Available online at www.isr-publications.com/mns
Math. Nat. Sci., 4 (2019), 13–25
Research Article

Online: ISSN 2600-7665
Mathematics in Natural Science
An International Journal
Journal Homepage: www.isr-publications.com/mns

The Ulam's types stability of non-linear Volterra integro-delay dynamic system with simple non-instantaneous impulses on time scales

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Abstract

This manuscript presents Hyers-Ulam stability and Hyers-Ulam-Rassias stability results of non-linear Volterra integro-delay dynamic system on time scales with non-instantaneous impulses. Picard fixed point theorem is used for obtaining existence and uniqueness of solutions. By means of abstract Grönwall lemma, Grönwall's inequality on time scales and applications of Grönwall's inequality on time scales, we establish Hyers-Ulam stability and Hyers-Ulam-Rassias stability results. There are some primary lemmas, inequalities and relevant assumptions that helps in our stability results.

Keywords: Hyers-Ulam stability, time scale, impulses, delay dynamic equation, Grönwall's inequality, abstract Grönwall lemma, Banach fixed point theorem.

2010 MSC: 34N05, 34G20, 34A37, 45J05, 45M10.

1. Introduction

In 1940, in a talk before the mathematics club at the university of Wisconsin, Ulam [33, 34] presented a famed question related to the stability of homomorphisms: *"When an approximate homomorphism from a group G_1 to a metric group G_2 can be approximated by an exact homomorphism?"*.

This question was answered by Hyers [13] for the case when G_1 and G_2 are assumed to be Banach spaces by using direct method. So this interesting stability, initiated by Ulam and Hyers, is called Hyers-Ulam stability. In 1978, Rassias [25] extended Hyers-Ulam stability concept by introducing new function variables and after that it famed for the Hyers-Ulam-Rassias stability. In fact, the most interesting result was of Rassias [25] that weakens the condition for the bound of the norm of Cauchy difference $f(x+y)-f(x)-f(y)$. For further details and discussions, we recommend the book by Jung [15].

At the end of 19th century, a large number of researchers contributed to the stability idea of Ulam's type for various types of differential equations. There are many advantages of Ulam's type stability in

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doi: 10.22436/mns.04.01.02

Received: 2018-09-24 *Revised:* 2019-03-26 *Accepted:* 2019-05-04

tackling problems, related to optimization techniques, numerical analysis, control theory and many more, in such situations to get an exact solution is challenging. For more details on Hyers-Ulam stability, see [2, 3, 14, 16–18, 22, 23, 28–32, 35, 38, 41–54].

There are several implications of simple differential equations. Anyhow, the circumstances rather change when a real world process undergoes with unexpected variations, like significant mechanical processes, blood flows, heart beats, changes in population, radio physics, pharmacokinetics, mathematical economy, chemical technology, electrical technology, chemistry, different engineering fields, control theory and so on, see [6, 7, 21]. Sufficiently many mathematical problems in such circumstances generate a polished differential equation, which is known as impulsive differential equation.

More precisely, there are three parts of differential equations with impulse impact: an instantaneous impulsive differential equation [38], in which the impulse action is defined at certain discrete points; non-instantaneous impulsive differential equation [47], it establishes the effect of impulse on an interval; and the third one is an impulse rule, in which we define a distinct and well defined collection of impulse events having an active impulse equation.

Fractional differential and integral equations [47] play a key role not only in mathematics but also in the modeling of various physical phenomena in physics, control systems and dynamical systems. In fact, fractional order derivatives and integrals are assumed to be more realistic and practical than derivatives and integrals of integral order. These are excellent tools to model genetic transformation and memory retention qualities of several systems and products.

It is to be noted that, the pioneer of the Ulam's type stability for impulsive ordinary differentiable equation is Wang et al. [36]. Following their own work, in 2014, they proved the Hyers-Ulam-Rassias stability and generalized Hyers-Ulam-Rassias stability of impulsive evolution equations on a compact interval [37] which then they extended for infinite impulses in the same paper. Wang and Zhang [39], initially offered nonlinear differential equations having fractional integrable impulses, which are more interesting. They presented four Bielecki-Ulam's type stabilities for this class of differential equations. Also Lin et al. [19] discussed the existence and stability results for impulsive integro-differential equations. The work of Wang et al. [39] was extended by Zada et al. [43] in which they discussed Hyers-Ulam stability of higher-order nonlinear differential equations with fractional integrable impulses. They established Bielecki-Ulam-Hyers-Rassias stability, generalized Bielecki-Ulam-Hyers-Rassias stability and Bielecki-Ulam-Hyers stability for this class of differential equations on a compact interval.

However, despite the situations where only impulsive factor is involved or delay effects happened, we have a wide variety of evolutionary processes together delay and impulsive effects exist in their state. To model such phenomena which are subject to impulsive perturbations as the time delays, an impulsive delay differential equation is used.

The theory of dynamic equations on time scales has been rising fast and has acknowledged a lot of interest in recent years. This theory was introduced by Hilger [12] in 1988, with the inspiration to provide a unification of continuous and discrete calculus. For more details on time scales, see [1, 4, 5, 8–11, 20, 24, 27, 29–31, 40, 48, 49].

Recently, Zada et al. [49] obtained very interesting results about the Hyers-Ulam stability of nonlinear impulsive Volterra integro-delay dynamic system on time scales. But as far as we know that, the stability observations of Ulam's type of non-linear Volterra integro-delay dynamic systems having non-instantaneous impulses are also not yet investigated .

