42) can be given as Table 5.

Table 5: Solutions of $c^2 - 4a^2 = 81$.

Solutions	Class 1 solutions	Class 2 solutions	Class 3 solutions
(a,c)	$(\pm20, \pm41)$	$(\pm6, \pm15)$	$(\pm0, \pm9)$

Putting solutions into the (3.43), we get $b^2 = 1254$, $b^2 = 162$ or $b^2 = 54$ which they are not integer solution of (3.43). Thus, there is no such $d \in Z^+$ and the $P_{-27} = \{3, 9, 12\}$ Diophantine triple isnt extendible to P_{-27} quadruple. □

Theorem 3.10. *Sets $P_{-27} = \{4, 7, 9\}$ and $P_{-27} = \{4, 9, 19\}$ are regular Diophantine triple but can not be extended to P_{-27}-Diophantine quadruple.*

Proof. Both of the P_{-27} sets in the Theorem 3.10 are regular since (2.8) condition is satisfied. Assume that $\{4, 7, 9, \alpha\}$ is a P_{-27} set. There are X, Y, Z integers such that,

$$4\alpha - 27 = X^2, \tag{3.44}$$

$$7\alpha - 27 = Y^2, \tag{3.45}$$

$$9\alpha - 27 = Z^2. \tag{3.46}$$

Eliminating α between (3.44) and (3.46), we obtain

$$-9X^2 + 4Z^2 = 135 \tag{3.47}$$

and similarly, dropping α between (3.44) and (3.45) we have

$$-7X^2 + 4Y^2 = 81. \tag{3.48}$$

By (3.47), we have $(2Z - 3X)(2Z + 3X) = 135$ and solutions are found as $(X, Z) = (\pm 7, \pm 12)$ or $(X, Z) = (\pm 1, \pm 6)$. Substituting solutions $X^2 = 49$ or $X^2 = 1$ into the (3.48), we get $Y^2 = 106$ or $Y^2 = 22$ in return. It proves that Y is not integer yields (3.48). So, there is no α integer and the $P_{-27} = \{4, 7, 9\}$ is non-extendible to quadruple. In a similar way, there are x, y, z integers such that

$$4\beta - 27 = x^2, \tag{3.49}$$

$$9\beta - 27 = y^2, \tag{3.50}$$

$$19\beta - 27 = z^2 \tag{3.51}$$

for $P_{-27} = \{4, 9, 19, \beta\}$ Diophantine quadruple. Dropping β between (3.49) and (3.50) we get the following equation same as solutions of (3.47) are obtained,

$$(x, y) = (\pm 7, \pm 12), \quad (x, y) = (\pm 1, \pm 6).$$

Eliminating β between (3.49) and (3.51), we get $4z^2 - 19x^2 = 405$. Substituting solutions $(x, y) = (\pm 7, \pm 12)$, $(x, y) = (\pm 1, \pm 6)$ into the $4z^2 - 19x^2 = 405$ then we have $z^2 = 334$ or $z^2 = 106$ such that z is not integer. Hence, there is no such positive β integer and the $P_{-27} = \{4, 9, 19\}$ set is non-extendible to Diophantine P_{-27} quadruple. □

Theorem 3.11. *There isn't any set P_{-27} including R elements satisfying any of following conditions:*

(i) $R \in \mathbb{Z}$, $R \equiv 0 \pmod 5$;
(ii) $R \in \mathbb{Z}, R \equiv 0 \pmod 8$;
(iii) $R \in \mathbb{Z}, R \equiv 0 \pmod{11}$;
(iv) $R \in \mathbb{Z}, R \equiv 0 \pmod{17}$.

Proof.

(i) If $R = 5u$ and t are elements of set P_{-27} for $u \in \mathbb{Z}$, then we obtain

$$5ut - 27 = X^2. \tag{3.52}$$

Applying (mod 5) on both sides of the (3.52), we have

$$X^2 \equiv 3 (\text{mod } 5).$$

Using (2.6) of Theorem 2.6, we get

$$(\frac{3}{5})(\frac{5}{3}) = (-1)^{\frac{3-1}{2}\frac{5-1}{2}} = -1.$$

And $(\frac{5}{3}) = (\frac{2}{3}) = -1$ from (2.4) in Lemma 2.4. So, we obtain

$$(\frac{3}{5}) = -1.$$

This is a contradiction. So, there is no P_{-27} set containing any elements such that $R \in \mathbb{Z}$, $R \equiv 0 \pmod 5$.

(ii) Assume that s and $8v$, $(v \in \mathbb{Z})$ are elements of set P_{-27}, then we have

$$8vs - 27 = Y^2$$

for integer Y. Also, $Y^2 \equiv 5 \pmod 8$ is satisfied. By Lemma 2.1 and considering residue classes $(\bmod\ 8)$, we get $Y^2 \equiv 0, 1, 4 (\bmod\ 8)$. So, 5 is non-quadratic residue (mod 8) and it is a contradiction. Hence, there isnt any set P_{-27} including any element satisfying $R \in \mathbb{Z}, R \equiv 0 (\bmod\ 8)$.

(iii) If m and $R \in \mathbb{Z}$, $R \equiv 0 \pmod{11}$ are elements of set P_{-27},

$$11tm - 27 = Z^2 \tag{3.53}$$

satisfies for integer Z. Applying (modulo 11), we get

$$Z^2 \equiv 6 (\bmod\ 11).$$

Considering property of Legendre symbol from Definition 2.3, we have

$$(\frac{6}{11}) = (\frac{2}{11})(\frac{3}{11}).$$

From (2.4) and (2.3), we obtain $(\frac{2}{11}) = -1$ and $(\frac{3}{11}) = +1$. So, we get $(\frac{6}{11}) = -1$. The equation (3.53) is not solvable. Thus, $R \in \mathbb{Z}$, $R \equiv 0 \pmod{11}$ can not be an element of P_{-27}.

(iv) Assume that t is an element of set P_{-27}, if $R \in \mathbb{Z}$, $R \equiv 0 (\bmod\ 17)$ is also an element of set P_{-27}, then

$$17tn - 27 = T^2 \tag{3.54}$$

for integer T. If we apply (mod 17) on the both sides of (3.54), we get

$$T^2 \equiv 7 (\bmod\ 17).$$

Using Theorem 2.6, we have

$$(\frac{7}{17})(\frac{17}{7}) = (-1)^{(\frac{7-1}{2})(\frac{17-1}{2})} = 1.$$

From the Legendre symbol's property, we can write $(\frac{17}{7}) = (\frac{3}{7})$. Using (2.3) in Lemma 2.4, we obtain that $(\frac{3}{7}) = -1$. So, we have

$$(\frac{7}{17}) = -1$$

This shows that there is no integer T satisfying (3.54). It is a contradiction. Therefore, there is no set P_{-27} including any elements such that $R \in \mathbb{Z}$, $R \equiv 0 (\bmod\ 17)$. □

Remark 3.12. Using similar technique, readers also can prove that $\{1, 36, 91\}$, $\{1, 171, 196\}$, $\{3, 12, 21\}$, $\{4, 27, 49\}$, $\{4, 49, 79\}$, $\{6, 18, 42\}$, $\{7, 9, 28\}$, and so on. They are regular $D(-3^3)$ Diophantine triples although they aren't extendible $D(-3^3)$ Diophantine quadruple.

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