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## Homotopy analysis method for solving MHD free convection flow from a cooling sheet



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### Abstract

In this paper, we investigate the problem of MHD free convection cooling of a low-heat-resistance sheet that moves downwards in a viscous fluid. The basic equations are converted into coupled ordinary differential equations via the similarity transformation, and solved analytically using homotopy analysis method (HAM). The obtained analytical solutions for both of the velocity and the temperature with different values of the Prandtl number  $Pr$  and the magnetic parameter  $M$  are plotted and discussed in detail.

**Keywords:** Homotopy analysis method, cooling, MHD flow, similarity solution.

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### 1. Introduction

Convective flows in porous media have a wide range of applications in engineering and science. Examples of these applications include solar energy systems, boilers, cooling of electronic devices, compact heat exchangers and the cooling core of nuclear reactors (see [3, 13, 15, 17, 21, 24]). Furthermore, Magnetohydrodynamics (MHD) which is the study of the interaction of conducting fluids with electromagnetic phenomena, is of a great importance due to its applications in various areas of technology and engineering such as designing cooling systems which are liquid metals, MHD generators, accelerators, pumps and flow meters.

The homotopy analysis method (HAM), proposed by Liao in his PhD thesis [28], is an analytic tool for handling nonlinear problems, it can be considered as one of the powerful method for solving nonlinear problems since it yields rapidly convergent series solutions in most cases, which lead usually to very accurate solutions only in few iterations. In recent years, this method has been successfully used to solve many types of nonlinear problems in science and engineering such as the magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet [1], boundary-layer flows over an impermeable

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stretched plate [2], nonlinear model of combined convective and radiative cooling of a spherical bodies, exponentially decaying boundary layers [9], unsteady boundary-layer flows over a stretching flat plate [10]. Also, the boundary-layer flow problem was considered by Ghotbi et al. [12] and Hashim et al. [24]. The present paper extends the results of previous authors by considering the effect of the magnetic field on the velocity and the temperature on velocity slip. The method we employed here is based on the homotopy analytical method (HAM).

More recently, Bataineh et al. [5] applied the standard HAM to solve some problems in engineering sciences. All of these successful applications verified the validity and the effectiveness of the HAM.

In this paper we aim to find the analytical solutions for the flow problem considered in [12, 24]. Analytic solutions for the velocity and temperature components are obtained using a powerful, east-to-use technique, namely the homotopy analysis method (HAM). The series solution is firstly computed and then its convergence is properly discussed. Finally, the graphs are plotted and discussed for the variation of Prandtl numbers and Hartmann numbers. The method gives rapidly convergent series with specific significant features for each scheme.

## 2. Formulation of the problem

Consider the problem of cooling of a low-heat-resistance sheet that moves downwards in a viscous fluid in presence of a magnetic field. This phenomenon can be modeled by (cf. [3, 11, 26])

$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + g\beta(T - T_0) - \frac{\sigma B^2 x}{\rho} u, \\ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \kappa \frac{\partial^2 T}{\partial y^2}. \end{cases} \quad (2.1)$$

with the conditions

$$\begin{cases} u(x, 0) = 0, & v(x, 0) = 0, \\ u(x, \infty) = 0, & T(x, \infty) = T_0, \end{cases} \quad (2.2)$$

where  $u$  and  $v$  are the velocity components in the  $x$  and  $y$  directions, respectively,  $T$  is the temperature,  $T_0$  is the temperature of the surrounding fluid,  $\nu$  is the kinematic viscosity,  $\kappa$  is the thermal diffusivity,  $g$  is the acceleration due to gravity, and  $\beta$  is the coefficient of thermal expansion.