Motivated by the work done in [30, 49], the utmost purpose of this manuscript is to find different Hyers-Ulam and Hyers-Ulam-Rassias outcomes of stability for the following non-linear Volterra integro-dynamic system of the form

$$z^{\Delta}(t) = A(t)z(t) + \int_{t_0}^{t} \mathcal{K}(t, s, z(s))\Delta s,\ z(t_0) = z_0, \tag{1.1}$$

and for the following nonlinear Volterra integro-delay dynamic system with non-instantaneous impulses of the form

$$\begin{cases} \omega^{\Delta}(t) = M(t)\omega(t) + \int_{t_0}^{t} \mathcal{K}(t,s,\omega(s),\omega(h(s)))\Delta s, \quad t \in (s_i, t_{i+1}] \cap T_S, \ i = 0,1,\dots,m, \\ \omega(t) = g_i(t,\omega(t),\omega(h(t))), \ t \in (t_i, s_i] \cap T_S, \ i = 1,2,\dots,m, \\ \omega(t) = \alpha(t), \quad t \in [s_0 - \lambda, s_0] \cap T_S, \\ \omega(t_0) = \alpha(t_0) = \omega_0, \end{cases} \tag{1.2}$$

where $\lambda > 0$, $A(t)$ and $M(t)$ are continuous and piecewise continuous on ${T_S}^0 := [t_0, t_f]_{T_S}$, respectively, $0 = t_0 = s_0 < t_1 \leqslant s_1 \leqslant t_2 < \cdots < t_m \leqslant s_m \leqslant t_m \leqslant t_{m+1} = t_f$ are pre-fixed numbers, $\mathcal{K}(t,s,z(s))$ and $\mathcal{K}(t,s,z(s),z(h(s)))$ are continuous and piecewise continuous operators on $\Gamma = \{(t,s,z) : t_0 \leqslant s \leqslant t \leqslant t_f, \ z \in R^n\}$, respectively, $g_i : (t_i, s_i] \cap T_S \times R^n \times R^n \to R^n$, $i = 1,2,\dots,m$ are continuous functions, and $\phi : [s_0 - \lambda, s_0] \cap T_S \to R^n$ is history function. Moreover, $h : [s_0 - \lambda, t_f] \cap T_S \to (s_i, t_{i+1}] \cap T_S$ is a delay function with the consumption of continuity, additionally $h(t) \leqslant t$.

2. Preliminaries

In this section, we recall the main definitions and some basic notations of time scales calculus.

An arbitrary non-empty closed subset of real numbers T_S is called a time scale. The forward jump operator $\Theta : T_S \to T_S$, backward jump operator $\rho : T_S \to T_S$, and graininess operator $\mu : T_S \to [0,\infty)$, are defined by:

$$\Theta(s) = \inf\{t \in T_S : t > s\}, \quad \rho(s) = \sup\{t \in T_S : t < s\}, \quad \mu(s) = \Theta(s) - s,$$

respectively. An arbitrary $t \in T_S$ is called left scattered (resp. left dense) when $t < \rho(t)$ (resp. $t = \rho(t)$). While, in case of $t < \Theta(t)$ (resp. $\Theta(t) = t$), we call t is right scattered (resp. right dense). For a time scale T_S, the set of all limiting points ${T_S}^z$ is called the derived set and illustrated as follows:

$${T_S}^z = \begin{cases} T_S \backslash (\rho(\sup T_S), \sup T_S], & \text{if } \sup T_S < \infty, \\ T_S, & \text{if } \sup T_S = \infty. \end{cases}$$

The function $\mathcal{W} : T_S \to R$ is called regressive (resp. positively regressive) if $1 + \mu(t)\mathcal{W}(t) \neq 0$, (resp. $1 + \mu(t)\mathcal{W}(t) > 0$) $\forall \ t \in {T_S}^z$. The set of all right-dense continuous regressive functions (resp. right-dense continuous positively regressive functions) will be denoted by $\mathcal{R}_{\mathcal{G}}(T_S)$ (resp. $\mathcal{R}_{\mathcal{G}}(T_S)^+$). The delta derivative of the function $W : T_S \to R$ on $t \in T_S^z$, is given by

$$W^{\Delta}(t) = \lim_{s \to t, \ s \neq \Theta(t)} \frac{W(\Theta(t)) - W(s)}{\Theta(t) - s}.$$

For a rd-continuous function $W : T_S \to R$, the Δ-integral is defined to be

$$\int_a^b W(t)\Delta t = w(b) - w(a), \text{ for all } a, b \in T_S,$$

where w is the anti-derivative of W, i.e., $w^{\Delta} = W$ on ${T_S}^z$.

For $p \in \mathcal{R}_{\mathcal{G}}(T_S)$, the generalized exponential function is defined by

$$e_p(a,b) = \exp\left(\int_a^b \alpha_{\mu(s)} p(s)\Delta s\right) \text{ for all } a, b \in T_S,$$

while

$$\alpha_{\mu(t)} p(t) = \begin{cases} \dfrac{\operatorname{Log}(1 + \mu(t)p(t))}{\mu(t)}, & \text{if } \mu(t) \neq 0, \\ p(t), & \text{if } \mu(t) = 0, \end{cases}$$

is the cylindrical transformation.

The fundamental matrix $\Psi_M(t, t_0)$ is the unique solution of the dynamic equation $\omega^{\Delta}(t) = M(t)\omega(t)$, $\omega(t_0) = \omega_0$, $t \in {T_S}^0$.

3. Basic concepts and remarks

Consider $C({T_S}^0, R^n)$ be the Banach space of continuous functions with norm $\|z\| = \sup_{t\in {T_S}^0} \|z(t)\|$. Let $C(J, R^n)$ (resp. $PC(J, R^n)$) be the Banach space of all continuous functions (resp. the Banach space of piecewise continuous functions) with the norm $\|z\|_\infty = \sup_J \|z(t)\|$, $J = [s_0 - \lambda, t_f] \cap T_S$ and R represents the set of real numbers. Finally, we denote by $PC^1(J, R^n) = \{z \in PC(J, R^n) : z^\Delta \in PC(J, R^n)\}$, the Banach space with norm $\|z\|_1 = \sup\{\|z\|_\infty, \|z^\Delta\|_\infty\}$. Here, we denote by $\|x\| = \sum_{i=1}^n |x_i|$ for $x = (x_1, \dots, x_n) \in R^n$. Consider the following inequalities

$$\left\|y^\Delta(t) - A(t)y(t) - \int_{t_0}^t \mathcal{K}(t, s, y(s))\Delta s\right\| \leqslant \epsilon;\ t \in {T_S}^0, \tag{3.1}$$

$$\left\|y^\Delta(t) - A(t)y(t) - \int_{t_0}^t \mathcal{K}(t, s, y(s))\Delta s\right\| \leqslant \varphi(t);\ t \in {T_S}^0, \tag{3.2}$$

$$\begin{cases} \left\|\phi^\Delta(t) - M(t)\phi(t) - \int_{t_0}^t \mathcal{K}(t, s, \phi(s), \phi(h(s)))\Delta s\right\| \leqslant \epsilon,\ t \in (s_i, t_{i+1}] \cap T_S,\ i = 0, 1, \dots, m, \\ \left\|\phi(t) - g_i(t, \phi(t), \phi(h(t)))\right\| \leqslant \epsilon,\ t \in (t_i, s_i] \cap T_S,\ i = 1, 2, \dots, m, \end{cases} \tag{3.3}$$

$$\begin{cases} \left\|\phi^\Delta(t) - M(t)\phi(t) - \int_{t_0}^t \mathcal{K}(t, s, \phi(s), \phi(h(s)))\Delta s\right\| \leqslant \varphi(t),\ t \in (s_i, t_{i+1}] \cap T_S,\ i = 0, 1, \dots, m, \\ \left\|\phi(t) - g_i(t, \phi(t), \phi(h(t)))\right\| \leqslant \kappa,\ t \in (t_i, s_i] \cap T_S,\ i = 1, 2, \dots, m, \end{cases} \tag{3.4}$$

where $\epsilon > 0$, $\kappa \geqslant 0$ and $\varphi \in PC(J, R^+)$ is an increasing function.

Definition 3.1. Equation (1.1) is Hyers-Ulam stable on ${T_S}^0$ if for every $y \in C({T_S}^0, R^n)$ satisfying (3.1), there exists a solution $y_0 \in C({T_S}^0, R^n)$ of (1.1) with $\|y_0(t) - y(t)\| \leqslant K\epsilon$, $K > 0$, $\forall\, t \in {T_S}^0$.

Definition 3.2. Equation (1.1) is Hyers-Ulam-Rassias stable on ${T_S}^0$ if for every $y \in C({T_S}^0, R^n)$ satisfying (3.2), there exists a solution $y_0 \in C({T_S}^0, R^n)$ of (1.1) with $\|y_0(t) - y(t)\| \leqslant K\varphi(t)$, $K > 0$, $\forall\, t \in {T_S}^0$.

Definition 3.3. Equation (1.2) is said to be stable in the sense of Hyers-Ulam, if for every $\epsilon > 0$ and $\phi \in PC^1(J, R^n)$ satisfying (3.3), there exists a solution $\phi_0 \in PC^1(J, R^n)$ of (1.2) such that $\|\phi_0(t) - \phi(t)\| \leqslant K\epsilon$ for all $t \in J$. Here K is a positive number that depends on ϵ.

Definition 3.4. Equation (1.2) is said to be stable in the sense of Hyers-Ulam-Rassias, provided for every $(\varphi, \kappa) \in PC(J, R^+) \times R^{\geqslant 0}$ and for each $\phi \in PC^1(J, R^n)$ satisfying (3.4), there exists a solution $\phi_0 \in PC^1(J, R^n)$ of (1.2) such that the inequality $\|\phi_0(t) - \phi(t)\| \leqslant M\varphi(t)$ is true for all $t \in J$. Here $M > 0$ depends on (φ, κ).

Definition 3.5. In a metric space $(X; d)$, a mapping $\Lambda : X \to X$ is said to be Picard operator if it has precisely a unique fixed point $x^* \in X$, so that for every $x \in X$, the sequence $\{\Lambda^{(n)}(x)\}_{n\in N}$ converges to x^*.

Lemma 3.6 ([8, Grönwall's inequality, Corollary 6.7])**.** *Let y be the rd-continuous function, $p \in \mathcal{R}_\mathcal{G}(\mathcal{T}_\mathcal{S})^+$, $p \geqslant 0$ and $\alpha \in \mathbb{R}$. Then*

$$y(t) \leqslant \alpha + \int_{t_0}^t y(u)p(u)\Delta u,\ \forall\, t \in \mathcal{T}_\mathcal{S},$$

implies

$$y(t) \leqslant \alpha e_p(t, t_0),\ \forall\, t \in \mathcal{T}_\mathcal{S}.$$

Lemma 3.7 ([20])**.** *Suppose $\tau \in T_S^+$, y, $b \in \mathcal{R}_\mathcal{G}({T_S}^+)$, $p \in \mathcal{R}_\mathcal{G}({T_S}^+)^+$ and c, $b_k \in R^+$, $k = 1, 2, \dots,$ so*

$$y(t) \leqslant c + \int_\tau^t p(s)y(s)\Delta s + \sum_{\tau<t_k<t} b_k y(t_k),$$

implies

$$y(t) \leqslant c \prod_{\tau<t_k<t} (1 + b_k)e_p(t, \tau),\ t \geqslant \tau.$$

Lemma 3.8 ([26, Abstract Grönwall lemma]). *Let $(X, d, \leqslant)$ be an ordered metric space and let x^* be a fixed point for the increasing mapping $\Lambda : X \to X$. So, being arbitrary $x \in X$, $x \leqslant \Lambda(x)$ entails $x \leqslant x^*$ and $x \geqslant \Lambda(x)$ entails $x \geqslant x^*$, where x^* denotes the fixed point in Λ.*

Remark 3.9. A function $y \in C({T_S}^0, R^n)$ satisfies (3.1) if and only if there is a function $h \in C({T_S}^0, R^n)$ such that $\|h(t)\| \leqslant \epsilon$ for all $t \in {T_S}^0$ and

$$y^{\Delta}(t) = A(t)y(t) + \int_{t_0}^{t} \mathcal{K}(t, s, y(s))\Delta s + h(t), \; y(t_0) = y_0.$$

We do similar remark for (3.2).