Using the similarity variables:

$$\begin{cases} \eta(x, y) = \left( \frac{g\beta(T_1 - T_0)x_0^3}{\nu^2} \right)^{1/4} \left( \frac{y}{x_0 - x} \right), \\ \psi(x, y) = \left( g\beta(T_1 - T_0)\nu^2 x_0^3 \right)^{1/4} f(\eta), \\ T(x, y) = T_0 + (T_1 - T_0) \left( \frac{x_0}{x_0 - x} \right)^3 \theta(\eta), \end{cases}$$

Where  $\psi$  is the stream function defined by  $u = \frac{\partial \psi}{\partial y}$  and  $v = -\frac{\partial \psi}{\partial x}$ ,  $f$  and  $\theta$  are the dimensionless stream function and temperature, respectively, and  $T(0, 0) = T_1$ . Problem (2.1)-(2.2) is transformed to the following

$$\begin{cases} \theta'' = 3Prf'\theta, \\ f''' - f'^2 - Mf' + \theta = 0, \end{cases} \quad (2.3)$$

subject to the boundary conditions

$$\begin{cases} f(0) = 0, & f'(0) = 0, & f'(\infty) = 0, \\ \theta(0) = 1, & \theta(\infty) = 0, \end{cases} \quad (2.4)$$

where the prime is for differentiation with respect to the similarity variable  $\eta$ ,  $Pr = \frac{\nu}{\kappa}$ , is the Prandtl

number,  $B(x) = \frac{\sigma}{\rho} B_0^2 \frac{(g\beta(T_1 - T_0)x_0^3)^{1/2}}{(x_0 - x)^2}$  is the magnetic field, and  $M = \frac{\sigma}{\rho} B_0^2$  is the Hartmann number.

In the remainder of this paper, we shall solve this two-point boundary value problem analytically using a Homotopy analysis method.

### 3. The HAM solutions of the problem

In this section, we apply the homotopy analysis method to solve the system (2.3)-(2.4). The first step in the solution process is to choose a proper initial guess approximations satisfying the boundary conditions and to find the linear operator. For the present problem, we choose the initial guess approximations for velocity and temperature as follows

$$\begin{cases} \theta_0(\eta) = e^{-\eta}, \\ f_0(\eta) = 0. \end{cases}$$

Besides, we select the auxiliary linear operators  $L_1$  and  $L_2$  for velocity and temperature functions, respectively, of the form

$$\begin{cases} L_1(f) = f''' - f', \\ L_2(\theta) = \theta'' - \theta, \end{cases}$$

satisfying the following properties

$$\begin{cases} L_1(C_1 + C_2e^\eta + C_3e^{-\eta}) = 0, \\ L_2(C_4e^\eta + C_5e^{-\eta}) = 0, \end{cases}$$

where  $C_i, i = 1, \dots, 5$ .

#### 3.1. Zeroth-order deformation problems

First we construct the zeroth-order deformation problem as the following

$$\begin{cases} (1 - p)L_1[f(\eta; p) - f_0(\eta)] = p\hbar_1 N_1[f(\eta; p), \theta(\eta; p)], \\ f(0; p) = 0, \quad f'(0; p) = 0, \quad f'(\infty; p) = 0, \\ (1 - p)L_2[\theta(\eta; p) - \theta_0(\eta)] = p\hbar_2 N_2[f(\eta; p), \theta(\eta; p)], \\ \theta(0; p) = 1, \quad \theta(\infty; p) = 0, \end{cases} \tag{3.1}$$

where

$$\begin{cases} N_1[f(\eta; p), \theta(\eta; p)] = \frac{\partial^3 f(\eta; p)}{\partial \eta^3} + \theta(\eta; p) - M \frac{\partial f(\eta; p)}{\partial \eta} - \left( \frac{\partial f(\eta; p)}{\partial \eta} \right)^2, \\ N_2[f(\eta; p), \theta(\eta; p)] = \frac{\partial^2 \theta(\eta; p)}{\partial \eta^2} - 3Pr \frac{\partial f(\eta; p)}{\partial \eta} \theta(\eta; p), \end{cases}$$

where  $p \in [0, 1]$  is the so-called embedding parameter,  $\hbar_1$  and  $\hbar_2$  are the non-zero auxiliary parameter. Notice that  $p = 0$  and  $p = 1$  give

$$f(\eta; 0) = f_0(\eta) \quad \text{and} \quad \theta(\eta; 0) = \theta_0(\eta).$$