Lemma 3.10. *Every $y \in C({T_S}^0, R^n)$ that satisfies (3.1) also comes out perfect on the following inequality*

$$\left\| y(t) - \Psi_A(t, t_0)y_0 - \int_{t_0}^{t} \Psi_A(t, \Theta(s)) \int_{t_0}^{s} \mathcal{K}(s, u, y)\Delta u \Delta s \right\| \leqslant C(t_f - t_0)\epsilon,$$

for $t \in {T_S}^0$. Here C is the bound of fundamental matrix $\Psi_A(t, \Theta(s))$.

Proof. If $y \in C({T_S}^0, R^n)$ satisfies (3.1), then by Remark 3.9, we have

$$y^{\Delta}(t) = A(t)y(t) + \int_{t_0}^{t} \mathcal{K}(t, s, y(s))\Delta s + h(t), \; y(t_0) = y_0.$$

Then

$$y(t) = \Psi_A(t, t_0)y_0 + \int_{t_0}^{t} \Psi_A(t, \Theta(s)) \int_{t_0}^{s} \mathcal{K}(s, u, y)\Delta u \Delta s + \int_{t_0}^{t} \Psi_A(t, \Theta(s))h(s)\Delta s.$$

So,

$$\begin{aligned} \left\| y(t) - \Psi_A(t, t_0)y_0 - \int_{t_0}^{t} \Psi_A(t, \Theta(s)) \int_{t_0}^{s} \mathcal{K}(s, u, y)\Delta u \Delta s \right\| &\leqslant \int_{t_0}^{t} \|\Psi_A(t, \Theta(s))\| \|h(s)\| \Delta s \\ &\leqslant C(t - t_0)\epsilon \leqslant C(t_f - t_0)\epsilon. \end{aligned}$$

□

We have similar remarks for (3.2).

Remark 3.11. A function $\phi \in PC^1(J, R^n)$ satisfies inequality (3.3) (resp. inequality (3.4)) if and only if there exist a function $f \in PC^1(J, R^n)$ and a finite sequence $\{f_k : k = 1, \ldots, m\} \subset R^n$ (dependent on ϕ) such that $\|f(t)\| \leqslant \epsilon$ for all $t \in J$ and $\|f_i\| \leqslant \epsilon$ (resp. $\|f_i\| \leqslant \kappa$) for every $i = 1, 2, \ldots, m$ and

$$\begin{cases} \phi^{\Delta}(t) = M(t)\phi(t) + \int_{t_0}^{t} \mathcal{K}(t, s, \phi(s), \phi(h(s)))\Delta s + f(t), \; t \in (s_i, t_{i+1}] \cap T_S, \; i = 0, 1, \ldots, m, \\ \phi(t) = g_i(t, \phi(t), \phi(h(t))) + f_i, \; t \in (t_i, s_i] \cap T_S, \; i = 1, 2, \ldots, m. \end{cases}$$

Lemma 3.12. *If $\phi \in PC^1(J, R^n)$ satisfies inequality (3.3) (resp. inequality (3.4)), then the following inequalities*

$$\begin{cases} \left\| \phi(t) - \phi_0 - \Psi_M(t, t_0)\phi_0 - \int_{s_i}^{t} \Psi_M(t, \Theta(s)) \int_{s_0}^{s} \mathcal{K}(s, u, \phi(u), \phi(h(u)))\Delta u \Delta s - g_i(t, \phi(t), \phi(h(t))) \right\| \\ \quad \leqslant (Ct_f - Cs_i + m)\epsilon, \; t \in (s_i, t_{i+1}] \cap T_S, \; i = 1, 2, \ldots, m, \\ \left\| \phi(t) - g_i(t, \phi(t), \phi(h(t))) \right\| \leqslant m\epsilon, \; (\text{resp. } m\kappa), \; t \in (t_i, s_i] \cap T_S, \; i = 1, 2, \ldots, m, \end{cases}$$

are true. Here C is the bound of fundamental matrix $\Psi_M(t, \Theta(s))$.

Proof. If $\phi \in PC^1(J, R^n)$ satisfies (3.3), then by Remark 3.11, we have

$$\begin{cases} \phi^{\Delta}(t) = M(t)\phi(t) + \int_{t_0}^{t} \mathcal{K}(t, s, \phi(s), \phi(h(s)))\Delta s + f(t), \ t \in (s_i, t_{i+1}] \cap T_S, \ i = 0, 1, \dots, m, \\ \phi(t) = g_i(t, \phi(t), \phi(h(t))) + f_i, \ t \in (t_i, s_i] \cap T_S, \ i = 1, 2, \dots, m. \end{cases} \tag{3.5}$$

Clearly the solution of (3.5) is given as

$$\phi(t) = \begin{cases} \phi_0 + \Psi_M(t, t_0)\phi_0 + \int_{s_i}^{t} \Psi_M(t, \Theta(s)) \Big(\int_{s_0}^{s} \mathcal{K}(s, u, \phi(u), \phi(h(u)))\Delta u + f(s) \Big) \Delta s + g_i(t, \phi(t), \phi(h(t))), \\ \quad t \in (s_i, t_{i+1}] \cap T_S, \ i = 1, 2, \dots, m, \\ g_i(t, \phi(t), \phi(h(t))) + f_i, \ t \in (t_i, s_i] \cap T_S, \ i = 1, 2, \dots, m. \end{cases}$$

For $t \in (s_i, t_{i+1}] \cap T_S, \ i = 1, 2, \dots, m$, we get

$$\begin{aligned} &\Big\| \phi(t) - \phi_0 - \Psi_M(t, t_0)\phi_0 - \int_{s_i}^{t} \Psi_M(t, \Theta(s)) \int_{s_0}^{s} \mathcal{K}(s, u, \phi(u), \phi(h(u)))\Delta u \Delta s - g_i(t, \phi(t), \phi(h(t))) \Big\| \\ &\leqslant \int_{s_i}^{t} \|\Psi_M(t, \Theta(s))\| \|f(s)\| ds + \sum_{i=1}^{m} \|f_i\| \leqslant (Ct - Cs_i + m)\epsilon \leqslant (Ct_f - Cs_i + m)\epsilon. \end{aligned}$$

Proceeding as above we derive

$$\Big\| \phi(t) - g_i(t, \phi(t), \phi(h(t))) \Big\| \leqslant m\epsilon, \ t \in (t_i, s_i] \cap T_S, \ i = 1, 2, \dots, m.$$

We have similar processions for (3.4). □

4. Hyers-Ulam stability of equation (1.1)

Now we are going to give our result on Hyers-Ulam stability. First we assume some of the following conditions:

(C_1) the function $\mathcal{K}$ is continuous with the Lipschitz condition $\|\mathcal{K}(t, s, x_1) - \mathcal{K}(t, s, x_2)\| \leqslant L_k \|x_1 - x_2\|$, $L_k > 0$, for $t_0 \leqslant s \leqslant t \leqslant t_f$ and for all $x_1, x_2 \in R^n$;

(C_2) $\sup_{t \in {T_S}^0} \int_{t_0}^{t} \|\Psi_A(t, \Theta(s))\| \int_{t_0}^{s} L_k \Delta u \Delta s < 1$;

(C_3) $\varphi \in C({T_S}^0, R^n)$ is an increasing function such that for some $\rho > 0$

$$\int_{t_0}^{t} \varphi(s)\Delta s \leqslant \rho\varphi(t).$$

Theorem 4.1. *If conditions* (C_1)-(C_2) *hold, then equation* (1.1) *has precisely a unique solution in* $C({T_S}^0, R^n)$.

Proof.

i) Define an operator $\Lambda : C({T_S}^0, R^n) \to C({T_S}^0, R^n)$ by

$$(\Lambda z)(t) = \Psi_A(t, t_0)z_0 + \int_{t_0}^{t} \Psi_A(t, \Theta(s)) \int_{t_0}^{s} \mathcal{K}(s, u, z)\Delta u \Delta s.$$

Now for any $z_1, z_2 \in C({T_S}^0, R^n)$, we have

$$\Big\| (\Lambda z_1)(t) - (\Lambda z_2)(t) \Big\| = \Big\| \int_{t_0}^{t} \Psi_A(t, \Theta(s)) \int_{t_0}^{s} (\mathcal{K}(s, u, z_1) - \mathcal{K}(s, u, z_2))\Delta u \Delta s \Big\|$$

$$\leqslant \sup_{t\in T_S{}^0} \int_{t_0}^{t} \|\Psi_A(t,\Theta(s))\| \int_{t_0}^{s} \|(\mathcal{K}(s,u,z_1)-\mathcal{K}(s,u,z_2))\|\Delta u\Delta s$$

$$\leqslant \sup_{t\in T_S{}^0} \int_{t_0}^{t} \|\Psi_A(t,\Theta(s))\| \int_{t_0}^{s} L_k\|z_1(u)-z_2(u)\|\Delta u\Delta s$$

$$\leqslant \sup_{t\in T_S{}^0} \|z_1(t)-z_2(t)\| \sup_{t\in T_S{}^0} \int_{t_0}^{t} \|\Psi_A(t,\Theta(s))\| \int_{t_0}^{s} L_k\Delta u\Delta s$$

$$\leqslant \|z_1-z_2\| \sup_{t\in T_S{}^0} \int_{t_0}^{t} \|\Psi_A(t,\Theta(s))\| \int_{t_0}^{s} L_k\Delta u\Delta s.$$

Following from (C_2), the operator is strictly contractive and hence a Picard operator on $C(T_S{}^0, R^n)$. From (3.1), it follows that the unique fixed point of this operator is in fact the unique solution of (1.1) in $C(T_S{}^0, R^n)$. □

Theorem 4.2. *If conditions* (C_1)-(C_2) *hold, then equation* (1.1) *has Hyers-Ulam stability on* $T_S{}^0$.

Proof. Let $y \in C(T_S{}^0, R^n)$ be a solution to (3.1). The unique solution $z \in C(T_S{}^0, R^n)$ of the equation (1.1) is given by

$$z(t) = \Psi_A(t,t_0)z_0 + \int_{t_0}^{t} \Psi_A(t,\Theta(s)) \int_{t_0}^{s} \mathcal{K}(s,u,z)\Delta u\Delta s.$$

Now by using Lemma 3.9,

$$\Big\|y(t)-z(t)\Big\| \leqslant \Big\|y(t)-\Psi_A(t,t_0)y_0 - \int_{t_0}^{t} \Psi_A(t,\Theta(s)) \int_{t_0}^{s} \mathcal{K}(s,u,y)\Delta u\Delta s\Big\|$$
$$+\int_{t_0}^{t} \|\Psi_A(t,\Theta(s))\| \int_{t_0}^{s} \|(\mathcal{K}(s,u,y)-K(s,u,z))\|\Delta u\Delta s$$
$$\leqslant C(t_f-t_0)\epsilon + \int_{t_0}^{t} \|\Psi_A(t,\Theta(s))\| \int_{t_0}^{s} L_k\|y(u)-z(u)\|\Delta u\Delta s.$$

Next, we show that the operator $T : C(T_S{}^0, R^+) \to C(T_S{}^0, R^+)$ given below is an increasing Picard operator,

$$(Tg)(t) = C(t_f-t_0)\epsilon + \int_{t_0}^{t} \|\Psi_A(t,\Theta(s))\| \int_{t_0}^{s} L_k g(u)\Delta u\Delta s. \tag{4.1}$$