Where  $f(\eta; p)$  (resp.  $\theta(\eta; p)$ ) varies from the known initial approximation  $f_0(\eta; p)$  (resp.  $\theta_0(\eta; p)$ ) to the unknown solution  $f(\eta)$  (resp.  $\theta(\eta)$ ) when the embedding-parameter  $p$  moves from 0 to 1.

Expanding  $f(\eta; p)$  and  $\theta(\eta; p)$  in Taylor series with respect to the embedding parameter  $p$ , we get

$$\begin{cases} f(\eta; p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) p^m, \quad f_m(\eta) = \frac{1}{m!} \frac{\partial^m f(\eta; p)}{\partial p^m}, \\ \theta(\eta; p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) p^m, \quad \theta_m(\eta) = \frac{1}{m!} \frac{\partial^m \theta(\eta; p)}{\partial p^m}. \end{cases} \tag{3.2}$$

Note that the auxiliary parameters  $\hbar_1$  and  $\hbar_2$  are properly chosen in such a way that these two series are convergent series (3.2) are convergent at  $p = 1$ . Therefore we have through equation (3.2)

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \quad \theta(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta).$$

### 3.2. $m^{\text{th}}$ -order deformation problems

Differentiating the zeroth-order deformation equations (3.1)  $m$ -times with respect to  $p$ , then setting  $p = 0$ , and finally dividing them by  $m!$ , we obtain the  $m^{\text{th}}$ -order deformation equations

$$\begin{cases} L_1[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar_1 R_m(\eta), \\ f_m(0) = f'_m(0) = f'_m(\infty) = 0, \\ L_2[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = \hbar_2 Q_m(\eta), \\ \theta_m(0) = \theta_m(\infty) = 0, \end{cases}$$

and

$$\begin{cases} R_m = f'''_{m-1} + \theta_{m-1} - \sum_{n=0}^{m-1} f'_{m-1-n} f'_n, \\ Q_m = \theta''_{m-1} - 3Pr \sum_{n=0}^{m-1} f'_{m-1-n} \theta_n - Mf'_{m-1}, \end{cases} \quad \chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

The first two deformations of the coupled solutions are computed using MAPLE software, we obtain the following solutions of problem (2.3)-(2.4) for the Hrtmann number  $M = 0.5$  and the Prandtl  $Pr = 1$ .

$$\left\{ \begin{array}{l} \bullet f_1(\eta) = -0.1666666667 \frac{h}{e^{2t}} + 1.416118082 \times 10^{-18} h e^t + 0.3333333333 \frac{h}{e^t} - 0.1666666667 h, \\ \bullet \theta_1(\eta) = 0.3333333333 \frac{h}{e^{2t}} + 3.119206702 \times 10^{-14} h e^t - 0.3333333334 \frac{h}{e^t}, \\ \bullet f_2(\eta) = -0.1666666667 \frac{h}{(e^t)^2} + 1.416118082 \times 10^{-18} h e^t + 0.3333333334 \frac{h}{e^t} - 0.1666666667 h \\ \quad - 1.559638754 \times 10^{-14} h^2 t - 0.06250000000 \frac{h^2}{(e^t)^3} - 1.559638754 \times 10^{-14} h^2 \ln(e^t), \\ \quad + 0.08333333332 \frac{h^2}{(e^t)^2} + 0.02083333340 \frac{h^2}{e^t} + 8.857167512 \times 10^{-20} h^2 e^t - 0.04166666672 h^2 \\ \bullet \theta_2(\eta) = 0.3333333333 \frac{h}{(e^t)^2} + 3.119206702 \times 10^{-14} h e^t - 0.3333333333 \frac{h}{e^t} + 0.2916666664 \frac{h^2}{(e^t)^3} \\ \quad - 3.119206702 \times 10^{-14} h^2 - 0.1138888885 \frac{h^2}{e^t} + 2.124177123 \times 10^{-18} \frac{h^2 t}{e^t} - 0.1111111112 \frac{h^2}{(e^t)^2} \\ \quad - 0.06666666670 \frac{h^2}{(e^t)^4} + 1.065730551 \times 10^{-14} h^2 e^t. \end{array} \right.$$