For any $g_1, g_2 \in C(T_S{}^0, R^+)$, we have

$$\Big\|(Tg_1)(t)-(Tg_2)(t)\Big\| = \Big\|\int_{t_0}^{t} \|\Psi_A(t,\Theta(s))\| \int_{t_0}^{s} L_k\big(g_1(u)-g_2(u)\big)\Delta u\Delta s\Big\|$$
$$\leqslant \sup_{t\in T_S{}^0} \int_{t_0}^{t} \|\Psi_A(t,\Theta(s))\| \int_{t_0}^{s} L_k\|g_1(u)-g_2(u)\|\Delta u\Delta s$$
$$\leqslant \sup_{t\in T_S{}^0} \|g_1(t)-g_2(t)\| \sup_{t\in T_S{}^0} \int_{t_0}^{t} \|\Psi_A(t,\Theta(s))\| \int_{t_0}^{s} L_k\Delta u\Delta s$$
$$\leqslant \|g_1-g_2\| \sup_{t\in T_S{}^0} \int_{t_0}^{t} \|\Psi_A(t,\Theta(s))\| \int_{t_0}^{s} L_k\Delta u\Delta s.$$

Since $\left(\sup_{t\in T_S{}^0} \int_{t_0}^{t} \|\Psi_A(t,\Theta(s))\| \int_{t_0}^{s} L_k\Delta u\Delta s\right) < 1$, so the operator is contractive on $C(T_S{}^0, R^+)$. Applying Banach contraction principle, T is Picard operator with unique fixed point $g^* \in C(T_S{}^0, R^+)$, i.e.,

$$g^*(t) = C(t_f-t_0)\epsilon + \int_{t_0}^{t} \|\Psi_A(t,\Theta(s))\| \int_{t_0}^{s} L_k g^*(u)\Delta u\Delta s.$$

For some $M_k > 0$, we have $\|\Psi_A(t,\Theta(s))\| = \sup_{t\in T_S{}^0} \|\Psi_A(t,\Theta(s))\| \leqslant M_k$, so

$$g^*(t) \leqslant C(t_f - t_0)\epsilon + \int_{t_0}^{t} M_k \int_{t_0}^{s} L_k g^*(u)\Delta u\Delta s.$$

By Lemma 3.6, we get

$$g^*(t) \leqslant C(t_f - t_0)\epsilon e_P(t,t_0),$$

where $P(s) = \int_{t_0}^{s} L_k M_k \Delta u$. If we set $g(t) = \|y(t) - z(t)\|$, then from (4.1), $g(t) \leqslant (Tg)(t)$ from which by using abstract Grönwall lemma, it follows that $g(t) \leqslant g^*(t)$, thus

$$\Big\|y(t) - z(t)\Big\| \leqslant C(t_f - t_0)\epsilon e_P(t,t_0).$$ □

Similarly, by following the same process, we can prove that:

Theorem 4.3. *If conditions* (C_1)-(C_3) *hold, then equation* (1.1) *has Hyers-Ulam-Rassias stability on* ${T_S}^0$.

5. Hyers-Ulam stability of equation (1.2)

Onward we will state our major results. The first solution to be establish is Hyers-Ulam stability. First we assume some of the following conditions:

(A_1) the function $\mathcal{K}$ is piecewise continuous with the Lipschitz condition $\|\mathcal{K}(t,s,x_1,x_2) - \mathcal{K}(t,s,y_1,y_2)\| \leqslant \sum_{k=1}^{2} L\|x_k - y_k\|$, $L > 0$, for all $t \in (s_i, t_{i+1}] \cap T_S$, $i = 0,1,\dots,m$ and $x_k, y_k \in R^n$, $k \in \{1,2\}$;

(A_2) $g_i : (t_i, s_i] \cap T_S \times R^n \times R^n \to R^n$ satisfies the Lipschitz condition $\|g_i(t,u_1,u_2) - g_i(t,v_1,v_2)\| \leqslant \sum_{k=1}^{2} L_{g_i}\|u_k - v_k\|$, $L_{g_i} > 0$, for all $t \in (t_i, s_i] \cap T_S$, $i = 1,2,\dots,m$ and $u_1,u_2,v_1,v_2 \in R$;

(A_3) $\left(\sum_{0<s_i<t} 2L_i + 2CL\int_{s_i}^{t}\int_{s_0}^{s}\Delta u\Delta s\right) < 1,\ i = 1,2,\dots,m;$

(A_4) $\varphi \in PC(J, R^+)$ is increasing so that for some $\rho > 0$

$$\int_{t_0}^{t} \varphi(r)\Delta r \leqslant \rho\varphi(t).$$

Theorem 5.1. *If conditions* (A_1)-(A_3) *hold, then equation* (1.2) *has precisely a unique solution in* $PC^1(J,\mathbb{R}^n)$.

Proof.

i) Determine an operator $\Lambda : PC(J,R^n) \to PC(J,R^n)$, as

$$(\Lambda\omega)(t) = \begin{cases} \alpha(t),\ t \in [s_0 - \lambda, s_0] \cap T_S, \\ g_i(s_i, \omega(s_i), \omega(h(s_i))),\ t \in (t_i, s_i] \cap T_S,\ i = 1,2,\dots,m,\ \alpha \in (0,1), \\ \alpha(t_0) + \Psi_M(t,t_0)\omega_0 + g_i(s_i,\omega(s_i),\omega(h(s_i))) \\ \quad + \int_{s_i}^{t} \Psi_M(t,\Theta(s)) \int_{s_0}^{s} \mathcal{K}(s,u,\omega(u),\omega(h(u)))\Delta u\Delta s, \\ \quad t \in (s_i, t_{i+1}] \cap T_S,\ i = 1,2,\dots,m,\ \alpha \in (0,1). \end{cases} \tag{5.1}$$

For any ω_1, $\omega_2 \in PC(J,R^n)$, $t \in (s_i, t_{i+1}] \cap T_S$, $i = 1,2,\dots,m$, we have

$$\begin{aligned}\Big\|(\Lambda\omega_1)(t) - (\Lambda\omega_2)(t)\Big\| \leqslant{}& \Big\|g_i(s_i,\omega_1(s_i),\omega_1(h(s_i))) - g_i(s_i,\omega_2(s_i),\omega_2(h(s_i)))\Big\| \\ &+ \int_{s_i}^{t} \Big\|\Psi_M(t,\Theta(s))\Big\| \Big\| \int_{s_0}^{s} \mathcal{K}(s,u,\omega_1(u),\omega_1(h(u)))\Delta u\Delta s\end{aligned}$$

$$
\begin{aligned}
&\quad - \int_{s_0}^{s} \mathcal{K}(s,u,\omega_2(u),\omega_2(h(u)))\Delta u\Delta s\Big\|\\
&\leqslant L_i\|\omega_1(s_i)-\omega_2(s_i)\| + L_i\|\omega_1(h(s_i))-\omega_2(h(s_i))\|\\
&\quad + L\int_{s_i}^{t} ||\Psi_M(t,\Theta(s))|| \int_{s_0}^{s} ||\omega_1(u)-\omega_2(u)||\Delta u\Delta s\\
&\quad + L\int_{s_i}^{t} ||\Psi_M(t,\Theta(s))|| \int_{s_0}^{s} ||\omega_1(h(u))-\omega_2(h(u))||\Delta u\Delta s\\
&\leqslant \sum_{0<s_i<t} L_i\|\omega_1(s_i)-\omega_2(s_i)\| + \sum_{0<s_i<t} L_i\|\omega_1(h(s_i))-\omega_2(h(s_i))\|\\
&\quad + 2CL\int_{s_i}^{t}\int_{s_0}^{s} \|\omega_1-\omega_2\|_\infty \Delta u\Delta s\\
&\leqslant \sum_{0<s_i<t} 2L_i\|\omega_1-\omega_2\|_\infty + 2CL\int_{s_i}^{t}\int_{s_0}^{s} \|\omega_1-\omega_2\|_\infty \Delta u\Delta s\\
&\leqslant \Big(\sum_{0<s_i<t} 2L_i + 2CL\int_{s_i}^{t}\int_{s_0}^{s}\Delta u\Delta s\Big)\|\omega_1-\omega_2\|_\infty.
\end{aligned}
$$

According to (A_3), we are dealing here with the strictly contractive operator on $(s_i, t_{i+1}] \cap T_S$, $i = 1,2,\dots,m$, and hence a Picard operator on $PC(J, R^n)$. Regarding to (5.1), it shows that the unique solution of equation (1.2) in $PC^1(J, R^n)$ is in fact the unique fixed point of this operator. □

Theorem 5.2. *If conditions* (A_1)-(A_3) *hold then equation* (1.2) *has Hyers-Ulam stability on* J.

Proof. Assume that (3.1) has a solution $PC^1(J, R^n)$. Then for dynamic equation (1.2), we have the unique solution

$$
\omega(t) = \begin{cases}
\alpha(t),\ t\in[s_0-\lambda, s_0]\cap T_S,\\
g_i(t,\omega(t),\omega(h(t))),\ t\in(t_i,s_i]\cap T_S,\ i=1,2,\dots,m,\ \alpha\in(0,1),\\
\alpha(t_0)+\Psi_M(t,t_0)\omega_0 + g_i(s_i,\omega(s_i),\omega(h(s_i))) + \int_{s_i}^{t}\Psi_M(t,\Theta(s))\int_{s_0}^{s}\mathcal{K}(s,u,\omega(u),\omega(h(u)))\Delta u\Delta s,\\
\quad t\in(s_i,t_{i+1}]\cap T_S,\ i=1,2,\dots,m,\ \alpha\in(0,1).
\end{cases}
$$

We observe that for all $t\in(s_i,t_{i+1}]\cap T_S$, $i=1,2,\dots,m$, using Lemma 3.12, we have

$$
\begin{aligned}
||\phi(t)-\omega(t)|| &\leqslant ||\phi(t)-\phi_0-\Psi_M(t,t_0)\phi_0 - \int_{s_i}^{t}\Psi_M(t,\Theta(s))\int_{s_0}^{s}\mathcal{K}(s,u,\phi(u),\phi(h(u)))\Delta u\Delta s\\
&\quad - g_i(t,\phi(t),\phi(h(t)))|| + ||g_i(s_i,\phi(s_i),\phi(h(s_i))) - g_i(s_i,\omega(s_i),\omega(h(s_i)))||\\
&\quad + \int_{s_i}^{t}||\Psi_M(t,\Theta(s))||\int_{s_0}^{s}||\mathcal{K}(s,u,\phi(u),\phi(h(u)))-\mathcal{K}(s,u,\omega(u),\omega(h(u)))||\Delta u\Delta s\\
&\leqslant (m+Ct_f-Cs_i)\epsilon + \sum_{0<s_i<t} L_i\|\phi(s_i)-\omega(s_i)\| + \sum_{0<s_i<t} L_i\|\phi(h(s_i))-\omega(h(s_i))\|\\
&\quad + CL\int_{s_i}^{t}\int_{s_0}^{s}||\phi(u)-\omega(u)||\Delta u\Delta s + CL\int_{s_i}^{t}\int_{s_0}^{s}||\phi(h(u))-\omega(h(u))||\Delta u\Delta s.
\end{aligned}
$$