The solutions  $f_i(\eta)$  and  $\theta_i(\eta)$  for  $i \geq 3$  were too long to be mentioned here, therefore, they are shown graphically. But it is easy to compute them by using Maple software.

## 4. Convergence of the HAM solutions

According to Liao [29], the convergence and the rate of approximation for the HAM solution is strongly dependent on the values of the auxiliary parameter  $h$ . Figures 1 and 3 show the  $h$ -curves of  $f''(0)$  at different orders of approximations, for various values of the Hartmann number  $M$  and the Prandtl number  $Pr$ , respectively. We remark that the convergence is strongly dependent on the values of involving parameters.

4.1. Effect of the Hartman number  $M$

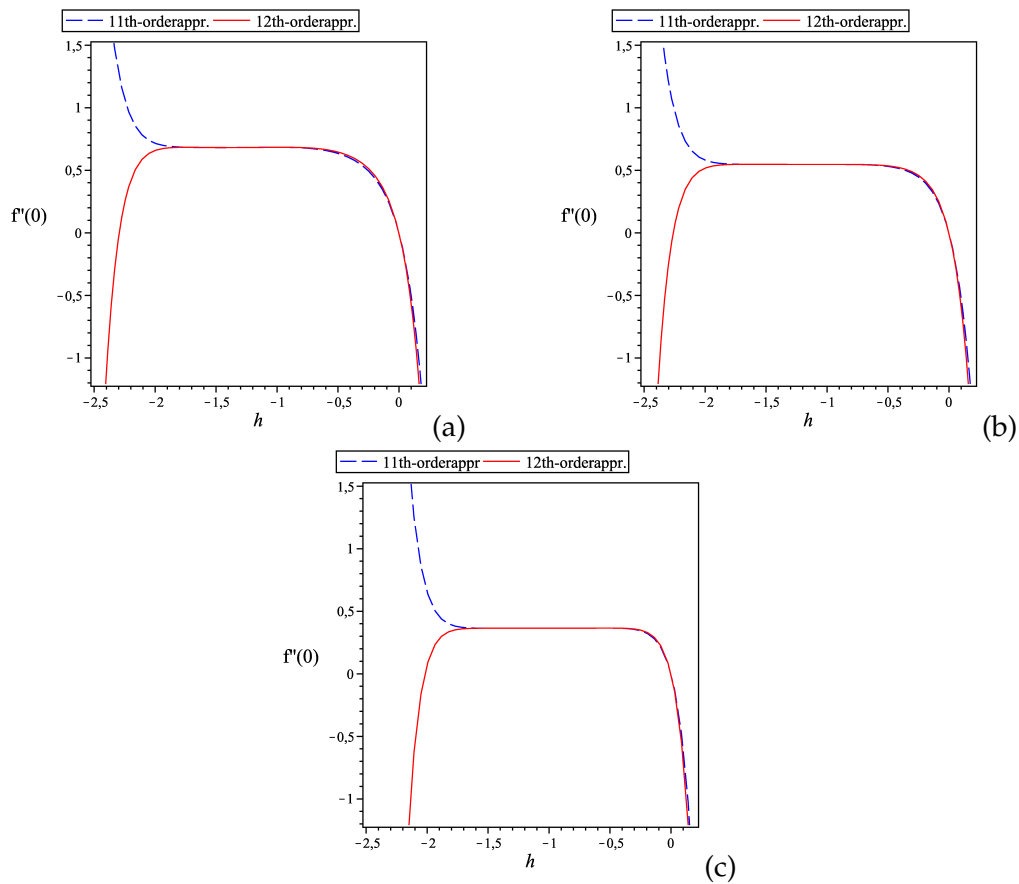
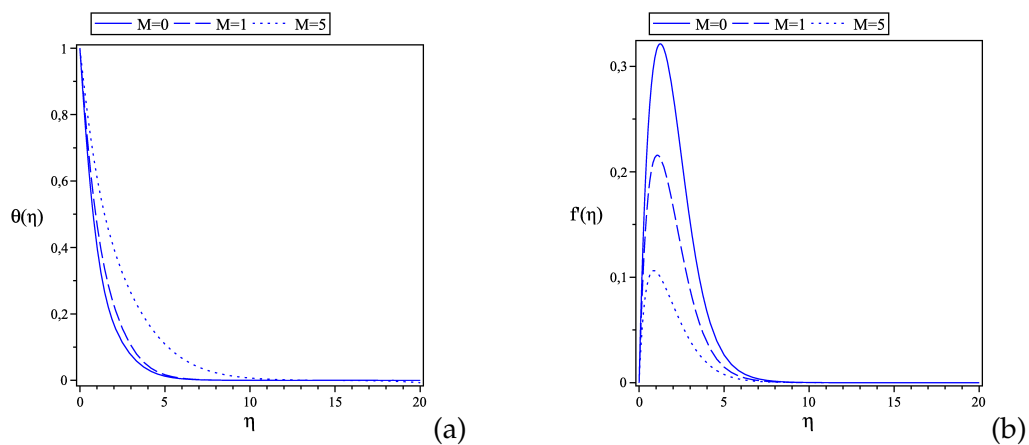