Next, we show that the operator $T: PC(J,R^+)\to PC(J,R^+)$ given below is an increasing Picard operator:

$$
(Tg)(t) = (m+Ct_f-Cs_i)\epsilon + \sum_{0<s_i<t} L_i g(s_i) + \sum_{0<s_i<t} L_i g(h(s_i))
$$

$$+ CL\int_{s_i}^{t}\int_{s_0}^{s} g(u)\Delta u\Delta s + CL\int_{s_i}^{t}\int_{s_0}^{s} g(h(u))\Delta u\Delta s.$$

For any $g_1, g_2 \in PC(J, R^+)$, $t \in (s_i, t_{i+1}] \cap T_S$, $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}\big\|(Tg_1)(t) - (Tg_2)(t)\big\| &\leqslant \sum_{0<s_i<t} L_i\|g_1(s_i) - g_2(s_i)\| + \sum_{0<s_i<t} L_i\|g_1(h(s_i)) - g_2(h(s_i))\| \\ &\quad + CL\int_{s_i}^{t}\int_{s_0}^{s} \big\|g_1(u) - g_2(u)\big\|\Delta u\Delta s + CL\int_{s_i}^{t}\int_{s_0}^{s} \big\|g_1(h(u)) - g_2(h(u))\big\|\Delta u\Delta s \\ &\leqslant \sum_{0<s_i<t} 2L_i\|g_1 - g_2\|_\infty + 2CL\int_{s_i}^{t}\int_{s_0}^{s} \|g_1 - g_2\|_\infty \Delta u\Delta s \\ &\leqslant \Big(\sum_{0<s_i<t} 2L_i + 2CL\int_{s_i}^{t}\int_{s_0}^{s} \Delta u\Delta s\Big)\|g_1 - g_2\|_\infty.\end{aligned}$$

Again according to (A_3), we are dealing here with the strictly contractive operator on $(s_i, t_{i+1}] \cap T_S$, $i = 1, 2, \ldots, m$ and hence a Picard operator on $PC(J, R^+)$. Banach fixed point theorem imply, T is Picard operator having unique fixed point $g^* \in PC(J, R^+)$, i.e.,

$$\begin{aligned}g^*(t) = (m + Ct_f - Cs_i)\epsilon + \sum_{0<s_i<t} L_i g^*(s_i) + \sum_{0<s_i<t} L_i g^*(h(s_i)) \\ + CL\int_{s_i}^{t}\int_{s_0}^{s} g^*(u)\Delta u\Delta s + CL\int_{s_i}^{t}\int_{s_0}^{s} g^*(h(u))\Delta u\Delta s.\end{aligned}$$

As, g^* is increasing, therefore $g^*(h(t)) \leqslant g^*(t)$, further we can write

$$g^*(t) \leqslant (m + Ct_f - Cs_i)\epsilon + \sum_{0<s_i<t} 2L_i g^*(s_i) + 2CL\int_{s_i}^{t}\int_{s_0}^{s} g^*(u)\Delta u\Delta s.$$

Using Lemma 3.7, we have

$$g^*(t) \leqslant (m + Ct_f - Cs_i)\epsilon \prod_{0<s_i<t}(1 + 2L_i)e_q(t, s_i),$$

where $q = 2CL\int_{s_0}^{s}\Delta u$. If we determine $g = \|\phi - \omega\|$, then $g(t) \leqslant (Tg)(t)$, which follows by utilizing abstract Grönwall lemma that $g(t) \leqslant g^*$, hence

$$\|\phi(t) - \omega(t)\| \leqslant (m + Ct_f - Cs_i)\epsilon \prod_{0<s_i<t}(1 + 2L_i)e_q(t, s_i). \qquad \square$$

Similarly we can establish the Hyers-Ulam-Rassias stability of (1.2) on J. Its proof will be omitted.

Theorem 5.3. *If conditions* (A_1)-(A_4) *hold, then, equation* (1.2) *has Hyers-Ulam-Rassias stability on* J.

Example 5.4. Consider the following semilinear Volterra integro-dynamic equation:

$$\omega^\Delta(t) = (t-2)\omega(t) + \int_{t_0}^{t} e_p(t, \omega(s))\Delta s,\ \omega(0) = 1, \quad t \in [0, 3]_{T_S}, \tag{5.2}$$

and its associated inequality

$$\Big|\phi^\Delta(t) - (t-2)\phi(t) - \int_{t_0}^{t} e_p(t, \phi(s))\Delta s\Big| \leqslant 1.5,\ t \in [0, 3]_{T_S}. \tag{5.3}$$

Setting $p(t) = (t-2)$ and $K(t,s,\omega(s)) = e_p(t,\omega(s)) = e_p(t,s)e_p(s,\omega(s))$ for $t \in T_S$ and put $\epsilon = 1.5$. If $\varphi \in P^1_C([0,3]_{T_S}, R)$ satisfies the inequality (5.3), then there exists $f \in P^1_C([0,3]_{T_S}, R)$ such that $|f(t)| \leqslant 1.5$ for $t \in T_S$. So we have

$$\varphi^{\Delta}(t) = (t-2)\varphi(t) + \int_{t_0}^{t} e_p(t,\varphi(s))\Delta s + f(t), \quad t \in T_S,$$

and the solution of Eq. (5.2) is given as

$$\omega(t) = e_p(t,0) + \int_0^t e_p(t,\Theta(s)) \int_0^s e_p(s,\omega(u))\Delta u \Delta s.$$

Based on our theoretical results, Eq. (5.2) has a unique solution in $P^1_C([0,3]_{T_S}, R)$ and is Hyers-Ulam stable on $[0,3]_{T_S}$.

6. Conclusion

This manuscript is about the establishment of Hyers-Ulam stability and Hyers-Ulam-Rassias stability of (1.1) and (1.2) with the utilization of fixed point approach. Furthermore, abstract Grönwall lemma, Lemma 3.6, and Lemma 3.7 presented a fruitful outcome to our end. Our work assures the existence of an exact solutions of (1.1) and (1.2) near to approximate solution. We added an example to show the validity of our results. In fact, our results are significant when finding exact solution is quite difficult and hence are important in approximation theory etc.

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