Figure 1:  $h$ -curves of  $f''(0)$  at different orders of approximations, when: (a)  $M = 0$ ,  $Pr = 1$ . (b)  $M = 1$ ,  $Pr = 1$ , and (c)  $M = 5$ ,  $Pr = 1$ .

Table 1: Variation of the Nusselt number and the skin friction for various values of the Hartmann number and  $Pr = 1$ .

| $M$   | $-\theta'(0)$ | $f''(0)$ |
|-------|---------------|----------|
| 00.00 | 0.75637       | 0.68176  |
| 01.00 | 0.65517       | 0.54707  |
| 05.00 | 0.47119       | 0.36412  |



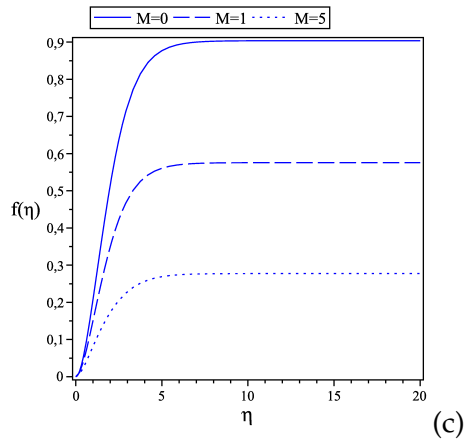


Figure 2: Effect of the Hartman number on: (a) non-dimensional temperature profiles, (b) non-dimensional velocity, (c) non-dimensional stream-function; when  $M = 0.5$ .

#### 4.2. Effect of the Prandtl number $Pr$

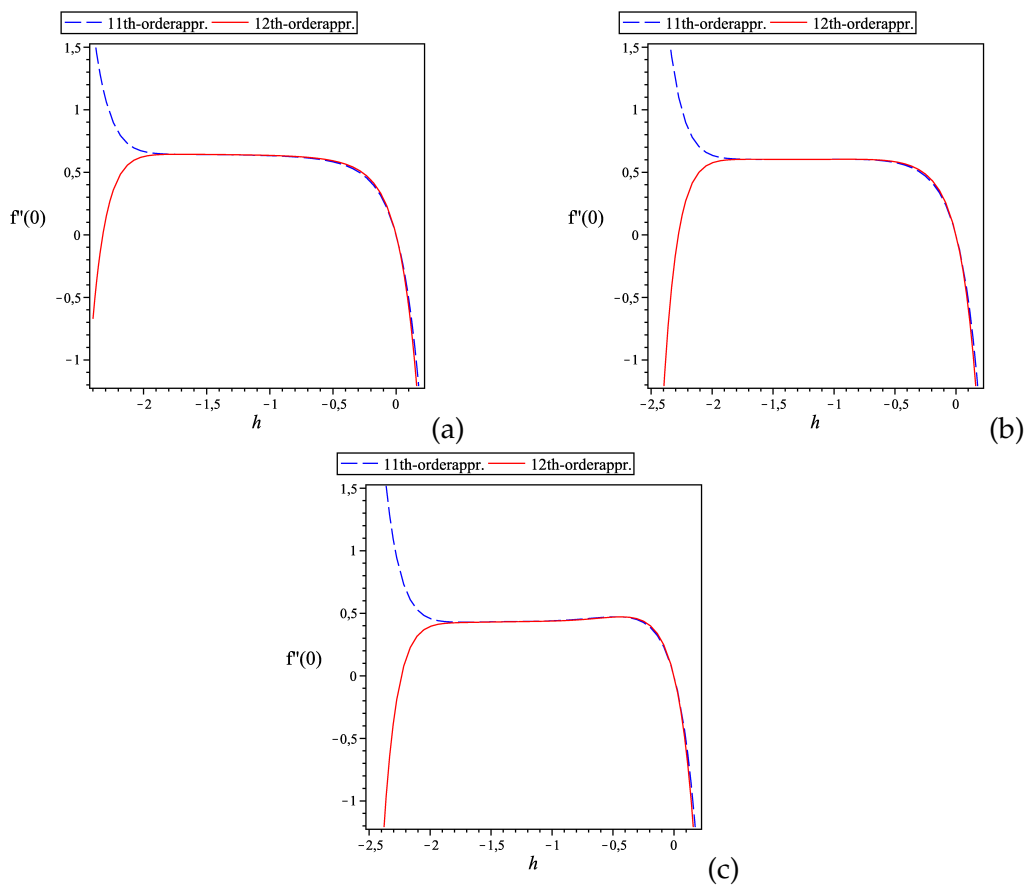


Figure 3: h-curves of  $f''(0)$  at different orders of approximations, when: (a)  $M = 0.5$ ,  $Pr = 0.1$ . (b)  $M = 0.5$ ,  $Pr = 1$ , and (c)  $M = 0.5$ ,  $Pr = 10$ .

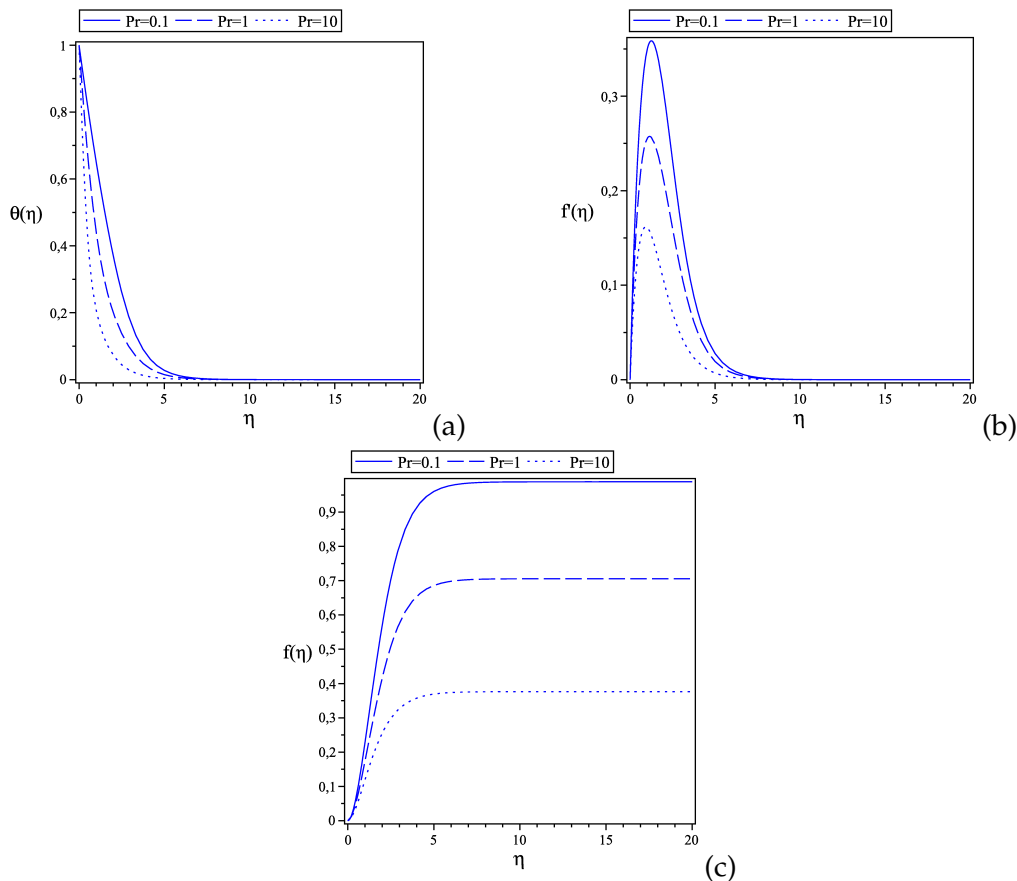


Figure 4: Effect of the Prandtl number on: (a) non-dimensional temperature profiles, (b) non-dimensional velocity, (c) non-dimensional stream-function; when  $M = 0.5$ .

Table 2: Variation of the Nusselt number and the skin friction for various values of the Prandtl number and  $M = 0.5$ .

| Pr   | $-\theta'(0)$ | $f''(0)$ |
|------|---------------|----------|
| 0.10 | 0.36906       | 0.72711  |
| 1.00 | 0.69841       | 0.60294  |
| 10.0 | 1.42065       | 0.42965  |

### 5. Summary and conclusion

A comprehensive computation is conducted for various physical parameters involved in current problem. To determine the accuracy in favor of present results, a comparison of obtained results for the skin friction and local Nusselt number are presented in Tables. It is seen from Table 1 that the increase in the Hartmann number decreases the skin-friction coefficient, which means the magnetic field retard the flow and reduces the flow velocity. Besides, increasing Hartmann numbers decreases the Nusselt number (rate of heat transfer), the reason of this is retarding effect of magnetic field on convection. On the other hand, it is observed from Fig. 2 that as magnetic parameter increases, the fluid velocity decreases. However, the fluid temperature increases with the increase in the magnetic parameter. So the magnetic field can therefore be used to control the flow characteristics. Table 2 and Figure 3 show the increase in Prandtl number causes reduction in the temperature gradient (i.e.,  $\theta'(0)$ ) at the surface of the plate. The temperature gradient reduces at the surface because low Prandtl fluid has high thermal conductivity, causing the fluid to attain higher temperature thereby reducing heat flux at the surface. Also, for such low Prandtl number, the velocity boundary layer is inside the thermal boundary layer and its thickness reduces as

Prandtl number decreases so the fluid motion is confined in more and more thinner layer near surface, which experiences increased drag (skin-friction) by the fluid. Thus higher Prandtl number leads to faster cooling of the plane sheet.

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